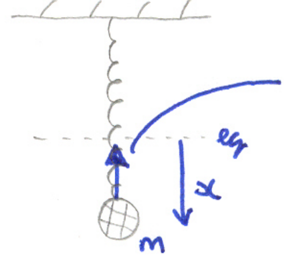
WAVES

Single, Driven, Damped 1D oscillator



Hookeian restoring force + Driving force + velocity dependant damping force.

General equation for linear damping force: $F_{damp} \propto -\dot{x}$ is:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t) \quad (*)$$

Notes: γ is a damping coefficient
$\frac{\omega_0}{2\pi}$ is frequency of oscillations when $f(t)=0$, $\gamma=0$. $f(t)$ = Driving force/m

LINEAR 2nd order differential equation. (ordinary).
Hence general solution is solution to $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$ + function usually similar to $f(t)$ that satisfies $(*)$. i.e, $x(t) = CF + PI$. Note CF will have 2 arbitrary constants determinable by b.c's. i.e, $x(0)$ = constant, $\dot{x}(0)$ = constant known.
Now interesting case occurs when $f(t) = \mathrm{Re}\left(\frac{F}{m}e^{i\omega t}\right)$ [m = oscillator mass]
For convenience write $x(t) = \mathrm{Re}(z(t)) \Rightarrow (*)$ becomes:

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = \frac{F}{m}e^{i\omega t}$$

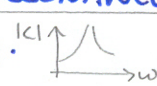
Now CF (= TRANSIENT solution) takes form $e^{-\gamma t}\left(Ae^{i\omega_1 t} + Be^{-i\omega_1 t}\right)$ with $\omega_1^2 = \omega_0^2 - \gamma^2$.
PI (= STEADY STATE) takes form $Ce^{i\omega t}$ which by substitution yields:

$$C = \frac{F}{m}\left\{\frac{(\omega_0^2-\omega^2) - 2i\gamma\omega}{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}\right\}$$

since $e^{-\gamma t}$ prefix will cause transients to die /decay with time steady state becomes most important in the long term.

In steady state $\mathrm{Re}(z(t)) = x(t) = |C|\cos(\omega t + \arg(C))$

So $|C|$, the steady state amplitude demonstrates the forced response of the oscillator. $|C|$ as a function of ω will demonstrate RESONANCE, i.e, a peak at some value of ω. (usually close to ω_0).



Note, since $z = Ce^{i\omega t}$ in steady state, velocity response, characterised by $\dot{z} = i\omega Ce^{i\omega t}$ will also exhibit resonant behaviour. Note phase change with z of $\pi/2$.

Now define IMPEDANCE ($|Z|$) as '$\left|\frac{\text{Force}}{\text{velocity}}\right|$' which in above case

$$|Z| = \left|\frac{Fe^{i\omega t}}{i\omega Ce^{i\omega t}}\right| = \left|\frac{F}{\omega C}\right|.$$

This clearly will exhibit resonant behaviour as well.

Not so useful in this case but very handy when considering waves on boundaries and more generally.

Other definitions (Applicable to $(*)$).

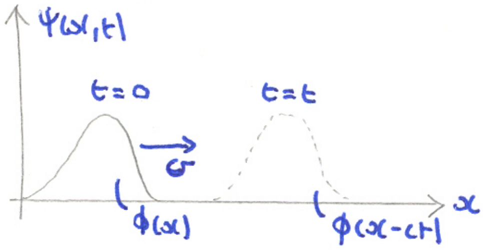
QUALITY FACTOR: $Q = \frac{\omega_0}{2\gamma}$ – used to characterise oscillation.

