DYNAMICS – (Non relativistic classical mechanics or "Newtonian mechanics")

Definitions I Classical mechanics is a VECTOR theory. Motion is described by vectors whose components are given w.r.t to REFERENCE FRAME being used. Dynamics describes how these vectors evolve with time.

POSITION is described by the vector $\underline{r}$ w.r.t some origin

VELOCITY $= \lim_{\delta t\to 0} \delta\underline{r}/\delta t = \dot{\underline{r}} \equiv \underline{v}$ ACCELERATION $= \lim_{\delta t\to 0} \delta\underline{v}/\delta t = \dot{\underline{v}} \equiv \ddot{\underline{r}} \equiv \underline{a}$

Definitions II Newton's 2nd law $m\underline{a} = \underline{F}$ [m = body mass, $\underline{F}$ = applied force]

↑Not really a definition but v. important!

MOMENTUM ($\underline{p}$) $\underline{p} = m\underline{v}$ Note $\dot{\underline{p}} = m\underline{a} = \underline{F}$ if $\frac{dm}{dt} = 0$.

IMPULSE ($\Delta\underline{p}$) $\Delta\underline{p} = \underline{p}_f - \underline{p}_i = \int_{t_i}^{t_f} \underline{F}\,dt$ ANGULAR MOMENTUM ($\underline{J}$) $\underline{J} = \underline{r}\times\underline{p}$

TORQUE ($\underline{G}$) $\underline{G} = \dot{\underline{J}}$ ANGULAR IMPULSE ($\Delta\underline{J}$) $\Delta\underline{J} = \underline{J}_f - \underline{J}_i = \int_{t_i}^{t_f} \underline{G}\,dt$

Note: $\dot{\underline{J}} = \frac{d}{dt}(\underline{r}\times\underline{p}) = \underline{r}\times\dot{\underline{p}} + \dot{\underline{r}}\times\underline{p}$ Now $\dot{\underline{r}}\times\underline{p} = m(\dot{\underline{r}}\times\dot{\underline{r}}) = 0$

So $\dot{\underline{J}} = \underline{r}\times\dot{\underline{p}} = \underline{r}\times\underline{F}$. So $\underline{G} = \dot{\underline{J}}$ is the net moment of force $\underline{F}$ about a body axis of rotation. WORK (W) $= \int_p \underline{F}\cdot d\underline{r}$ [p = path]

Now $d\underline{r} = \underline{v}\,dt \Rightarrow W = \int_{t_1}^{t_2} \underline{F}\cdot\underline{v}\,dt = m\int_{t_1}^{t_2} \frac{d\underline{v}}{dt}\cdot\underline{v}\,dt = \frac{1}{2}m\int_{t_1}^{t_2} \frac{d}{dt}(\underline{v}\cdot\underline{v})\,dt$

↑Power $= \underline{F}\cdot\underline{v}$

$= \frac{1}{2}m\int_{t_1}^{t_2} dv^2 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$ So define KINETIC ENERGY (T) as $T = \frac{1}{2}mv^2$

Conservative force fields and potential energy If a particle moves in a conservative field of force $\underline{F}$: * $\underline{F} = \underline{F}(\underline{r})$ * $\oint_C \underline{F}\cdot d\underline{r} = 0$ for any closed curve C.

$\Rightarrow \int_A^B \underline{F}\cdot d\underline{r} = \phi(B) - \phi(A)$ for scalar function ϕ of positions A, B. i.e. work integral is path independent. Now $\phi(B) - \phi(A) = \int_A^B d\phi \Rightarrow \underline{F}\cdot d\underline{r} = d\phi \Rightarrow \underline{F} = \nabla\phi$.

[In practice ϕ is chosen s.t. $\underline{F} = -\nabla\phi$ for a conservative field].

POTENTIAL ENERGY (U) is defined for a conservative force field as $\underline{F} = -\nabla U$

Hence $\int_A^B \underline{F}\cdot d\underline{r} = U(A) - U(B)$. Now by LAW OF CONSERVATION OF ENERGY if no external forces act on a body net work $= 0$. So $0 = U(A) - U(B)$ ↙gain in PE

$+ T(B) - T(A) \Rightarrow T(B) + U(B) = T(A) + U(A) = $ constant (E) i.e. $E = T + U$

↑gain in KE

Now 'zero' of U is arbitrarily chosen. i.e for gravity, $U = 0$ is for $|\underline{r}| = \infty$. [Note all CENTRAL forces and constant forces are conservative]

Hence $U(r)$ can be defined $U(r) = \int_r^{r_0} \underline{F}\cdot d\underline{r}$ where $U(r_0) = 0$

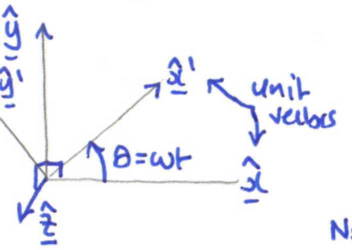
Frames of reference

INERTIAL frames are those where $\underline{F} = m\underline{a}$ holds. i.e. for any one inertial frame all possible others will be moving at some relative velocity $\underline{u}$ to this frame. Transform between inertial frame via GALILEAN transforms. $(\underline{r}, t) \to (\underline{r}', t')$ $t' = t$ ↙if origins don't coincide. $\underline{r}' = \underline{r} - \underline{R}_0 - \underline{u}t$ $\dot{\underline{r}}' = \dot{\underline{r}} - \underline{u}$

NON INERTIAL FRAMES

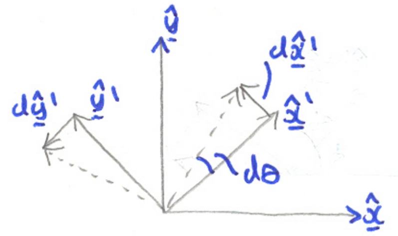
In these frames 'extra' forces are felt by all bodies due to acceleration of the frame in some way. e.g., if one frame is accelerating from an inertial frame at rate $\underline{a}$, all masses in the accelerating frame will have the force $-m\underline{a}$ acting on them. This is simply because we have to make the acceleration transform $-\underline{a}$ to go from the inertial frame to the accelerating one. (So by definition, all objects with acceleration $\underline{a}$ in the inertial frame appear stationary in the accelerating frame).

