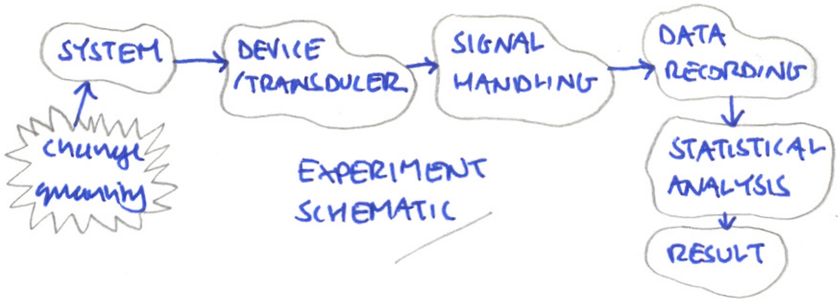
EXPERIMENTAL METHODS

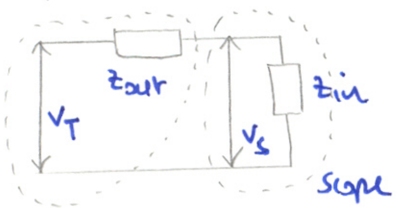
im to measure / observe a ystem quantity without disrupting e system by the measurement rocess.



EXPERIMENT SCHEMATIC

Many systems convert physical system effects into electrical signals. A TRANSDUCER performs the conversion and the signals are usually recieved by some sort of scope – i.e CRO. Output impedance of transducer (Z_{out}) and input impedance of scope (Z_{in}) must be chosen s.t the transducer produced signal is unperturbed.

$V = IZ$



Transducer

By ohm's law*, current conservation

$$\boxed{V_S = V_T \frac{Z_{in}}{Z_{in} + Z_{out}}}$$

So for ideal measurement ($V_S = V_T$)

$\Rightarrow$ Z_{in} V. HIGH Z_{out} V. LOW

For CURRENT measurement: least disturbs system if device takes ALL current. i.e LOW Z_{in}.

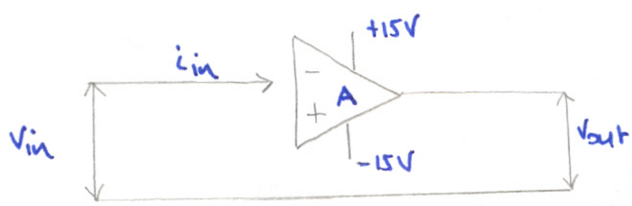
For POWER transfer: max power transfer if $Z_{in} = Z_{out}$. (compare $V_T I$ to $I^2 Z_{in}$)

input power

'Dissipated' power. were with complex Z.

Op-Amps are used to amplify signals and form the basis of useful circuits.

General



$\boxed{V_{out} = AV_{in}}$ ← open loop.

[A -ve if input to '–'
A +ve if input to '+']

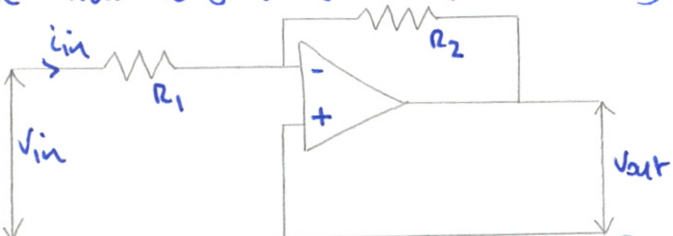
$\boxed{\text{Gain} = \left| V_{out}/V_{in} \right|}$ ← general.

Ideal Op-Amp

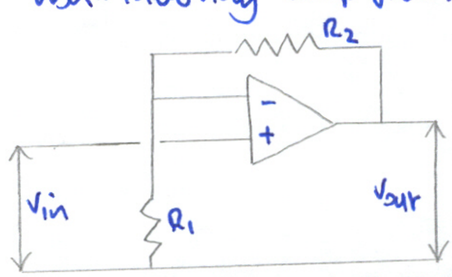
* Inputs draw no current (V. high Z_{in})
* With output shorted can draw lots of current ($Z_{out} \sim 0$)
* Voltages on +, – terminals must be equal.