[For $f(t)=0$, Q = number of radians the oscillation goes through for the energy to fall by $1/e$; 2π times the energy stored in the system / energy lost per cycle]

INSTANTANEOUS POWER: $P = \mathrm{Re}(Fe^{i\omega t})\,\mathrm{Re}(\dot{z})$ MEAN POWER $= \frac{1}{2}\mathrm{Re}(Fv^*)$
$\quad\quad F \quad\quad v$

The wave equation

- consider a function $\phi(x)$ propagating at velocity v in the +ve x direction s.t. it's profile does not change.



i.e, $\psi(x,t) = \phi(x - vt)$ let $u = x - vt$

$\Rightarrow \psi(x,t) = \phi(u)$. By chain rule: $\frac{\partial \psi}{\partial x} = \frac{d\phi}{du}\frac{\partial u}{\partial x} = \frac{d\phi}{du} \Rightarrow \frac{\partial^2 \psi}{\partial x^2} = \frac{d^2\phi}{du^2}\frac{\partial u}{\partial x} = \frac{d^2\phi}{du^2}$

Similarly: $\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{d^2 f}{du^2}$ $\Rightarrow$ 1D wave equation: $\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$

which generalises in 3D to $\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$

Note LINEAR EQUATION - g. solution $f(x-ct) + g(x+ct)$.

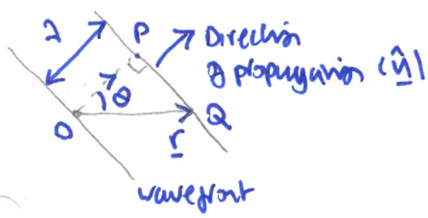
Plane and spherical Harmonic waves

In above example $f(x)$ is shifted by ct in x direction via transform of $x \to x - ct$. For a time function i.e, $e^{i\omega t}$, same propagation can be explained by a time transform of $t \to t - x/v$ i.e, $\psi(x,t) = Ae^{i\omega(t - x/v)}$ $\Rightarrow$ $\psi(x,t) = Ae^{i(\omega t - kx)}$

with k being ω/v. Note ωt and kx are non dimensional angles or PHASES. kx altered by 2π when $x = \lambda$ (wavelength) so $k = 2\pi/\lambda$

Similarly $\omega T = 2\pi \Rightarrow \omega = 2\pi f$ (T = period, f = frequency) so as $k = \omega/v$

$\Rightarrow v \cdot \frac{2\pi}{\lambda} = 2\pi f \Rightarrow v = f\lambda$. Note by convention a 1D wave (travelling) is written as $\psi(x,t) = Ae^{i(kx - \omega t)}$ for +ve x propagation. For a 3D plane wave:

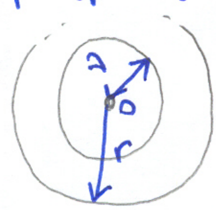


* P→Q line has same phase relative to O.
* Phase at P = $kr\cos\theta = \underline{k}\cdot\underline{r}$ defining wave-vector $\underline{k}$ as $k\hat{\underline{n}} = \frac{2\pi}{\lambda}\hat{\underline{n}}$.

Hence $\psi(\underline{r},t) = Ae^{i(\underline{k}\cdot\underline{r} - \omega t)}$ for a plane wave.

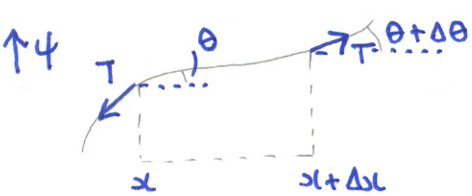
For a spherical wave: since wave energy $\propto |\psi|^2$, and energy flow across a sphere of radius $r \propto 4\pi r^2|\psi|^2$; assuming no losses: $4\pi r^2|\psi|^2 = \text{constant}$ $\Rightarrow \psi \propto \frac{1}{r}$ Hence for spherical wave:

$$\psi(r,t) = \frac{A}{r}e^{i(kr - \omega t)}$$



Waves (Transverse) on a Tense String

- similar method used for ELASTIC WAVES



let ψ be displacement (transversely) of a uniformly tense string under tension T. $\Rightarrow$ regarding forces on segment Δx (in ψ direction), if string has mass / unit length ρ by Newton's 2nd law: (assume θ is small)

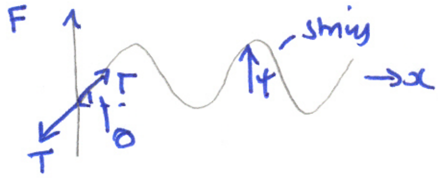
$$\rho\Delta x \frac{\partial^2 \psi}{\partial t^2} = T\sin(\theta + \Delta\theta) - T\sin\theta \simeq T\tan(\theta+\Delta\theta) - T\tan\theta$$