Rotating frames



If $\hat{x}', \hat{y}'$ basis vectors of rotating S' frame rotate about $\hat{z}$ axis of S frame ($\hat{x}, \hat{y}, \hat{z}$ basis vectors) with angular speed ω $(= \dot{\theta})$ then from diagram on the right:

$$\dot{\hat{x}}' = \omega \hat{y}' \qquad \dot{\hat{y}}' = -\omega \hat{x}' \quad (*)$$



Now since vectors are frame independent quantities the vector $\underline{u}$ in the S frame can be equated to vector $\underline{u}$ in the S' frame – only components change.

i.e. $\underline{u} = u_x' \hat{x}' + u_y' \hat{y}' + u_z' \hat{z}'$, $\quad \dot{\underline{u}} = \dot{u}_x' \hat{x}' + u_x' \dot{\hat{x}}' + \dot{u}_y' \hat{y}' + u_y' \dot{\hat{y}}' + \dot{u}_z' \hat{z}' + u_z' \dot{\hat{z}}'$

Now since S' rotates about $\hat{z}$, $\dot{\hat{z}}' = 0 \Rightarrow \dot{\underline{u}} = \left(\frac{d\underline{u}}{dt}\right)_{S' \text{ frame}} + \omega(u_x' \hat{y}' - u_y' \hat{x}')$ using $(*)$

So defining $\underline{\omega} = \omega \hat{z} = \omega \hat{z}' \Rightarrow \dot{\underline{u}} = \left.\frac{d\underline{u}}{dt}\right|_{S'} + \underline{\omega} \times \underline{u}$

Now since $\dot{\underline{u}} = \left.\frac{d\underline{u}}{dt}\right|_{S}$ we can make rotating frame differential operator relationship on all vectors:

$$\left.\frac{d}{dt}\right|_S = \left.\frac{d}{dt}\right|_{S'} + \underline{\omega} \times \quad \Rightarrow \quad \left.\frac{d^2 \underline{r}}{dt^2}\right|_S = \left(\left.\frac{d}{dt}\right|_{S'} + \underline{\omega} \times\right)\left(\left.\frac{d\underline{r}}{dt}\right|_{S'} + \underline{\omega} \times \underline{r}\right)$$

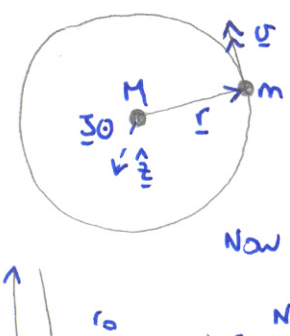
$$\Rightarrow \left.\frac{d^2 \underline{r}}{dt^2}\right|_S = \left.\frac{d^2 \underline{r}}{dt^2}\right|_{S'} + 2\underline{\omega} \times \left.\frac{d\underline{r}}{dt}\right|_{S'} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

Now $m \left.\frac{d^2 \underline{r}}{dt^2}\right|_S = \underline{F}$ ← true applied force.

So: $$m \left.\frac{d^2 \underline{r}}{dt^2}\right|_{S'} \equiv m \ddot{\underline{r}}\big|_{S'} = \underbrace{\underline{F}}_{\text{true force}} - \underbrace{2m\underline{\omega} \times \dot{\underline{r}}\big|_{S'}}_{\text{'Coriolis force'}} - \underbrace{m\underline{\omega} \times (\underline{\omega} \times \underline{r})}_{\text{'centrifugal force'}}$$

Orbits

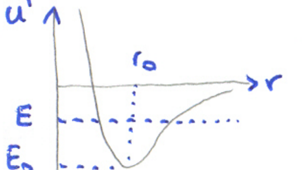
Consider a particle moving under the influence of a central force [⇒ conservative field, $\underline{J}$ is constant]. Now particle is thus in a scalar potential $U(r)$ and total energy E is given by $E = \frac{1}{2}mv^2 + U(r)$. Now for 2D polars: $\underline{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ so $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + U(r)$.



Note also $\underline{J} = m\underline{r} \times \underline{v} = mr^2\dot{\theta}\hat{z} \Rightarrow$ as J is a constant, $J^2 = m^2 r^4 \dot{\theta}^2$

$\Rightarrow (*)$ $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + U(r) = \frac{1}{2}m\dot{r}^2 + U'(r)$ ← 'Effective potential'.

Now for example of Gravitation: $U(r) = -\frac{GMm}{r} \equiv -\frac{A}{r}$ ← general inverse square force law.



Now define: $r_0 = J^2/mA$, $E_0 = -mA^2/2J^2$ } Clearly from $U'(r)$ graph $E < E_0$ is impossible, $E = E_0 \Rightarrow r = r_0$ ⇒ circular orbit, $E_0 < E < 0$ ⇒ Elliptic, bound orbit (r_{max}, r_{min}), $E \geq 0$ ⇒ unbound orbit orbits, cont... Now from (*), $\dot{r} = \frac{J}{m}\sqrt{\frac{2}{r r_0} - \frac{1}{r^2} - \frac{E}{E_0}\cdot\frac{1}{r_0^2}}$ ← 'Elliptic function'

⇒ $\dot{r} = \frac{J}{m}\sqrt{\frac{e^2}{r_0^2} - \left(\frac{1}{r} - \frac{1}{r_0}\right)^2}$ Defining 'eccentricity' $e = \sqrt{1 - E/E_0}$ Now as $\dot{\theta} = \frac{J}{mr^2}$

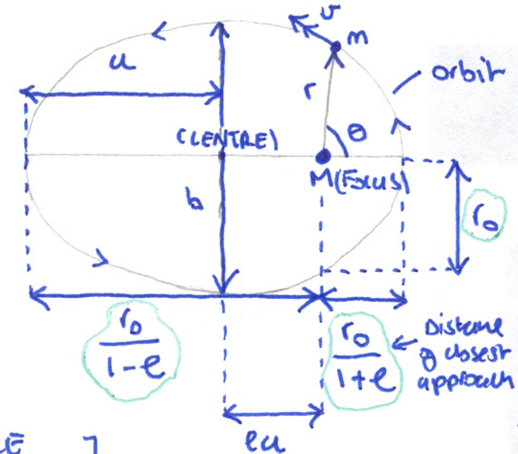
and $\frac{dr}{d\theta} = \frac{dr}{dt}\cdot\frac{dt}{d\theta} = \dot{r}/\dot{\theta}$:

⇒ $\frac{dr}{d\theta} = r^2\sqrt{\frac{e^2}{r_0^2} - \left(\frac{1}{r} - \frac{1}{r_0}\right)^2}$ ⇒ (using substitution $u = \frac{1}{r}$) $r = \frac{r_0}{1 + e\cos\theta}$

↑ Elliptic orbit. (if $e<1$)

$= \frac{r_0}{1-e} + \frac{r_0}{1+e}$

$\frac{2r_0}{1-e^2} = -\frac{A}{E}$



a = 'semi-major axis'
b = 'semi-minor axis'
r_0 = 'semi-latus rectum ($\theta = \pi/2, 3\pi/2$)

Now in cartesians (origin 'centre', not 'focus')

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\left(a = 2r_0/1-e^2,\ b = r_0/\sqrt{1-e^2}\right)$

→ If origin is focus: $x \to x - ea$

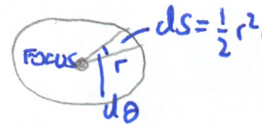
i.e. $\frac{(x-ea)^2}{a^2} + \frac{y^2}{b^2} = 1$

[NOTE HERE $E<0$ so $e<1$, $e^2 = 1 - E/E_0$]

Keplers laws

1) Planets move in Ellipses with sun at one focus (✓ see above)
2) Radius vector sweeps out equal areas in equal times
3) Square of orbital period is ∝ to cube of major axis.