[$V_{out} \leq 30V$ in above case, $A \sim \infty$ (or V. high) so for V_{out} not to saturate $V_+ - V_- \sim 0$ (= V_{in} in open loop example)] ← NEGATIVE FEEDBACK allows this to occur.

Use these rules to find gain for inverting and non-inverting amplifiers. (ideal).



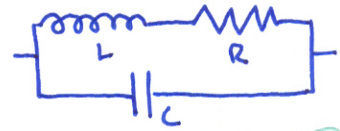
INVERTING AMPLIFIER $\boxed{\frac{V_{out}}{V_{in}} = -\frac{R_2}{R_1}}$

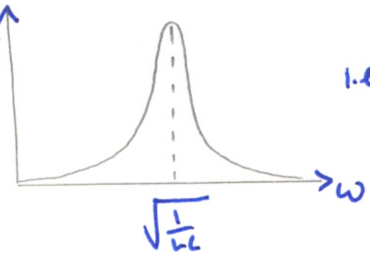


$\boxed{\frac{V_{out}}{V_{in}} = 1 + \frac{R_2}{R_1}}$

NON INVERTING AMPLIFIER

If R_1, R_2 are replaced by more general COMPLEX impedances - i.e from an LCR array the inverting and non inverting circuitry above becomes frequency dependent. So if an AC input is used a circuit could be designed to have high gain at a specific frequency range - i.e a filter.

e.g, for the inverting amplifier - if $R_2 \rightarrow$  $\left[\text{i.e, } Z_2 = \frac{1}{\frac{1}{R+i\omega L}+i\omega C}\right]$

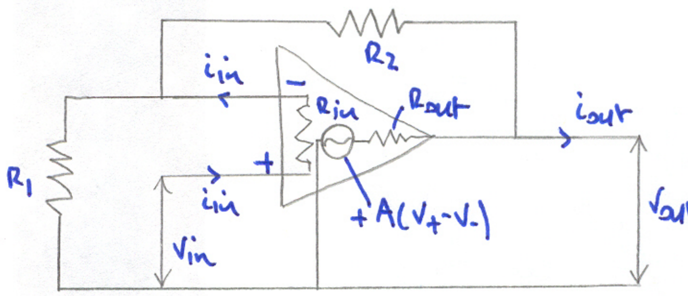
then: $\frac{-V_{out}}{V_{in}}$ 

i.e, a 'gain resonance' at $\omega_0 = \sqrt{\frac{1}{LC}}$. ← often crops up.

This filter picks out a desired frequency at ω_0 from a noise / frequency spectrum.

For REAL op-amps: * A not ∞ but $10^4 - 10^6$ * Z_{in} not ∞ but high * Z_{out} not 0 but low * A may be frequency dependent

Non inverting amplifier becomes:



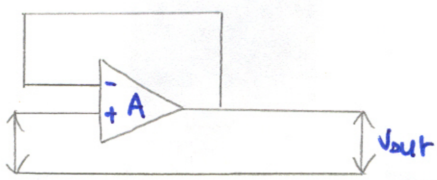
For $A \to \infty$, $Z_{in} \to \infty$, $Z_{out} \to 0$

gain $\rightarrow 1 + R_2/R_1$ as in 'ideal' case.

If $R_2/R_1 = 0$ (well R_1 and $R_2 = 0$)

$\rightarrow$ unity gain buffer.

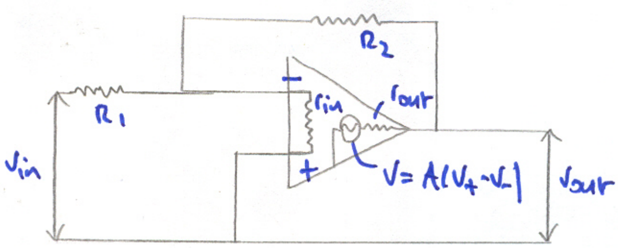
$V_{in} = V_{out}$



Good device for connecting circuitry - no 'effective' impedance as $V_{in} = V_{out}$

Note// $Z_{in} = r_{in} A$ and $Z_{out} = r_{out}/A$ - so will draw current.

