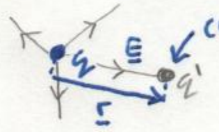
ELECTROMAGNETISM

– Theory of electric and magnetic fields generated by charged particles.

Electrostatics ⇒ no time variation of charge distribution i.e. no currents or magnetic fields.

Now for a charged particle q (stationary), Force $\underline{F}$ on it results from the local electric field $\underline{E}$ (resulting from other charged particles in space). $\underline{F} = q\underline{E}$. Now $\underline{E}$ for point source charge at $-\underline{r}$ from our observed charge is given by Coulomb's law. (↑ generated by)

$$\underline{E} = \frac{q}{4\pi\varepsilon_0}\frac{\underline{r}}{r^3}$$



charge feels force $\underline{F} = q'\underline{E}$

Now $\underline{E}$ is related to a scalar function of position, V (the 'potential') s.t.

$$\underline{E} = -\nabla V$$

↑ ⇒ $\underline{E}$ is conservative

Now Gauss' theorem states:

$$\oint_A \underline{E}\cdot d\underline{S} = \frac{1}{\varepsilon_0}\int_V \rho(\underline{r})\,dV \Rightarrow \nabla\cdot\underline{E} = \frac{\rho(\underline{r})}{\varepsilon_0}$$ by divergence theorem.

{ $d\underline{S}$ is vector area element of area A which encloses volume V. $\rho(\underline{r})$ is charge distribution.

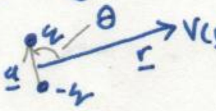
This leads to $\nabla^2 V = -\rho(\underline{r})/\varepsilon_0$ (Poisson's equation)

and $\nabla^2 V = 0$ (Laplace's equation) when $\rho(\underline{r}) = 0$.

Solutions of these yield any possible electrostatic field distribution if b.c's are specified.

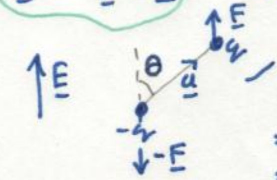
Now charges close to one another form multipoles which can be treated as single objects when Coulomb's law is approximated in the far field limit. For a dipole of charges + and − q separated by vector $\underline{a}$, Define DIPOLE MOMENT

$\underline{p} = q\underline{a}$ and $V = \dfrac{|\underline{p}|\cos\theta}{4\pi\varepsilon_0 r^2}$

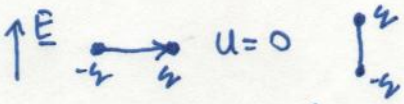


$\left[\text{so } V = \dfrac{\underline{p}\cdot\underline{r}}{4\pi\varepsilon_0 r^3}\right]$

Now in a uniform field, no net force on a dipole BUT there will be a couple $\underline{G} = \underline{p}\times\underline{E}$ and dipole will acquire potential energy $U = -\underline{p}\cdot\underline{E}$



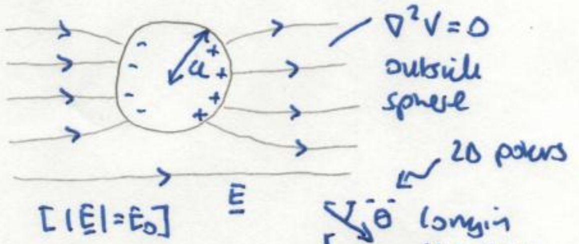
net force = 0
but couple $|\underline{G}| = aF\sin\theta$
$= aqE\sin\theta = pE\sin\theta$
$= |\underline{p}\times\underline{E}|$ so $\underline{G} = \underline{p}\times\underline{E}$



So intermediate case, $U = -Eqa\cos\theta = -\underline{p}\cdot\underline{E}$

$U = -$ Force × displacement $= -Eqa = -pE$

Dipoles are useful in solving electrostatic field problems. i.e. conducting sphere in uniform $\underline{E}$ field. Good example of guessing solution to $\nabla^2 V = 0$ (outside the sphere in this case) and applying UNIQUENESS THEOREM which states if guess fits b.c.s then it must be the only solution.