Proof: 1) (Above) 2) rate of sweeping out area is $\frac{dS}{dt} = \frac{1}{2}r^2\dot{\theta}$



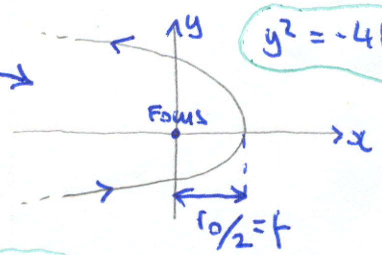
$= \frac{J}{2m}$ = constant. [$\dot{\theta} = \frac{J}{mr^2}$]

3) orbital period $T = \text{Area}/\frac{dS}{dt} = \pi ab/J/2m$. Now $b^2 = ar_0 = aJ^2/mA$

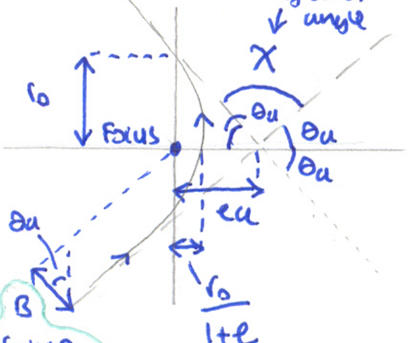
∴ $T^2 = (2\pi ma/J)^2 \cdot aJ^2/mA = 4\pi^2 ma^3/A$. i.e. $T^2 \propto a^3$. [For gravity $T^2 = \left(\frac{4\pi^2}{GM}\right)a^3$]

Unbound orbits

Parabola, $E=0$, $e=1$ $y^2 = -4f(x-f)$



Hyperbola $E>0$ $e>1$



$r_0\cos\theta_a$

cartesian equation is similar to ellipse: $\frac{(x-ea)^2}{a^2} - \frac{y^2}{b^2} = 1$

$a = \frac{r_0}{e^2-1}$ $b = \frac{r_0}{\sqrt{e^2-1}}$ ← Different.

[Note $r = \frac{r_0}{1+e\cos\theta}$ describes any orbit.

∴ need to know $r_0 = J^2/mA$, $e^2 = 1 - E/E_0$ ONLY.]

Now $\chi = \pi - 2\theta_a$ Assymptotes ($r=\infty$) when $\cos\theta = -\frac{1}{e}$

i.e. when $\theta = \theta_a + \chi = \pi - \theta_a$ so $\theta_a = \pi - \cos^{-1}(-1/e)$ and $\chi = 2\cos^{-1}(-1/e) - \pi$

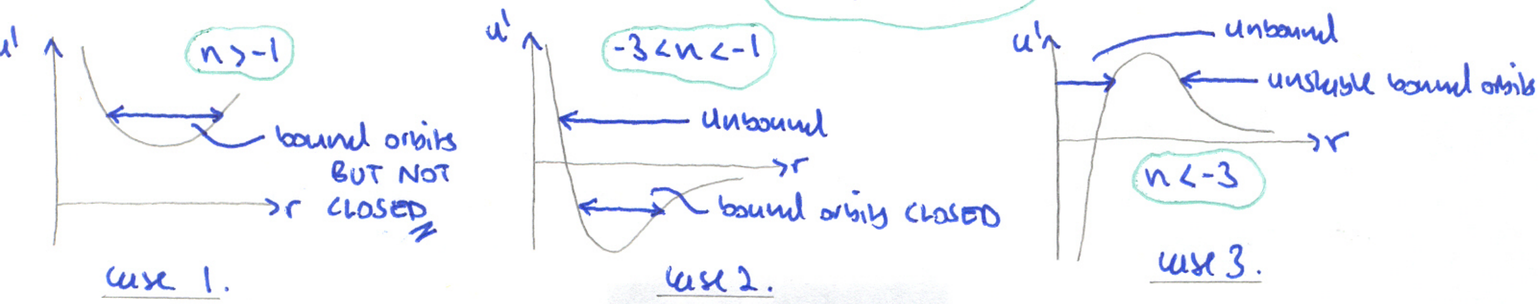
and $\theta = 2\pi - (\theta_a + \chi)$

Note: $\sin\left(\frac{\chi}{2}\right) = \frac{1}{e}$ ⇒ $\cot\frac{\chi}{2} = mv_0^2 B/A$ ← considering initial E, J to find $1/e$

(v_0 = velocity at ∞ away from planet) $J = mv_0 B$

Other force laws (central) let $F \propto r^n \Rightarrow \underline{F} = -Ar^n \hat{\underline{r}}$, $U(r) = \dfrac{A r^{n+1}}{n+1}$ (using $U(\infty)=0$)

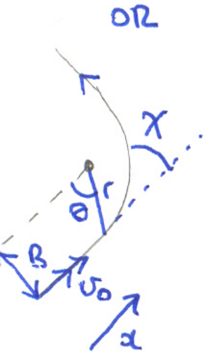
$\Rightarrow$ Effective potential $U'(r) = \dfrac{Ar^{n+1}}{n+1} + \dfrac{J^2}{2mr^2}$ $\rightarrow$ 3 cases.



Two body problem - planets etc orbit about their common centre of mass. Transform to zero momentum frame / centre of mass frame to analyse motion.

Note on an easy way of deriving $\cot\frac{\chi}{2} = mv_0^2 B/A$ $\nearrow$ $e = \sqrt{1 - E/E_0}$ $\quad E = \frac{1}{2}mv_0^2$, $E_0 = -\dfrac{mA^2}{2J^2}$ *, $J = mv_0 B$

You can use orbit equation $\Rightarrow \sin\left(\frac{\chi}{2}\right) = \frac{1}{e}$



OR consider momentum change along direction of initial motion:

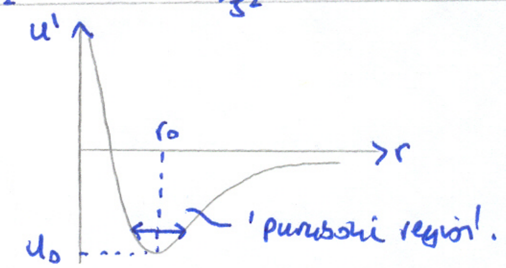
$$\Delta p_x = mv_0(\cos\chi - 1) = \int F\,dt = \int_0^{\pi+\chi} \frac{A\cos\theta}{r^2} \cdot \frac{dt}{d\theta} \cdot d\theta = A\int_0^{\pi+\chi} \frac{\cos\theta}{r^2} \cdot \frac{mr^2}{J}\,d\theta$$

$$= -\frac{A}{v_0 B}\sin\chi \Rightarrow \frac{mv_0^2 B}{A} = \frac{\sin\chi}{1-\cos\chi} = \cot\frac{\chi}{2}$$