Inverting amplifier becomes:



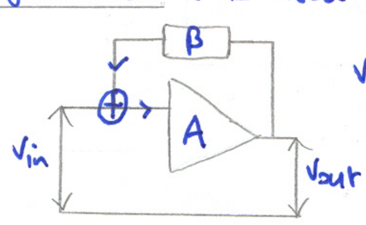
In ideal limit gain $\rightarrow -R_2/R_1$ as before.

[Note - In non ideal case we can only conserve current and use Ohm's law - $V_+ = V_-$ no longer applies!]

Overview of negative feedback (As used in above examples)

Idealised system:

[β is fraction feedback $\rightarrow$ -ve for -ve feedback]



$V_{out} = A(V_{in} + \beta V_{out}) \Rightarrow$ Gain $= \frac{V_{out}}{V_{in}} = \frac{A}{1-A\beta}$

$= \frac{1}{\frac{1}{A} - \beta}$ So if $A \gg 1 \Rightarrow$ Gain $= -\frac{1}{\beta}$

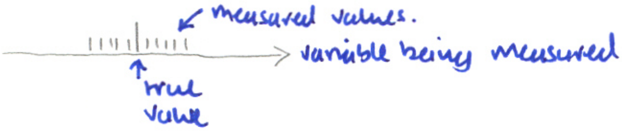
i.e independent of A.

$\rightarrow \beta$ function of components used i.e for non inverting amplifier: $\beta = -\frac{R_1}{R_1 + R_2}$

If $\beta > 0$ - +ve feedback. If $A\beta \rightarrow 1$ unstable output as gain $\rightarrow \infty$. If β is frequency dependent this can be used to create oscillations as the op-amp saturates periodically.

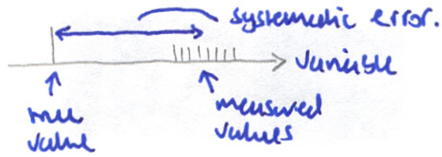
Types of experimental error: Random and systematic

Repeating a single measurement N times will reduce the random error in a measurement. i.e. the deviation around the true value caused by various random perturbations.



Random errors can be treated statistically easily.

Systematic errors are experimentally induced 'offsets' from the true value. In these cases the random error cluster will be offset from the true value $\Rightarrow$ mean $\neq$ true value as $N \to \infty$.



Systematics can only be minimised by careful consideration of experimental procedure.

For random errors, often use that true value $= \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i$. $\frac{1}{N}\sum_{i=1}^{N} x_i = \bar{x}$, the mean of discrete variable x.

For finite data sets 'true value' is approximated by mean $\pm$ error in mean.

error in mean, σ_m is given by $\frac{\sigma}{\sqrt{N}}$, N is number of measurements, σ is error in each measurement

Now σ can be evaluated by using the Standard Deviation for a data set without explicit errors. $\sigma = \sqrt{V(x)} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} x_i^2 - \bar{x}^2} \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}$

In this case $\sigma_m = \frac{\sigma}{\sqrt{N-1}}$ since the mean is used in calculating σ.

Note, $V(x)$ (= 'variance') obeys following result for more than one independent quantity (i.e. $x, y, z \ldots$)

$$V(x+y+z\ldots) = V(x) + V(y) + V(z) + \ldots$$

Proof, $V(x+y+z\ldots) = \frac{1}{N}\sum_{i=1}^{N}(x_i + y_i + z_i + \ldots)^2 - \bar{x}^2 - \bar{y}^2 - \bar{z}^2 \ldots$

$V(x) + V(y) + V(z) \ldots = \frac{1}{N}\sum_{i=1}^{N}(x_i^2 + y_i^2 + z_i^2 + \ldots) - \bar{x}^2 - \bar{y}^2 - \bar{z}^2$