Now $\tan\theta = \frac{\partial \psi}{\partial x}$ and $\tan(\theta + \Delta\theta) = \frac{\partial \psi}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)\Delta x = \frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial x^2}\Delta x$

so $\rho\Delta x\frac{\partial^2 \psi}{\partial t^2} = T\frac{\partial^2 \psi}{\partial x^2}\Delta x$ i.e, wave equation, $v = \sqrt{\frac{T}{\rho}}$

More waves on a string

To produce a transverse travelling wave must apply driving force which balances the transverse component of string tension at the point of driving.



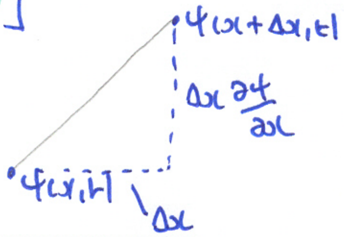
i.e. driving force $= F = -T\sin\theta \simeq -T\tan\theta \simeq -T\frac{\partial\psi}{\partial x}$

So 'impedance' of string $= \left|\frac{\text{'Force'}}{\text{'velocity'}}\right|$

$= \left|\frac{-T\,\partial\psi/\partial x}{\partial\psi/\partial t}\right|$ Now $\psi(x,t) = \phi(x \pm vt)$ $\Rightarrow$ $Z_{string} = \frac{T}{v} = \sqrt{T\rho} = \rho v$

[Note: for elastic waves: Pressure waves have $Z_p = \left|\frac{\tau_{xx}}{\dot{u}_x}\right| = \left|\frac{M_L\,\partial u_x/\partial x}{\partial u_x/\partial t}\right| = \sqrt{M_L\rho}$

Shear waves have $Z_s = \left|\frac{\tau_{xy}}{\dot{u}_x}\right| = \left|\frac{\mu\,\partial u_x/\partial y}{\partial u_x/\partial t}\right| = \sqrt{\mu\rho}$]



Now for waves on a string: Work done on segment Δx in passing of wave (= PE density Δx)

$= T.\text{extension} = T\left\{\left(\Delta x^2 + \left(\Delta x\frac{\partial\psi}{\partial x}\right)^2\right)^{\frac{1}{2}} - \Delta x\right\}$

$\simeq \frac{T}{2}\left(\frac{\partial\psi}{\partial x}\right)^2\Delta x$ (expanding binomially) $\Rightarrow$ PE density $= \frac{T}{2}\left(\frac{\partial\psi}{\partial x}\right)^2$

Now KE density $= \frac{1}{2}\rho\left(\frac{\partial\psi}{\partial t}\right)^2$ But since $\psi(x,t) = \phi(x \pm vt)$, $\left(\frac{\partial\psi}{\partial t}\right)^2 = v^2\left(\frac{\partial\psi}{\partial x}\right)^2$

[use $u = x \pm ct$ and chain rule] $\Rightarrow$ PE density $= \frac{T}{2}\cdot\frac{1}{v^2}\left(\frac{\partial\psi}{\partial t}\right)^2$ Now $v^2 = \frac{T}{\rho}$

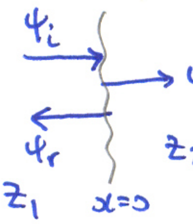
So PE density $= \frac{\rho}{2}\left(\frac{\partial\psi}{\partial t}\right)^2$ i.e. PE density = KE density. So TOTAL energy density = PE + KE $= \rho\left(\frac{\partial\psi}{\partial t}\right)^2$. For Harmonic wave $\psi(x,t) = Ae^{i(kx-\omega t)}$

$\langle\text{energy density}\rangle = \rho\left\langle\frac{\partial\psi}{\partial t}\right\rangle^2 = \rho\langle -i\omega Ae^{i(kx-\omega t)}\rangle^2 = \frac{1}{2}\rho\omega^2A^2$

so after time Δt, extra length $v\Delta t$ is oscillating so average wave power input $= \frac{1}{2}\rho\omega^2A^2v\Delta t/\Delta t = \frac{1}{2}\rho\omega^2A^2v = \frac{1}{2}Z\omega^2A^2$

These results are GENERAL wave phenomena / properties.