* You would have to know this (or follow derivation above to find $dU'/dr = 0$) OR work out E for circular orbit with $J = mv_0B$. i.e. $E = \frac{1}{2}mv^2 - \frac{A}{r}$ $= r^2\omega^2 m/2 - mr^2\omega^2$ (By Newton 2 $mr\omega^2 = A/r^2$) so $E = -\frac{1}{2}mr^2\omega^2$. Now $\frac{A^2}{J^2} = r^2\omega^2$ since $J = mr^2\omega$

Nearly circular orbits

For $F \propto r^n$ force law ($n > -3$, < -1) and orbit near minimum ($r = r_0$) of effective potential U'
$\hookrightarrow$ assume U' is parabolic in this region:



i.e. $U' \approx U_0 + \frac{1}{2}(r-r_0)^2 \left.\dfrac{d^2U}{dr^2}\right|_{r=r_0}$ Now $U' = \dfrac{Ar^{n+1}}{n+1} + \dfrac{J^2}{2mr^2}$, $\dfrac{dU'}{dr} = Ar^n - \dfrac{J^2}{mr^3} = 0$ at $r = r_0$

$\dfrac{d^2U'}{dr^2} = nAr^{n-1} + \dfrac{3J^2}{mr^4} = (n+3)\dfrac{J^2}{mr_0^4}$ at $r = r_0$. So $E = \frac{1}{2}m\dot{r}^2 + U' \approx \frac{1}{2}m\dot{r}^2 + U_0 + \frac{1}{2}(r-r_0)^2(n+3)\frac{J^2}{mr_0^4}$

Try SHM solution, $r = r_0 + a\cos\omega t$. $\therefore E - U_0 = \text{constant} = \frac{1}{2}m\omega^2 a^2 \sin^2\omega t + \frac{1}{2}a^2(n+3)\frac{J^2}{mr_0^4}\cos^2\omega t$

Now since $\sin^2\omega t + \cos^2\omega t = 1 \Rightarrow$ coefficients of $\sin^2$, $\cos^2$ must be the same:

$\Rightarrow \omega = \sqrt{n+3}\,\dfrac{J}{mr_0^2} = \sqrt{n+3}\,\Omega \leftarrow$ orbit frequency. [$J = mr_0^2\omega$ for circular orbit - let $\omega = \Omega$ for this special case!]

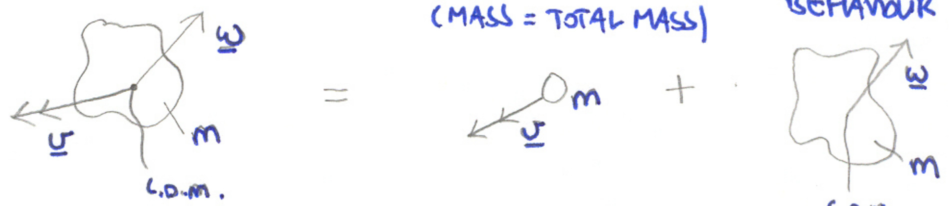
when $\sqrt{n+3}$ is not rational i.e. $n+3$ is not a square number
$\rightarrow$ NON CLOSED ORBITS, i.e. precession.

Many particle systems - no general solution to dynamics of more than two interacting bodies but there are a few general statements:

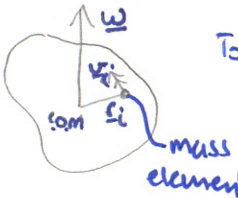
① Motion of centre of mass is that of a particle with the total mass of the system due to the total external force. ② Rate of change of total angular momentum = total external torque. ③ For a RIGID BODY all internal distances are fixed so only EXTERNAL forces do work (= KE change of body)

Rigid body Dynamics

GENERAL MOTION OF RIGID BODY = PARTICLE MOTION OF C.O.M. (MASS = TOTAL MASS) + ZERO MOMENTUM FRAME BEHAVIOUR



Now for a general rigid body spinning about some axis (i.e. with angular velocity vector $\underline{\omega}$), in Z.M.F:



Total angular momentum ($\underline{J}$) is $\underline{J} = \sum_i m_i \underline{r}_i \times \underline{v}_i$

$= \sum_i m_i \underline{r}_i \times (\underline{\omega} \times \underline{r}_i)$ since $\underline{v}_i = \underline{\omega} \times \underline{r}_i$ ← $\left.\begin{matrix}\underline{v}_i \cdot \underline{r}_i = 0 \\ \underline{v}_i \cdot \underline{\omega} = 0\end{matrix}\right\}$ For any pure rot.

$= \sum_i m_i (\underline{\omega} r_i^2 - \underline{r}_i(\underline{\omega} \cdot \underline{r}_i)) = \underline{\underline{I}}\,\underline{\omega}$ ~~where $I_{ij} = \sum ...$~~

INERTIA TENSOR $\underline{\underline{I}}$ is thus defined

$$\sum_i m_i \begin{pmatrix} r_i^2 - x_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & r_i^2 - y_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & r_i^2 - z_i^2 \end{pmatrix}$$

For a continuous medium*, $\underline{J} = \underline{\underline{I}}\,\underline{\omega}$ still but

*volume $J_i = \iiint_V \rho(\underline{r})(r_k r_k \delta_{ij} - r_j r_i)\omega_j \, dV$ i.e.

↑ mass distribution

$I_{ij} = \iiint_V \rho(\underline{r})(r_k r_k \delta_{ij} - r_j r_i)\, dV$ since $\underline{\omega}$ is known.

Now $\underline{\underline{I}}$ is symmetric and can be diagonalised into form $[\underline{\underline{I}}]'_{ij} = I_{(i)}\delta_{ij}$ by using $\underline{\underline{I}}' = \underline{\underline{U}}^T \underline{\underline{I}}\, \underline{\underline{U}}$. $\underline{\underline{U}}$ is a matrix of the eigenvectors of $\underline{\underline{I}}$ and $\{I_{(i)}\}$ are the set of eigenvalues given by the equation $\underline{\underline{I}}\,\underline{e} = \lambda \underline{e}$. The set of eigenvalues of $\underline{\underline{I}}$ are the moments of inertia w.r.t a body's PRINCIPLE AXIS. These are usually the symmetry axis of a body. Hence if BODY AXIS in ZMF are chosen to be the principle axis, $\underline{\underline{I}}$ will be diagonalised. HANDY! (Note $\underline{\omega}$ must be described w.r.t these axis, and hence $\underline{J}$ also).