So if $V(x+y+z+\ldots) = V(x) + V(y) + V(z) \ldots \Rightarrow \sum_{i=1}(x_i^2 + y_i^2 + z_i^2 \ldots) = \sum_{i=1}(x_i + y_i + z_i + \ldots)^2$

$\Rightarrow$ cross terms, $\sum_{ijk\ldots} x_i y_j z_k \ldots = 0$. Now this is true if $x, y, z \ldots$ are independent. Hence lemma is proven. We can use this proof to prove $\sigma_m = \frac{\sigma}{\sqrt{N}}$.

(1) $V(x)$ (sum of measurements) $= \sum$ (variance for each measurement)

$\quad = \text{Var}(N \cdot \text{mean}) = N^2 \text{var}(\text{mean}) = N^2 \cdot \text{error}^2(\text{mean})$

$\quad = N \times$ average variance for each measurement $= N \cdot$ average error^2 for each measurement

$\Rightarrow$ error in mean $= \dfrac{\text{error in each measurement}}{\sqrt{N}}$

Note errors must be random for measured variable x.

Combination of random errors / propagation of random errors in formulae

For a function f of independent random variables $x, y, z \ldots$ which have Gaussian distributed errors $\sigma_x, \sigma_y, \sigma_z \ldots$

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial f}{\partial z}\right)^2 \sigma_z^2 + \ldots$$

So if f is calculated from measurements $x = \bar{x} \pm \sigma_x$, $y = \bar{y} \pm \sigma_y$, $z = \bar{z} \pm \sigma_z, \ldots$

$\Rightarrow$ $f = f(\bar{x}, \bar{y}, \bar{z} \ldots) \pm \sigma_f$. Now if f is complicated the approximation:

[$\frac{\partial f}{\partial \ldots}$ can be estimated using this method]
$$\frac{\partial f}{\partial x} \approx \frac{f(\bar{x} + \sigma_x, \bar{y}, \bar{z} \ldots) - f(\bar{x}, \bar{y}, \bar{z} \ldots)}{\sigma_x}$$
can be useful.

If $f = x^a y^b z^c \ldots\ldots$ for some indexes $a, b, c \ldots\ldots$ $\{a, b, c \ldots\} \in \mathbb{R}$

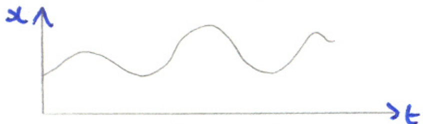
Above result $\Rightarrow$
$$\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{a\sigma_x}{x}\right)^2 + \left(\frac{b\sigma_y}{y}\right)^2 + \left(\frac{c\sigma_z}{z}\right)^2 + \ldots$$

And if $f = x + y + z + \ldots$, $\sigma_f^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 \ldots\ldots$

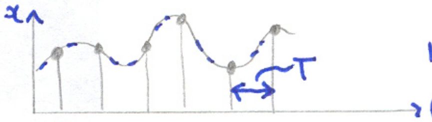
Deals with most functions

Digital sampling of analogue signals

— Desirable for storage (i.e. digital devices vs non-linear magnetic tape — price, preservability, durability....)

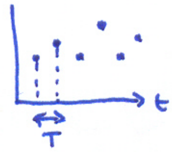


original, analogue signal.



sampled signal (sampling frequency $1/T$).

i.e. $s(t) =$



Now errors associated with sampling are two fold: * Digitisation error (— increase reproduction ↑ quality by increasing sampling rate. More measurements, less error i.e. $1/\sqrt{N}$)

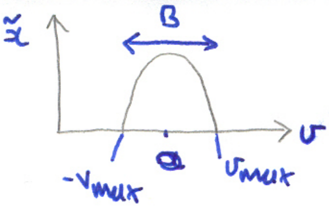
* Sampling error. (Higher frequency Fourier components of $x(t)$ than sampling frequency are removed i.e. ∿∿ looks like ∼ for ∿ sampling)