Guess: $V(\text{outside sphere}, r \geq a) = \left(-E_0 r + \dfrac{p}{4\pi\varepsilon_0 r^2}\right)\cos\theta$ (uniform field; dipole)

B.C's are: $V = 0$ at $r = a$ (conductor) ⇒ $E_{\parallel}$ to surface $= 0$ ⇒ $p = 4\pi\varepsilon_0 a^3 E_0$

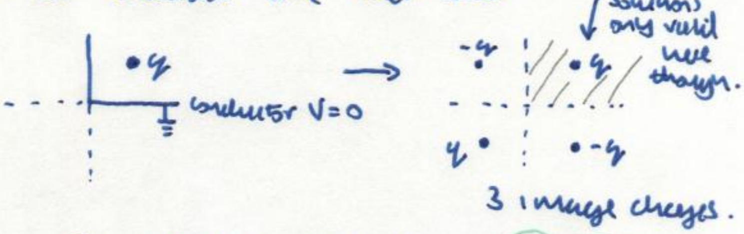
[Define Polarisability for conducting sphere α as $p/E_0 = 4\pi\varepsilon_0 a^3$] Hence $V(r \geq a) = \left(\dfrac{a^3}{r^2} - r\right)E_0\cos\theta$

We can now apply $\underline{E} = -\nabla V$ $\left(= -\left(\dfrac{\partial V}{\partial r}, \dfrac{1}{r}\dfrac{\partial V}{\partial\theta}\right)\right.$ in 2D polars) to find field. Note SURFACE CHARGE DENSITY σ is given by $\varepsilon_0 E_\perp$ (Gauss' law) $= 3\varepsilon_0 E_0\cos\theta$ in this case ($E_\perp = E_r$ here).

Note: for a dipole in a NON UNIFORM field there will be a net force of components $dF_x = q\,dE_x$, in general: $\underline{F} = (\underline{p}\cdot\nabla)\underline{E}$

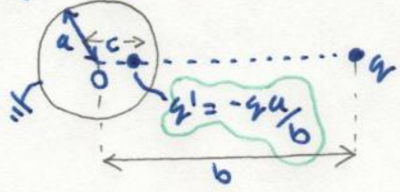
uniqueness theorem can also be applied to solutions of Poisson's equation for regions of space containing charges. For point charges near conductors use METHOD OF IMAGE CHARGES to guess field distribution. i.e. replacing conductor by 'image' charge s.t. b.c.s at conductor are met still.

For plane conductors mirror real charge inside conductor. i.e.

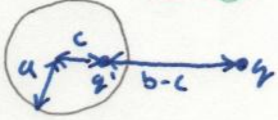


For spheres place image charge q' at inverse point.

$\boxed{c = a^2/b}$



For cylinders put charge $q' = -q$ at $c = a^2/b$

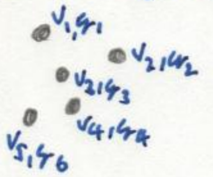


Electrostatic energy U present in an $\underline{E}$ field is given by

$U = \frac{1}{2}\sum_i q_i V_i$ (discrete distribution of charge) or $\frac{1}{2}\int_{\text{All space}} \rho V \, d\tau$ for continuous charge distribution.

Note $U = \frac{1}{2}\int_{\text{All space}} \underline{E}\cdot\underline{D}\, d\tau$ also – see below for definition of $\underline{D}$ field.

Some justification: Imagine charge system

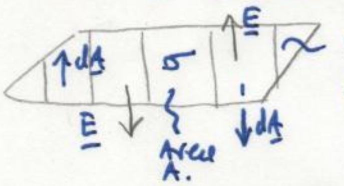


i.e. each charge has charge q_i and is in local potential V_i.