[Must be cartesian]

Now rotational K.E is given by $T = \frac{1}{2}\sum_i m_i(\underline{\omega} \times \underline{r}_i)(\underline{\omega} \times \underline{r}_i) = \frac{1}{2}\underline{\omega} \cdot \underline{J} = \frac{1}{2}\underline{\omega} \cdot \underline{\underline{I}}\,\underline{\omega}$

so find $\underline{\underline{I}}$ and $\underline{\omega}$ w.r.t principle axis and then $T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$

Now principle axis (special case of BODY AXIS) are fixed with body but body rotates with $\underline{\omega}$. So principle axis, body axis are a rotating frame (Non inertial).

Hence using principle axis coordinate system:

$$\underline{G} = \left.\frac{d\underline{J}}{dt}\right|_{\text{space frame}} = \left.\frac{d\underline{J}}{dt}\right|_{\text{body frame}} + \underline{\omega} \times \underline{J}$$

torque that would be felt by body in inertial frame

↑ torque felt in body frame

NOTE THIS MEANS $\underline{\omega}$ vector will change with time in body frame. This ⇒ tumbling in space frame (inertial). Nothing actually HAPPENS physically in body frame if body is rigid except we can monitor $\underline{\omega}$ changes conveniently here.

w.r.t PRINCIPLE BODY AXIS → EULERS EQUATIONS.

$\underline{\omega}$ defines ROTATION.
[Transform to non rotating BODY frame to find $\underline{\omega}$ etc]
↑ similar with $\underline{v}$ c.f rotating frames.

$$\left.\begin{aligned} G_1 &= I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) \\ G_2 &= I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) \\ G_3 &= I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) \end{aligned}\right\}$$

completely defines rigid body motion when coupled with particle dynamics of c.o.m. Hence $\underline{v}$ and $\underline{\omega}$ are the only really important quantities.

Elasticity – An elastic material is one which applied stresses do not induce permanent strains. i.e. removal of stress causes material to return to original shape.

Definitions In 1D STRESS is defined as $\tau = F/A$ (force per unit area) and STRAIN as $e = \delta L / L$ (extension / original length). In general these are symmetric 2nd rank tensors with components computable w.r.t some body axis. Hooke's law states 'stress $\propto$ strain' so in tensor notation this extends to:

Note: $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right)$

[Note: $\underline{\underline{\tau}}\,\underline{A} = \underline{F}$  'area (dA) element.]

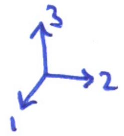
$$\tau_{ij} = k_{ijkl} e_{kl}$$

with k being a 4th rank tensor.

$\tau_{ij} \Rightarrow$ i – force direction, j area vector direction – same, symmetry, definition

Now both $\underline{\underline{\tau}}$ and $\underline{\underline{e}}$ can be DIAGONALISED w.r.t a set of principle axis (usually symmetry). i.e. $\underline{\underline{\tau}} = \tau_{(i)}\delta_{ij}$, $\underline{\underline{e}} = e_{(i)}\delta_{ij}$. ($e_{(i)} = {}'\delta l_i / l_i{}'$).

Now if medium is ISOTROPIC, stress and strain principle axis COINCIDE. We can express stress in terms of strain (and vice versa) w.r.t these axis.



$$e_1 = \frac{1}{Y}\left[\tau_1 - \sigma(\tau_2 + \tau_3)\right]$$
$$e_2 = \frac{1}{Y}\left[\tau_2 - \sigma(\tau_1 + \tau_3)\right]$$
$$e_3 = \frac{1}{Y}\left[\tau_3 - \sigma(\tau_1 + \tau_2)\right]$$

For $\underline{\underline{e}} = \begin{pmatrix} e_1 & & 0 \\ & e_2 & \\ 0 & & e_3 \end{pmatrix}$

$\underline{\underline{\tau}} = \begin{pmatrix} \tau_1 & & 0 \\ & \tau_2 & \\ 0 & & \tau_3 \end{pmatrix}$

Y is the YOUNG'S MODULUS given by $Y = \dfrac{\text{"longitudinal, unconstrained stress}}{\text{resultant strain"}}$

σ is the POISSON RATIO given by $\dfrac{\delta w}{w} = -\sigma \dfrac{\delta L}{L}$ ← (i.e. width change when lengthened)

(L = length of material, w = width).

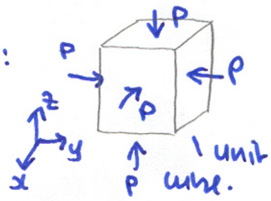
All other elastic moduli can be expressed in terms of σ and Y.

BULK MODULUS (B) For a volume V under pressure P (All isotropic)

$$B = -P \Big/ \left(\frac{\delta V}{V}\right)$$

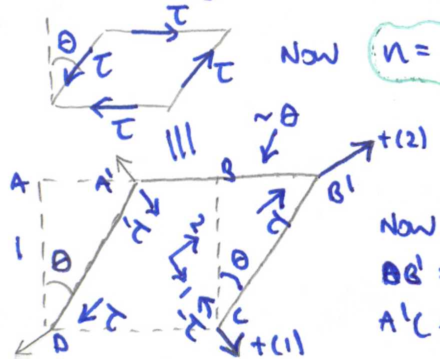
Also: $B = \dfrac{Y}{3(1-2\sigma)}$

Proof:



For each direction P contributes same to strain. → take x direction. P in x direction causes following strains: $\delta x = -P/Y$, $\delta y = \delta z = \sigma P/Y$. Hence $\delta z = -P/Y$, $\delta y = \delta x = \sigma P/Y$ for z direction P etc... Net result $\delta x = \delta y = \delta z = -\frac{P}{Y}(1-2\sigma)$. Now $\frac{\delta V}{V} = (1+\delta x)(1+\delta y)(1+\delta z) - 1 \approx \delta x + \delta y + \delta z$. Hence $B = Y/3(1-2\sigma)$ as required from B's definition.

SHEAR MODULUS (n) A shear occurs when stress acts along the sides of a body ($\tau \parallel A$) though no net torque is exerted or any acceleration of the c.o.m occurs.



Now $n = \tau/\theta$, also $n = \dfrac{Y}{2(1+\sigma)}$.