For DIGITISATION errors — error reduces by $\sqrt{N}$, i.e. take more measurements ($\Rightarrow T \to 0$) and signal quality (sampled) increases

For SAMPLING errors — obey NYQUIST'S CRITERION. i.e. sample $\geq 2\times$ bandwidth of frequency spectrum (at least). To be safe sample $\geq 2\times$ highest frequency Fourier component. Proof// sampled signal is $x(t) \times \ldots$ ⊥⊥⊥⊥⊥⊥ $\ldots$ ← set of δ functions (T)

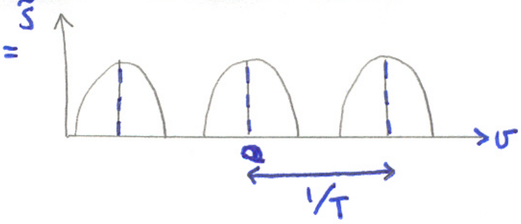
Now $FT(s(t)) = FT(x(t) \times \delta \text{ function array}) = FT(x(t)) * FT(\delta \text{ function array})$ by CONVOLUTION THEOREM. Now $FT(x(t))$ is the frequency spectrum and $FT(\delta \text{ function array}) =$ array of δ functions with spacing $1/T$.

↓ cont...

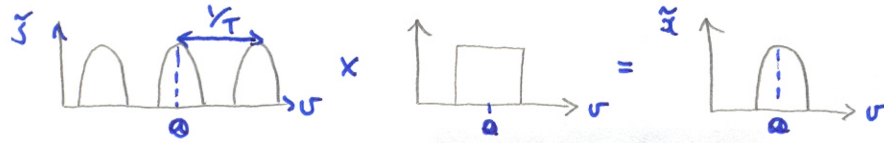
i.e. FT(x(t)) = 

[B = Bandwidth]

Nyquist's theorem cont....

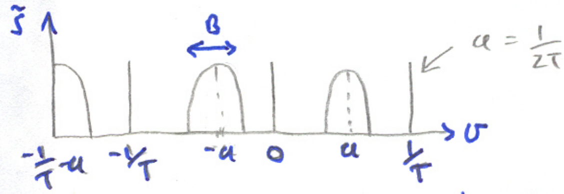
Hence FT(S(t)) = 

Now to recover FT(x(t)) from $\tilde{S}$ evaluate $\tilde{S}$ × top hat (width $1/T$)

i.e. 

So $FT^{-1}(\tilde{x}(t)) = x(t)$

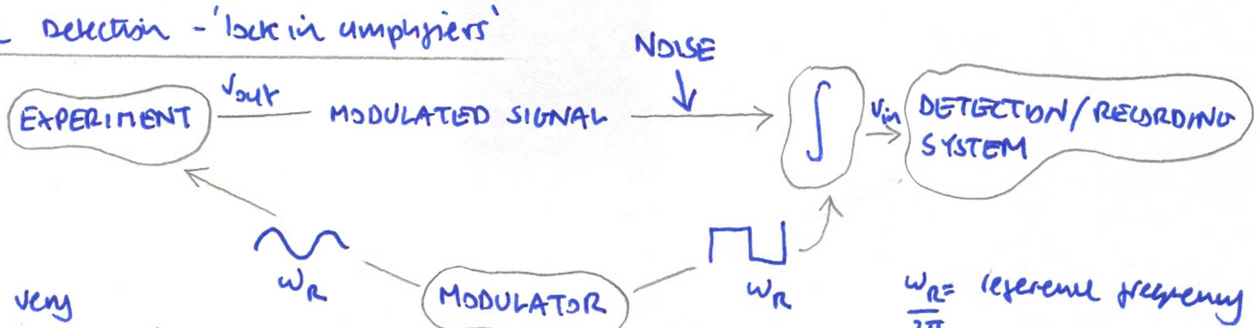
Now last stage can be done electronically but errors will creep in if $\tilde{S}$ 'convolution image' has overlapping $\tilde{x}$ spectra. Clearly this will occur when $1/T < 2\nu_{max}$. So Nyquist's criterion for minimum sampling is justified.