Since U is a state function we can pick a process where we have a charge system but gradually 'turn up' the charge from $0 \to q_i$. So at some point 'q_i' $= \alpha q_i$ ($0 \le \alpha \le 1$). Also 'V_i' $= \alpha V_i$. Now work done $\alpha \to \alpha + d\alpha = \sum_i \alpha V_i q_i \, d\alpha$ ← Add charge $q_i d\alpha$ to charge $q_i\alpha$ in potential αV_i

So Total work $= \int_0^1 \sum_i V_i q_i \alpha \, d\alpha = \frac{1}{2}\sum_i q_i V_i$ // ⚠ Must assume local potential is constant as 'q_i' goes from αq_i to $(\alpha + d\alpha) q_i$

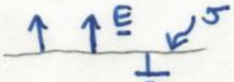
Force on surface of charged conductor – consider charged surface.



charged surface, charge density σ

Gauss: $\int \underline{E}\cdot d\underline{A} = Q/\varepsilon_0 = \sigma A/\varepsilon_0$

$\Rightarrow 2EA = \sigma A/\varepsilon_0 \Rightarrow E = \sigma/2\varepsilon_0$ Now for conductor

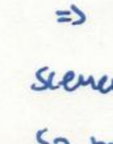


– get $\uparrow \hat{\underline{x}}$ scenario by adding $\underline{E}_{ext} = \frac{\sigma}{2\varepsilon_0}\hat{\underline{x}}$ to charged surface. (Removes ↓ part).

So for conductor, force on charges $= A\sigma \underline{E}_{ext} = A\frac{\sigma^2}{2\varepsilon_0}\hat{\underline{x}}$ so force per unit area F' has magnitude $F' = \frac{\sigma^2}{2\varepsilon_0}$

Dielectrics and polarisation charges – Insulators can become polarised by electric fields. Induced dipole $\underline{p}\,(= q\underline{a})$ per atom, $\underline{P} = N\underline{p}$ per unit volume.

Now for volume V with surface S in electric field s.t. V's contents are polarised:



(No external charges really)

Surface S has surface charge density $\sigma = \underline{P}\cdot\hat{\underline{n}} = \underline{P}\cdot d\underline{S}/dS$. Now surface

$q\,dx = P_x\, dx\,dy\,dz \quad \sigma = P_x = q/dydz$

polarisation charge must have come from V so by charge conservation:

$\oint_S \sigma \, dS + \int \rho_p \, d\tau = 0$ (ρ_p = polarisation charge density) $\Rightarrow \oint_S \underline{P}\cdot d\underline{S} = \int \rho_p \, d\tau$

Now by divergence theorem: $\oint_S \underline{P}\cdot d\underline{S} = \int_V \nabla\cdot\underline{P}\, d\tau$ so by equating integrands $\Rightarrow \rho_p = -\nabla\cdot\underline{P}$

→ Modify Gauss' theorem for dielectrics: $\oint \underline{E}\cdot d\underline{S} = \frac{1}{\varepsilon_0}\int d\tau(\rho_f + \rho_p)$ (ρ_f = free charges)

$\Rightarrow \oint \underline{E}\cdot d\underline{S} = \frac{1}{\varepsilon_0}\int d\tau(\rho_f - \nabla\cdot\underline{P}) = \int d\tau \nabla\cdot\underline{E}$ so $\rho_f = \nabla\cdot(\varepsilon_0\underline{E} + \underline{P})$ Define $\underline{D} = \varepsilon_0\underline{E} + \underline{P}$

$\Rightarrow \nabla\cdot\underline{D} = \rho_f$ (MAXWELL 1) so $\oint \underline{D}\cdot d\underline{S} = \int \rho_f \, d\tau$

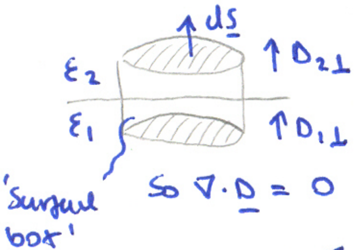
Now for isotropic medium $\underline{P} \parallel \underline{E}$ so: $\underline{D} = \varepsilon_0\underline{E} + \underline{P} = \varepsilon\varepsilon_0\underline{E}$ $\quad \varepsilon$ = 'permittivity'

Define susceptibility χ by: $\underline{P} = \chi\varepsilon_0\underline{E}$ $\Rightarrow$ $\varepsilon = 1+\chi$. Note/ Poisson's equation becomes $\nabla^2 V = -\rho_f/\varepsilon\varepsilon_0$ for uniform dielectrics so replace ε_0 for $\varepsilon\varepsilon_0$ in all normal no non dielectric field solutions. (Though $\underline{E} = -\nabla V$ still holds since $\underline{E}$ is sourced by ρ_f and ρ_p).