Proof: consider unit shear w.r.t principle axis (2D drawing on right) Assume θ small. (Radians)

$\underline{\underline{\tau}} = \begin{pmatrix} -\tau & & 0 \\ & \tau & \\ 0 & & 0 \end{pmatrix}$

Now $AC = BD = \sqrt{2}$

$DB' = \sqrt{1+(1+\theta)^2} \approx \sqrt{2}\left(1+\frac{\theta}{2}\right)$

$A'C = \sqrt{1+(1-\theta)^2} \approx \sqrt{2}\left(1-\frac{\theta}{2}\right)$

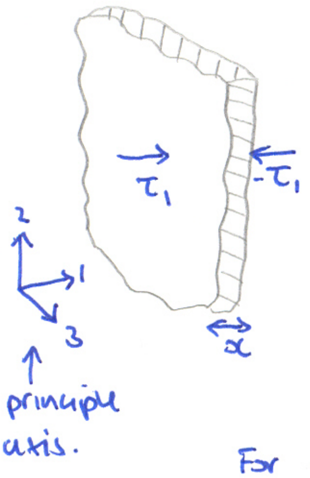
strains, choose '2'

$\Rightarrow e_2 = \theta/2$, $e_1 = -\theta/2$ [Now $\tau_3, e_3 = 0$]

using $e_1 = \frac{1}{Y}\left[\tau_1 - \sigma(\tau_2 + \tau_3)\right]$

$\Rightarrow -\frac{\theta}{2} = \frac{1}{Y}\left[-\tau - \sigma\tau\right] \Rightarrow \frac{\tau}{\theta} = \frac{Y}{2(1+\sigma)} = n$

LONGITUDINAL MODULUS (M_L) → for compression of an infinite sheet: (isotropic)



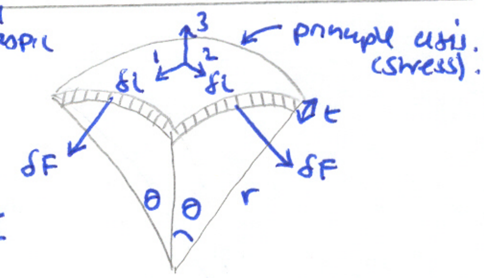
$e_1 = \frac{\delta x}{x}$, $e_2 = e_3 = 0$ ⇒ using strain, stress relations:

$$\tau_2 = \tau_3 = \frac{\sigma}{1-\sigma}\tau_1 \quad \text{and} \quad \tau_1/e_1 = \frac{Y(1-\sigma)}{(1+\sigma)(1-2\sigma)} = M_L$$

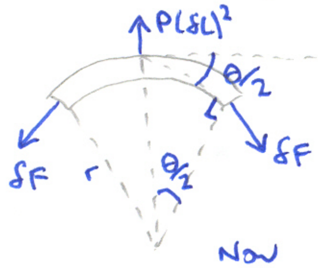
Note $$M_L = \frac{Y}{3(1-2\sigma)} + \frac{4}{3}\frac{Y}{2(1+\sigma)} = B + \tfrac{4}{3}n.$$

FRACTIONAL VOLUME CHANGE OF TENNIS BALL DUE TO EXCESS INTERNAL PRESSURE P

isotropic



For surface element in e.g. $(\delta L \times \delta L)$



→ if $\theta \ll 1$, $\sin\theta/2 \simeq \theta/2$

So resolving forces: $P(\delta L)^2 = 4\delta F\,\theta/2$

Now $\theta = \delta L/r$ ⇒ $\delta F = \frac{1}{2}Pr\delta L$

Now $\tau_1 = \tau_2 = \frac{\delta F}{t\delta L} = \frac{P}{2}\frac{r}{t} \gg P$. $\tau_3 \sim P$ which we can ignore.

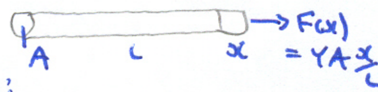
Now $e_1 = \frac{1}{Y}[\tau_1 - \sigma(\tau_2+\tau_3)] = \frac{(1-\sigma)}{Y}\frac{Pr}{2t} = e_2$

Now area: $(\delta L)^2 \to (\delta L)^2(1+e_2)(1+e_1)$ Hence $4\pi r^2 \to 4\pi r^2\left[1+\frac{(1-\sigma)}{Y}\frac{Pr}{2t}\right]^2$

So volume: $\frac{4\pi r^3}{3} \to \frac{4\pi r^3}{3}\left[1+\frac{(1-\sigma)}{Y}\frac{Pr}{2t}\right]^3$ i.e. $\frac{\delta V}{V} = \frac{3Pr}{2t}\cdot\frac{(1-\sigma)}{Y}$ to 1st order.

Elastic Energy

For a 1D elastic filament of length L:



Work done (W) in stretching wire to achieve a strain $e = \frac{x}{L}$ is:

$$\Delta E = W = \int_0^{eL} F(x)dx = \tfrac{1}{2}Ye^2AL = \tfrac{1}{2}\tau e V = \tfrac{1}{2}\times(\text{Final stress})\times(\text{Final strain})\times(\text{Volume})$$

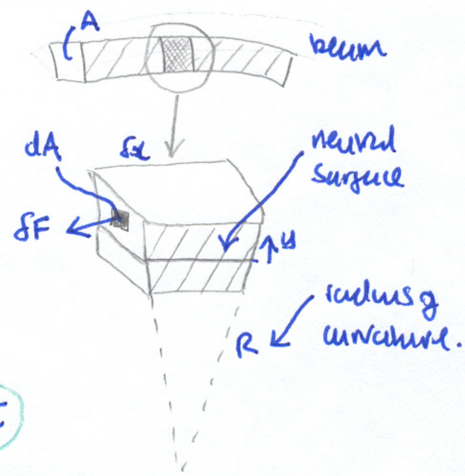
↳ Generalise for tensors $\underline{\underline{e}}$ and $\underline{\underline{\tau}}$: $\underset{\text{per unit vol.}}{W} = \frac{1}{2}\sum_{ij}\tau_{ij}e_{ij} = \frac{1}{2}\mathrm{Tr}(\underline{\underline{\tau}}\,\underline{\underline{e}})$

$= \frac{1}{2}(\tau_1e_1 + \tau_2e_2 + \tau_3e_3)$ for isotropic medium. $W = U$, stored elastic energy per unit volume. i.e. for Ball above: $\Delta E = 4\pi r^2 t\cdot U = 4\pi r^2 t\left(\frac{1}{2}(\tau_1e_1 + \tau_2e_2)\right)$

$= \frac{(1-\sigma)}{Y}\left(\frac{Pr}{2t}\right)^2\cdot 4\pi r^2 t = \frac{(1-\sigma)}{Y}\cdot\frac{\pi r^4P^2}{t}$ But $\frac{\Delta V}{V} = \frac{(1-\sigma)}{Y}\cdot\frac{3Pr}{2t}$ ⇒ $\Delta V = \frac{(1-\sigma)}{Y}\cdot\frac{2\pi r^4 P}{t}$

⇒ $\Delta E = \frac{1}{2}P\Delta V$

Bending Beams, cantilevers and 2nd Moment of Area.



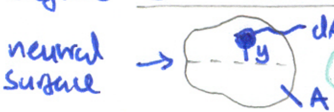
* Stress and strain vary ~ LINEARLY across cross section of bent beam.
* There is a NEUTRAL SURFACE where stress and strain = 0.