Now if $\tilde{x}$ is not centered on 0 but a Hz we can be more subtle. $\tilde{S}$ convolution image is: 

Now when $\frac{1}{T} \leq 2B$ the images of $\tilde{x}$ will start to overlap - hence 'desampled' x(t) will not look like original. Hence sampling rate $\geq 2B$ is absolute minimum if sampling errors are to be avoided.

Phase sensitive detection - 'lock in amplifiers'

Schematic



$\frac{\omega_R}{2\pi}$ = reference frequency.

Now a very good V_{out} reproduction is V_{in} because since the noise enters $\int$ as random perturbations over the modulated signal - integrator 'filters' these out to zero.

↑ well reference oscillator - modulator/mixers are at 'experiment' and before integrator.

Net integrated result is the integral of the sine modulated signal × square wave which is a function of V_{out} (and not noise) only.

ϕ is phase of ∿ with V_{out}.

e.g. for DC V_{out}: $$V_{in} = \frac{1}{T}\left[\int_0^{\pi/\omega} V_{out}\sin(\omega_R t + \phi)\,dt - \int_{\pi/\omega}^{2\pi/\omega} V_{out}\sin(\omega_R t+\phi)\,dt\right]$$

Noise terms cancel.

$$= \frac{1}{T}\frac{V_{out}}{\omega}\left\{\left[\cos(\omega_R t + \phi)\right]_0^{\pi/\omega} - \left[-\cos(\omega_R t+\phi)\right]_{\pi/\omega}^{2\pi/\omega}\right\} = \frac{2}{\pi}V_{out}\cos\phi$$

Now if $V_{out} = Ae^{i\omega_s t} \Rightarrow \omega_s t$ is the equivalent ϕ so $\frac{2|V_{out}|}{\pi}\cos(\omega_s t) = V_{in}$ (A)

when noise is present or not. So not only amplitude of V_{out} is recovered but it's 'phase' as well. So $V_{out} = \frac{\pi V_{in}}{2\cos(\omega_s t)} e^{i\omega_s t}$ ← ω_s can be found from period of V_{in}.

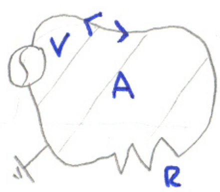
↑ well frequency!

Filtering and shielding to remove noise

* Circuit filter (i.e. notch filter, low pass filter) – good to extract certain parts of a signal spectra. Hence useful if a signal spectra is the output of the experiment.
* Phase sensitive detection – great for filtering a signal from external noise. (see above)
* Vibrational filtering – make resonant frequencies of experiment >> environmental frequencies. i.e. rest experiment on air cushion. Also damp environment surrounding experiment to reduce any oscillations (unwanted!) that may occur.
* Thermal shielding – use Stephan's law $\dot{q} = \varepsilon \sigma T^4$ to show that multiple reflective coatings are very efficient. [$\varepsilon = 1 -$ reflectivity].
* Make a Differential measurement – absolute measurements can have big systematics – Differential amplifier and twisted cable make common mode interference disappear.
* Electric and magnetic shielding – $E_{in} = 0$ in conductor, metal boxes can reduce B_{in} as well. (Though not completely). Avoid earth loops that can induce EMF's. For a wire $B \propto I$:

$EMF = -\frac{\partial}{\partial t}(BA)$

if A large:



i.e. voltage induced over wire loop – get noise across R

Solution: reduce A $\rightarrow$

EMF much smaller. Same circuit.

Probability Distributions

Random errors will follow a probability distribution (Gaussian if N is large). In assessing the validity of a model / likelihood of an outcome probability distributions are helpful.

General

For continuous variable x and probability distribution $P(x)$

* $dx\,P(x) =$ probability of an event occurring in range $x \rightarrow x + dx$

$\rightarrow$ so probability of event occurring for $x < X$ (X fixed x) is $P(x < X) = \int_{-\infty}^{X} p(x)\,dx$.