Reflection and Transmission of INCIDENT PLANE WAVES on a boundary



Consider boundary separating two media of impedances Z_1 Z_2.
Boundary conditions are: ψ continuous and $\frac{\partial\psi}{\partial x}$ continuous.
(i.e. no 'kinks' and finite transverse force $= -T\,\partial\psi/\partial x$)
$\Rightarrow$ ω is constant but since $Z = \text{'}T/v\text{'}$ $\Rightarrow$ k, v must vary.

let $\psi_i = Ie^{i(k_1x-\omega t)}$ $\psi_r = Re^{i(-k_1x-\omega t)}$ $\psi_t = Te^{i(k_2x-\omega t)}$ so applying b.c's at $x=0$: ① $I + R = T$ ② $k_1I - k_1R = k_2T$ Now $k = \omega/v$

so ②: $\frac{I}{v_1} - \frac{R}{v_1} = \frac{T}{v_2}$ Now 'T/v' (T= Tension) $= Z$ so since T must be constant: $\Rightarrow$ $Z_1I - Z_1R = Z_2T$ ② Now $Z_2$① − ② $\Rightarrow$ $I(-Z_1+Z_2) + R(Z_2+Z_1) = 0$

$\Rightarrow$ $\frac{R}{I} = \frac{Z_1-Z_2}{Z_1+Z_2}$, and $Z_1$① + ② $\Rightarrow$ $2Z_1I = T(Z_1+Z_2)$ $\Rightarrow$ $\frac{T}{I} = \frac{2Z_1}{Z_1+Z_2}$

R/I and T/I are REFLECTION COEFFICIENTS.

Reflection and Transmission of energy - Power of Harmonic wave (average) = $\frac{1}{2}Z\omega^2A^2$ ($A = |\psi|$). $\therefore$ on our boundary: energy input $= \frac{1}{2}Z_1\omega^2I^2$, reflection $= \frac{1}{2}Z_1\omega^2R^2$, transmission $= \frac{1}{2}Z_2\omega^2T^2$

$\Rightarrow \dfrac{\text{reflected energy}}{\text{incident energy}} = \dfrac{Z_1}{Z_1}\cdot\dfrac{R^2}{I^2} = \left(\dfrac{Z_1-Z_2}{Z_1+Z_2}\right)^2$; $\dfrac{\text{transmitted}}{\text{incident}} = \dfrac{Z_2}{Z_1}\cdot\dfrac{T^2}{I^2} = \dfrac{4Z_1Z_2}{(Z_1+Z_2)^2}$

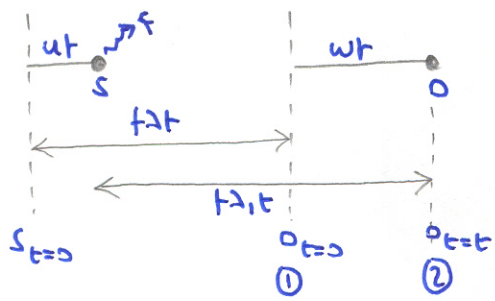
Note $Z = \left|\text{Force}/\text{velocity}\right|$ is not always true and Z (i.e. in EM) can be COMPLEX. In this case replace above by $\left|\dfrac{Z_1-Z_2}{Z_1+Z_2}\right|^2$ and $1-\left|\dfrac{Z_1-Z_2}{Z_1+Z_2}\right|^2$ since $\left|\dfrac{R}{I}\right|^2 = \dfrac{R}{I}\cdot\dfrac{R^*}{I^*}$ and Power $= \frac{1}{2}\mathrm{Re}(Z)\omega^2|A|^2$