At y above neutral surface: $e = y/R$, ∴ $\tau = Ye = Yy/R$

∴ Torque on element $= y\,dF = y\tau\,dA = Yy^2/R\,dA$

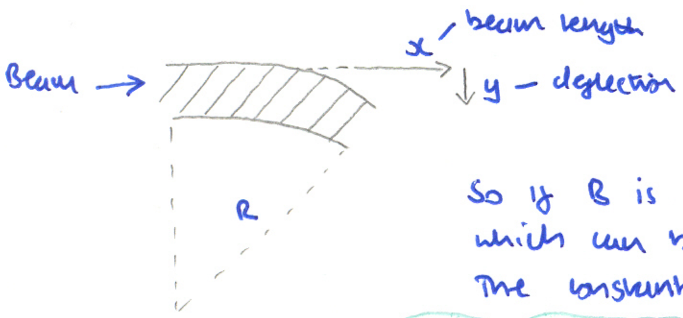
∴ Total torque = BENDING MOMENT (B) $= \frac{Y}{R}\int y^2\,dA$

Define I = 2nd moment of area: $I = \int y^2\,dA$ ⇒ $BR = YI$



$I = \int_A y^2\,dA$ [A = Beam cross section - by definition of a Beam, A is constant throughout length].

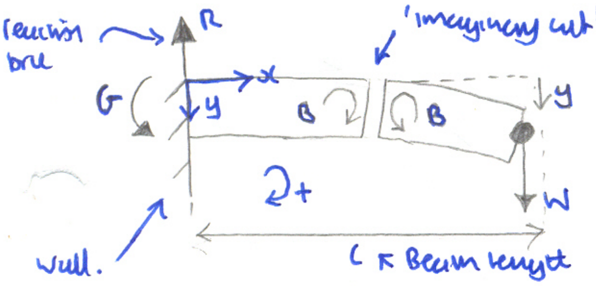
Now a CANTILEVER is a beam constrained at one or more points along it's length. Can use $\boxed{BR = IY}$ to predict shape of bent beam using



$$\boxed{\frac{1}{R} = \frac{d^2y/dx^2}{\left[1+(dy/dx)^2\right]^{3/2}} \approx \frac{d^2y}{dx^2}}\ (y \text{ small}).$$

So if B is found as a function of x; $\boxed{YI\frac{d^2y}{dx^2} = B(x)}$ which can be integrated to yield $y(x)$.
The constants of integration are determined by BOUNDARY CONDITIONS, e.g. $\boxed{y(x_0) = \text{constant}}$, $\boxed{\left.\frac{dy}{dx}\right|_{x_0} = \text{constant}}$ etc...

B(x) can be found by following construction (or one like it): Note at any one point on static beam: Bending moment must balance torque due to external forces. Can also use e.g. considerations on beam as a whole.

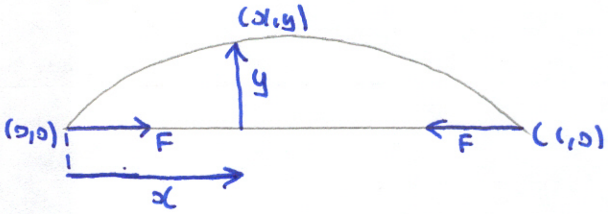


So $B = W(L-x)$ and $B - G + Rx = 0$ using sections
$R = W$, $G = WL$ using whole beam.
↳ confirms $B = W(L-x)$. OFTEN NEED ONLY CONSIDER ONE 'SECTION'. E.g. will ensure consistency.

So in this case $\boxed{BR = IY}$

$$\Rightarrow \boxed{YIy = -\tfrac{1}{6}Wx^3 + \tfrac{1}{2}WLx^2 + Cx + D}$$

For a BOWED (BUCKLING) BEAM



$$\boxed{B(x) = -Fy \approx YI\frac{d^2y}{dx^2}}$$

$\Rightarrow \boxed{y = C\sin kx + D\cos kx}$ where $\boxed{k^2 = F/YI}$. Now b.c's are $y(0) = y(L) = 0$
$\Rightarrow D = 0$ and C is arbitrary! [We also need $k = \pi/L$]. So in this case define CRITICAL BOWING FORCE, $\boxed{F_c = YI\frac{\pi^2}{L^2}}$ (EULER FORCE)

For $F < F_c$, no bowing. $F > F_c$ beam buckles. Now in this regime FRACTURE will soon occur and C arbitrariness is further added to by small y approximation being invalid as C becomes large. Hence do not consider $F > F_c$ regime for most purposes as beam will break soon!

Normal Modes The general motion of a set of linear (Hookean) oscillators can be described as a superposition of NORMAL MODES. i.e a LINEAR COMBINATION of harmonic functions, each with a characteristic frequency of the normal mode. Each oscillator will of course have different amplitudes for it's normal modes but by definition, a normal mode is one characteristic to the entire oscillator set.

i.e if displacements from equilibrium of the oscillator set are $x_1, x_2, x_3 \ldots$
then: $x_1 = \boxed{\sum_i a_i\cos(\omega_i t + \phi_i)}$, $x_2 = \sum_i b_i\cos(\omega_i t + \theta_i)$ ...... [coefficients determined by b.c's i.e $x(0) =, \dot{x}(0) = \ldots$
with $\frac{\omega_i}{2\pi}$ being ith normal mode frequency.

For convienience we can group $x_1, x_2 \dots$ into the vector $\underline{x}$ where $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}$

Hence: (*) $\underline{x} = \sum_i \underline{e}_i \cos\omega_i t + \sum_i \underline{f}_i \sin\omega_i t$ ← phase '$\cos(\omega t+\phi)$' formalism not appropriate here.

So how do we find $\{\omega_i\}$ and $\{\underline{e}_i\}$, $\{\underline{f}_i\}$? → TWO METHODS.

1) Analyse the forces on each oscillator and write down Newton's 2nd law for each coordinate of $\underline{x}$. Will find mathematical relationship that yields desired results (solve simultaneous equations generated by substituting $\underline{x}$ for (*)). Writing results in MATRIX FORM makes this easier → equations like $\underline{\underline{A}}\,\underline{e} = 0$ ($\Rightarrow \underline{e} = 0$ if $\underline{\underline{A}}$ has inverse, $|\underline{\underline{A}}| = 0$ if $\underline{e} \neq 0$ ⇒ $\underline{\underline{A}}$ does not have an inverse) ⇒ $|\underline{\underline{A}}| = 0$ for non trivial solutions.

2) Write down ENERGY equations. Write in matrix form.

i.e. $T = \sum_i \frac{1}{2} m_i \dot{x}_i^2 = \frac{1}{2}\dot{\underline{x}} \cdot \underline{\underline{M}}\,\dot{\underline{x}}$ with $\underline{\underline{M}} = m_{(i)}\delta_{ij}$ [kinetic energy]

[Potential energy] $U = \sum_i \frac{1}{2} k_i (\text{combination of } x \text{ terms})^2 = \frac{1}{2}\underline{x} \cdot \underline{\underline{K}}\,\underline{x}$ with $\underline{\underline{K}}$ being some matrix of relevant constants for each problem.