Note $\int_{-\infty}^{\infty} p(x)\,dx = 1$. Now taking limits of discrete mean and variance definitions: for continuous variables: mean of $p(x)$, is:

$E(x) \equiv \langle x \rangle \equiv \bar{x} = \int_{-\infty}^{\infty} x\,p(x)\,dx$. Variance $= V(x) \equiv \langle x^2 \rangle - \langle x \rangle^2 = \int_{-\infty}^{\infty} x^2 p(x)\,dx - \langle x \rangle^2$

Binomial Distribution:

n trials with binary outcomes (i.e. true or false). One outcome has probability p. For r of these outcomes and $n - r$ of the other outcome [probability $1-p$] : $E(r) = \langle r \rangle = np$ $\quad V(r) = np(1-p)$

DISCRETE distribution defined by $P(r \mid p, n) = \frac{p^r (1-p)^{n-r}\, n!}{r!(n-r)!}$

{ widest for $p = 0.5$
$\rightarrow$ Poisson as $p \rightarrow 0$, $n \rightarrow \infty$

Poisson Distribution : outcome 'rate' known but no. of trials is not.

DISCRETE distribution : $P(r|\lambda) = \dfrac{\lambda^r}{r!}e^{-\lambda}$, r = outcome rate.

$E(r) = \lambda$, $V(r) = \lambda$. Poisson always broader than Binomial distribution.

No. of events can exceed mean event rate. If $r = \lambda + x$ with $\lambda \to \infty$ then Poisson $\to$ Gaussian distribution.

Gaussian Distribution

CONTINOUS DISTRIBUTION

$$P(x \to x+dx \,|\, \mu, \sigma) = \frac{dx}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$E(x) = \mu$, $V(x) = \sigma^2$

let $z = \dfrac{x-\mu}{\sigma}$ $(dx = \sigma\, dz)$ $\Rightarrow$ $P(z < X) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{X} e^{-\frac{1}{2}z^2}\,dz$ which is tabulated.

Central limit theorem : N samples, each own probability distribution.

① $\sum$ sample means = new mean

Note: $\overline{aX+bY} = a\bar{X} + b\bar{Y}$
For independent variables X, Y and constants a, b.

② $\sum$ variances = new variance

Note: $V[aX+bY] = a^2V[X] + b^2V[Y]$

③ Resulting probability distribution is Gaussian as $N \to \infty$.

Bayes Theorem. In statistical analysis - want parameter a given some data set of measurements $\{x_1, x_2, x_3 \ldots\}$ i.e. $P(a|\text{data})$ is the probability of attaining this. Hence maximising $P(a|\text{data})$ gives best estimate of a. Now when we make a measurement we calculate $P(\text{data}|a) = P(x_1|a)P(x_2|a)\ldots$ since truth of a is assumed. BAYES THEOREM STATES

$$P(a|\text{data}) = P(\text{data}|a) \cdot P(a)/P(\text{data})$$

[$P(\text{data})$ always turns out to be a normalising constant

$P(a)$ is called the 'prior probability'.

Proof/

Intersection of sets $P(A)$, $P(B)$ can be viewed from 'A|B' side or 'B|A' side. Hence $P(A|B)P(B) = P(B|A)P(A) \Rightarrow P(A|B) = P(B|A).P(A)/P(B)$

Example: observer see's a Blue escort. Probability? let H = car was blue escort $\bar{H}$ = car wasn't, D = data - what observer says. Say observer is 99% likely to be right. Want $P(H|D)$. Have $P(D|H) = 0.99$, $P(D|\bar{H}) = 0.01$, $P(H) = 0.02$
↑ stats on cars.

so by Bayes theorem
$$P(H|D) = P(D|H) \cdot P(H)/P(D)$$
$$P(\bar{H}|D) = P(D|\bar{H}) \cdot P(\bar{H})/P(D)$$