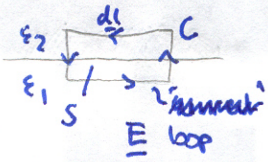
Properties of $\underline{D}$ and $\underline{E}$: * field lines begin and end on FREE CHARGES.

* At boundaries: $D_\perp$ is continuous, $E_\parallel$ is continuous $\Leftrightarrow$ V is continuous

no free charges



So $\nabla\cdot\underline{D} = 0$ in 'box'. $\rightarrow$ Applying Gauss' $\oint d\underline{S}\cdot\underline{D} = 0 \Rightarrow D_{1\perp} = D_{2\perp}$



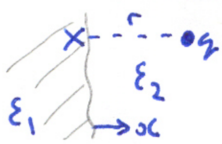
Now since V must be continuous (it is single valued at each point in space) and since $\underline{E} = -\nabla V$, consider
$$\int_S d\underline{S}\cdot\nabla\times\underline{E} = -\int_S d\underline{S}\cdot\nabla\times\nabla V = 0 \quad (\nabla\times\nabla = 0)$$

By Stokes theorem:
$$\int_S d\underline{S}\cdot\nabla\times\underline{E} = \oint_C \underline{E}\cdot d\underline{l} \quad \text{so} \quad \oint_C \underline{E}\cdot d\underline{l} = 0$$
$\Rightarrow E_{1\parallel} = E_{2\parallel}$ since 'sides' can be shrunk to zero.

These results in conjunction with 'uniqueness theorem' guesses at V allow us to look at field distributions in and outside of dielectrics in uniform fields. $\rightarrow$ If uniform field is assumed inside dielectric: (and outside external field has magnitude E_0) - inside field should be $\parallel$ to $\underline{E}_0$ and has in general magnitude: $E_i = \dfrac{E_0}{1+n\chi}$ $\left\{\begin{array}{ll}\text{Thin rod} & n=0 \\ \text{Slab} & n=1\end{array}\right.$ $\quad$ sphere $n=\frac{1}{3}$, cylinder $n=\frac{1}{2}$

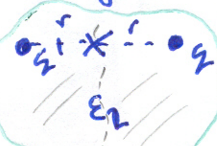
($\Rightarrow P_i = \varepsilon_0\chi E_i$). Useful result: for dielectric sphere $E_i = E_0 - \dfrac{P_i}{3\varepsilon_0}$

Image charges in dielectrics



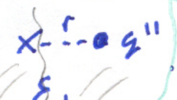
Want to replace screens by image charges.

$\rightarrow$ Solution: in region 2: replace by



i.e at reflection point

in region 1 replace by



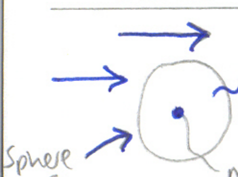
Now match b.c's at boundary (x=0) [Point X] for each region's viewpoint

$\Rightarrow$ V continuity at X: $q + q' = q''$ ①
$\left(V = -\dfrac{q}{4\pi\varepsilon_0 x}\right)$ (same as $E_\parallel$ cont.) Now $D_\perp$ cont yields:
coulomb's law $\rightarrow$ use $\underline{E} = -\nabla V$

$$\varepsilon_1\varepsilon_0 E_{\perp(1)} = \varepsilon_2\varepsilon_0 E_{\perp(2)} \Rightarrow \varepsilon_1\left(\frac{-q''}{4\pi\varepsilon_0 r}\right) = \varepsilon_2\left(\frac{q'}{4\pi\varepsilon_0 r} - \frac{q}{4\pi\varepsilon_0 r}\right) \Rightarrow \varepsilon_1 q'' = \varepsilon_2(q-q') \quad ②$$
region observing from

So using ① and ② $\Rightarrow$ $q' = q\dfrac{(\varepsilon_2-\varepsilon_1)}{\varepsilon_1+\varepsilon_2}$ $\quad q'' = \dfrac{2\varepsilon_2}{\varepsilon_1+\varepsilon_2}q$

Clausius-Mossotti equation / local fields in dielectrics



consider cavity inside material of polarisation $\underline{P}$ per mole. $\leftarrow$ Now it can be shown that E_{local} felt by molecule is due to objects OUTSIDE a sphere of molecules surrounding it - so model this by a cavity.