↑ a la Hookes law elastic energy $\frac{1}{2}kx^2$

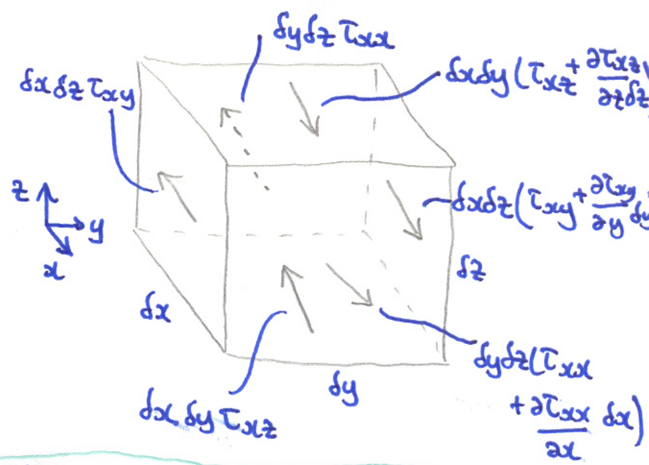
Note any of linear terms in U will dissapear by results of next analysis ← No coefficient of $\sin^2\omega t \dots$ for linear terms. Next substitute (*) for $\underline{x}$, and use fact that $\sin^2 ax + \cos^2 ax = 1$ to match coefficients since $T + U = E =$ constant. (Assume system is lossless). {The vector representation ⇒ each mode has constant energy so we deduce NO ENERGY TRANSFER BETWEEN MODES for a lossless system. (This uses trig identity above → find $E_n \equiv E_i$ (oops.) by substituting (*) into $E = T + U$).}

Now from coefficient matching yield '$\underline{\underline{A}}\,\underline{e} = 0$' like equation which yields $\{\omega_i\}$ from $|\underline{\underline{A}}| = 0$ and 'un-normalised' $\{\underline{e}\}$. B.C's will normalise $\{\underline{e}\}$.

As number of coupled oscillators → infinite continuum, NORMAL MODES → STANDING WAVES. i.e Behaviour of oscillator continuum will be governed by the WAVE EQUATION, $\frac{\partial^2\psi}{\partial t^2} = c^2\nabla^2\psi$ [c = wave speed]. Note vector operator ∇^2 used in general (3D) → in this case for discrete oscillators care must be taken to not confuse $\underline{x}$ with REAL displacement vectors!

Now for MOLECULES of N atoms → 3N normal modes. (Some may have zero frequency). In THERMODYNAMICS, each DEGREE OF FREEDOM yields $\frac{1}{2}kT$ to total mean energy (classical limit). So TRANSLATIONAL normal modes (zero frequency) yield $\frac{kT}{2}$, so do PURE ROTATIONAL modes, and Vibrating and Twisting modes yield kT (KE and PE).

Elastic waves consider pres on elastic element $\delta x\delta y\delta z$ in x direction: Applying Newton's 2nd law: [Assume isotropic, uniform density ρ]



$$\delta x\delta y\delta z\,\rho\ddot{u}_x = \delta F_x = \delta x\delta y\delta z\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}\right)$$

displacement u_x

$$\Rightarrow \rho\ddot{u}_x = \frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}$$

Similarly: $\rho\ddot{u}_y = \frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}$ and $\rho\ddot{u}_z = \frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}$

Above are generalisations of special cases which lead to WAVE EQUATIONS.

* PRESSURE WAVES (Longitudinal): No shear stresses i.e for x displacement

$$\rho\ddot{u}_x = \frac{\partial\tau_{xx}}{\partial x}$$

Now for a Bulk medium: $\tau_{xx} = M_L e_{xx}$ and $e_{xx} = \frac{\partial u_x}{\partial x}$

$$\Rightarrow \rho\frac{\partial^2 u_x}{\partial t^2} = M_L\frac{\partial^2 u_x}{\partial x^2}$$

$\Rightarrow$ wave equation, wave speed $c = \sqrt{\frac{M_L}{\rho}}$

↓ i.e. $u_x = f(x \pm ct)$.

* SHEAR WAVES (Transverse): No longitudinal stresses i.e for wave propagation in y direction, displacement in x direction: $\Rightarrow u_y = 0$ $e_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right) = \frac{1}{2}\frac{\partial u_x}{\partial y}$

and $\tau_{xy} = \frac{Y}{1+\sigma}e_{xy} = 2n\,e_{xy}$ since $\tau_{xx} = \tau_{xz} = 0$ (stress, strain relation).

Now from above, $\rho\ddot{u}_x = \partial\tau_{xy}/\partial y$

$$\Rightarrow \rho\frac{\partial^2 u_x}{\partial t^2} = n\frac{\partial^2 u_x}{\partial y^2}$$

$\Rightarrow$ wave equation, wave speed $c = \sqrt{\frac{n}{\rho}}$ ← same for TORSIONAL OSCILLATIONS.

Notes: Shear waves could be for $\rho\ddot{u}_x = \partial\tau_{xz}/\partial z$ in x direction as well.

Hence: For each direction of propagation, 3 'polarisations' of elastic wave caused by some disturbance.

1 (longitudinal) PRESSURE WAVES $c = \sqrt{M_L/\rho}$

2 (Transverse) SHEAR WAVES $c = \sqrt{n/\rho}$

Note assume non dispersive waves

Waves in fluids (i.e liquid or gas) NO SHEAR STRESSES $\Rightarrow n = 0$.

$\therefore$ only longitudinal waves. $c = \sqrt{\frac{M_L}{\rho}}$ BUT, since $M_L = B + \frac{4}{3}n \Rightarrow M_L = B$ for fluids. Hence $c = \sqrt{B/\rho}$ for a fluid. Now if fluid is an IDEAL GAS: $\rightarrow$ pressure changes are faster than heat fluxes hence for a pressure wave $\Rightarrow$ ADIABATIC CHANGE in gas. i.e. $PV^\gamma =$ constant. $\Rightarrow \frac{\Delta p}{p} = -\gamma\frac{\Delta V}{V}$ with $\gamma = C_p/C_v$ [$C_p = C_v + R$] so $B = \gamma p$ (Δp is 'excess pressure')

so $c_{gas} = \sqrt{\frac{\gamma p}{\rho}}$ Now $p = \frac{nRT}{V} = \frac{mRT}{m_{mol}V} = \frac{\rho RT}{m_{mol}} \Rightarrow c_{gas} = \sqrt{\frac{\gamma RT}{m_{mol}}}$