Now $P(\bar{H}|D) = 1 - P(H|D)$ (Normalisation). So $P(D) = P(D|\bar{H}) \cdot P(\bar{H}) / (1 - P(H|D))$

$$\Rightarrow P(H|D) = \frac{P(D|H) \cdot P(H)\,(1 - P(H|D))}{P(D|\bar{H}) \cdot P(\bar{H})} = \frac{(0.99)(0.02)}{(0.01)(0.98)}\left(1 + \frac{0.99 \cdot 0.02}{0.01 \cdot 0.98}\right)^{-1} = 0.67$$

less than you would think

χ^2 test — Assume uniform prior $\Rightarrow$ $p(a|data) \propto p(data|a)$ for parameter a estimation. Assume gaussian errors so for measurement set $\{y_i\}$ and theoretical model $f(x_i|a)$
$$P(y_i|a) = \frac{1}{\sigma_i\sqrt{2\pi}} e^{-[y_i - f(x_i|a)]^2/2\sigma_i^2} \quad (f(x_i|a) \simeq \text{mean})$$

Now likelihood $L = \prod_i P(y_i|a)$. Now maximising L is best done by maximising $\ln L$. $\Rightarrow$ minimising
$$\sum_i \left[\frac{y_i - f(x_i|a)}{\sigma_i}\right]^2$$
is the crucial part.

Define:
$$\chi^2 = \sum_i \left[\frac{y_i - f(x_i|a)}{\sigma_i}\right]^2.$$
lowest χ^2 gives best indication of fit.

For data sets with no error i.e, $y_i = y_i$ not $y_i = \bar{y}_i \pm \sigma_i$

$\rightarrow$ use
$$\chi^2 = \sum_i \left[\frac{y_i - f(x_i|a)}{f(x_i|a)^{\frac{1}{2}}}\right]^2 = \sum_i \frac{(y_i - f(x_i|a))^2}{f(x_i|a)}$$

χ^2 values are tabulated $\rightarrow$ probability distribution of whether $f(x)$ does actually fit data or 'correlation' is pure chance.

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linear regression — For $\{x_i\}$ and $\{y_i\}$ data sets want to find parameters m, c to fit $y = mx + c$. Now $\chi^2 = \sum_i \left[\frac{y_i - mx_i - c}{\sigma_i}\right]^2$ and χ^2 minimum for parameters m and c is found by $\frac{\partial \chi^2}{\partial m} = 0$ and $\frac{\partial \chi^2}{\partial c} = 0$. using 'divide by N' trick yields

$$m = \frac{\overline{xy} - \bar{x}\bar{y}}{\overline{x^2} - \bar{x}^2} \qquad c = \frac{\overline{x^2}\,\bar{y} - \bar{x}\,\overline{xy}}{\overline{x^2} - \bar{x}^2} \quad (c = \bar{y} - m\bar{x})$$

for $\sigma_i = \sigma =$ constant for all x, y. using laws of error propagation:

$$\sigma_m^2 = \frac{\sigma^2}{N(\overline{x^2} - \bar{x}^2)}, \quad \sigma_c^2 = \frac{\sigma^2\,\overline{x^2}}{N(\overline{x^2} - \bar{x}^2)}$$

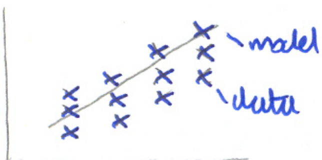
Now if $\sigma_i \neq$ constant for all y, x:

Define:
$$\bar{x} = \frac{\sum_i x_i/\sigma_i^2}{\sum_i 1/\sigma_i^2}, \quad \bar{y} = \frac{\sum_i y_i/\sigma_i^2}{\sum_i 1/\sigma_i^2}$$
rather than $\bar{x} = \sum_i x_i/N$.

'weighted averages'.

Non parametric statistics — i.e, where prior may not be uniform, errors not gaussian ..... i.e, χ^2 test does not apply.

e.g;



Important names of tests:

* "run test" / "sign test"
* Mann-Whitney
* Kolmogorov-Smirnov

Find in books.