$\underline{P} = \varepsilon_0\chi\underline{E}$

Now $\underline{P} = \varepsilon_0\chi\underline{E} = \varepsilon_0(\varepsilon-1)\underline{E}$ and $\underline{P} = N\alpha\underline{E}_{local}$

If $\underline{P}_{molecule} = \alpha\underline{E}_{local}$ (α = molecular polarisability) But for molecule in dielectric sphere $E_i = E_0 - P_i/3\varepsilon_0$ so remove sphere by adding $P_i/3\varepsilon_0$ and make cavity by adding another $P_i/3\varepsilon_0$ (or swap $\varepsilon_i \rightarrow \varepsilon_0$) $\Rightarrow$ $\underline{E}_{local} = \underline{E} + \underline{P}/3\varepsilon_0$

So: $\left(\dfrac{\varepsilon-1}{\varepsilon+2}\right) = \dfrac{N\alpha}{3\varepsilon_0}$

Magnetostatics — charge motion or CURRENTS are sources of magnetic field $\underline{B}$

$d\underline{l} \to \underline{F} \uparrow \underline{B}$

A current element in a uniform $\underline{B}$ field will interact with the field s.t. it feels a force $d\underline{F}$ given by $d\underline{F} = I d\underline{l} \times \underline{B}$. Now if $Id\underline{l}$ is resulting from a moving charge: $I d\underline{l} = \frac{dq}{dt}\underline{v}\, dt = \underline{v}\, dq$. Hence $d\underline{F} = dq\, \underline{v} \times \underline{B}$ ⇒ leads to more general electrodynamical result of Lorentz force $\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$

Biot Savart law gives $\underline{B}$ field due to a current element $Id\underline{l}$

{μ_0 defined as $4\pi \times 10^{-7}\,\mathrm{Hm^{-1}}$ - Amp defined as current flowing in 2 // conductors of ∞ length 1m part s.t. force between them $= 4\pi \times 10^{-7}\,\mathrm{Hm^{-1}}$}

$$d\underline{B} = \frac{\mu_0 I}{4\pi r^3} d\underline{l} \times \underline{r}$$

$Id\underline{l}$ $\underline{r}$ $d\underline{B}$

Now there are NO sources of $\underline{B}$ ⇒ $\underline{B}$ must form closed loops. i.e. $\oint_S d\underline{S} \cdot \underline{B} = 0 \Rightarrow \nabla \cdot \underline{B} = 0$ MAXWELL 3

Now a small current loop in a uniform $\underline{B}$ field will not experience any net force (Biot Savart) but will feel a couple $\underline{G}$. Defining magnetic dipole $\underline{m} = I d\underline{S}$

$\underline{G} = \underline{m} \times \underline{B}$ (Note loop could be any size since internal loop elements will cancel except on the outside i.e. $\underline{m} = I\int d\underline{S}$). Now stored magnetic energy of dipole $U = -\underline{m} \cdot \underline{B} = -I\int d\underline{S} \cdot \underline{B} = -I\Phi$. $\Phi = \int d\underline{S} \cdot \underline{B}$ = magnetic flux.