Proof: Power $= \frac{1}{2}\mathrm{Re}(Fv^*) = \frac{1}{2}\mathrm{Re}(Zvv^*) = \frac{1}{2}\mathrm{Re}(Z)|v|^2$ Now $v = -i\omega\psi$ so $|v|^2 = \omega^2|\psi| = \omega^2|A|^2$

Standing waves - If some boundary condition $\Rightarrow \psi(\underline{r},t) = 0$ for some $\underline{r}$ $\Rightarrow$ standing wave solutions by virtue of no transmission past the boundary. (Effective $Z_2 = \infty$). i.e. for clamped string of length L, $\psi = 0$ at $0, L$ $\Rightarrow$ conditions of allowed ω's of vibration. For standing wave solutions one may try SEPERABLE ψ's and substitute into WAVE EQUATION and then solve subject to b.c's (i.e. $\psi(\underline{r},t) = X(x)Y(y)Z(z)T(t)$) OR use 'no ~~reflection~~ transmission' principle and superpose permutations of $\psi = Ae^{i(\underline{k}\cdot\underline{r}-\omega t)}$ for different $\underline{k}$'s. Either way solution will be QUANTISED by b.c's - for a string clamped, $\psi(x,t) = 2A\sin kx\sin\omega t$, $kL = n\pi$ by b.cs $\Rightarrow k = \frac{n\pi}{L}$ ($n \in \mathbb{Z}$) $\Rightarrow \omega = \frac{vn\pi}{L}$. For a 2D plate ($a\times b$) $\omega = v\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}$, for a 3D box ($a\times b\times c$) $\omega = v\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{l^2}{c^2}}$ with $m, n, l \in \mathbb{Z}$.

For Longitudinal waves see 'Elastic waves' (Dynamics) section. Note ACOUSTIC IMPEDANCE for pressure waves in a gas = 'Excess pressure/velocity' $= \sqrt{\gamma P\rho} = \rho v = \gamma P/v$ [$v_{gas} = \sqrt{\frac{\gamma P}{\rho}}$ P = pressure, ρ = density, $\gamma = c_p/c_v$]

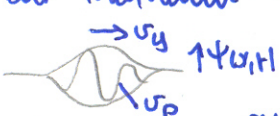
Doppler effect for sound waves consider source moving at speed u, observer at speed w with source emitting sound at frequency f. Consider stationary observer at points ① and ②.



Now ① and ② both measure speed of sound (in their frame) at $v = f\lambda$. From $S_{t=0}$ ① is a distance $f\lambda t$ away and ② a distance $ut + f\lambda_1 t$ away. However, ① and ② measure sound frequencies different, denote $f_①$ and $f_②$. $f_① = f$, $f_② = v/\lambda_1$ Now since: $ut + f\lambda_1 t = f\lambda t + wt \Rightarrow v + w = u + f\lambda_1$

$\Rightarrow \lambda_1 = \dfrac{v+w-u}{f}$ Hence $f_② = \dfrac{vf}{v+w-u}$ $\therefore f_②/f_① = \dfrac{v}{v+w-u}$ Now observer at O should measure the same f as ② since they are at the same position except O will measure v as $v-w$ since it is moving at $+w$ relative to ②. So $\dfrac{f_O}{f} = \dfrac{v-w}{v-u}$

Dispersive waves A disturbance or WAVE PACKET may be a Fourier synthesis of many different frequency Harmonic waves. Packet moves at GROUP VELOCITY v_g but individual waves may move at a different, PHASE VELOCITY, v_p. Hence DISPERSION.



Now in packet point of max amplitude corresponds to point x at time t where all Fourier components have SAME PHASE $= kx - \omega t + \phi$. i.e. PHASE is independent of ω. Hence $\frac{d}{d\omega}(kx - \omega t + \phi) = 0 \Rightarrow t - \left(\frac{dk}{d\omega}\right)_{\omega_0} x = 0$ So need dispersion relation $\omega(k)$

[ω_0 is AVERAGE group frequency] So $\frac{x}{t} = \left(\frac{d\omega}{dk}\right)_{\omega_0} = v_g(\omega_0)$ Note since $v_p k = \omega \Rightarrow v_g = v_p - \lambda\frac{dv_p}{d\lambda}$