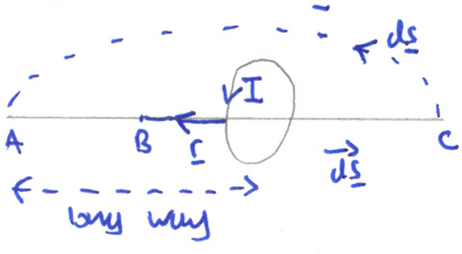
[Derived by electrostatic analogy with dipoles $\underline{p}$ in $\underline{E}$ fields]. Also, force on magnetic dipole in non uniform $\underline{B}$ field is $\underline{F} = (\underline{m} \cdot \nabla)\underline{B}$ - continuing with electrostatic analogy can define MAGNETIC SCALAR POTENTIAL ϕ_m s.t. $\underline{B} = -\mu_0 \nabla \phi_m$

By considering magnetic dipole and electric dipole: $V = \frac{p\cos\theta}{\varepsilon_0 4\pi r^2} = \frac{\underline{p} \cdot \underline{r}}{\varepsilon_0 4\pi r^3}$ $\underline{p}$ $\underline{r}$

So let $\phi_m = \frac{d\underline{m} \cdot \underline{r}}{4\pi r^3}$ Now solid angle of area $d\underline{S}$ at distance $\underline{r}$ is $d\underline{S} \cdot \underline{r}/r^3 = d\Omega$ ⇒ $\phi_m = \frac{I\,d\Omega}{4\pi}$ →

So for larger loops: $\phi_m = \frac{I\Omega}{4\pi}$. Allows us to derive Ampere's Theorem:

$\oint \underline{B} \cdot d\underline{l} = I\mu_0$ $\underline{B}$ I (really tells us how $\underline{B}$ fields are generated - could derive using BIOT SAVART LAW though).



at A (large distance) $\Omega_A = 0$ → go through loop gain a Ω of -2π ($d\underline{S} \parallel -\underline{r}$) ∴ at C net Ω gain is -4π. Now along return path assume big arc so Ω is not changed. Hence $\int_{loop} d\Omega = -4\pi \Rightarrow \int_{loop} d\phi_m = \frac{I}{4\pi}\int d\Omega$

$= -I$. Hence as $\underline{B} = -\mu_0 \nabla \phi_m$ and $\nabla\phi_m \cdot d\underline{s} = d\phi_m$ ⇒ $\int_{loop} -\frac{\underline{B} \cdot d\underline{s}}{\mu_0} = -I$

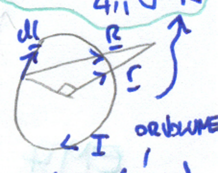
⇒ $\int_{loop} \underline{B} \cdot d\underline{s} = \mu_0 I$ so since $d\underline{s} \equiv d\underline{l}$ ⇒ $\oint \underline{B} \cdot d\underline{l} = \mu_0 I$ QED.

Now applying Stokes' theorem: ⇒ $\nabla \times \underline{B} = \mu_0 \underline{J}$ ($I = \int d\underline{S} \cdot \underline{J}$ $\underline{J}$ = current density).

Now ϕ_m turns out to be multivalued (use for long thin wire - apply Ampere)

↑ only use for non intersecting loops

→ better to generate $\underline{B}$ using magnetic VECTOR potential $\underline{A}$. $\underline{B} = \nabla \times \underline{A}$, $\underline{A} = \frac{\mu_0 I}{4\pi}\oint \frac{d\underline{l}}{R}$



Now certain media are magnetic - i.e. apply external $\underline{B}$ field and align magnetic dipoles in medium. i.e. create net circulation of current in medium

Need to relate $\underline{J}_m$ to $\underline{M}$ (magnetisation current density to net magnetic dipole/mole)

$\underline{J}_m$ can be related to $\underline{M}$ as follows: consider volume of magnetised medium V bounded by surface S:

Let $\underline{J}_m$ be magnetisation current density in V.

Now by continuity of current surface currents must come from V. NO NET FLOW. i.e. $\oint_S |d\underline{s}| \underline{J}_s = -\int_V d\tau \underline{J}_m$

Now $\underline{J}_s = \underline{M} \times \frac{d\underline{s}}{|d\underline{s}|}$

$\left(\underline{m} = \frac{IA}{AB} = \frac{I}{B}\hat{\underline{A}} \quad \underline{J}_s = I/_B\,\hat{\underline{J}}_s \text{ so } |\underline{M}| = |\underline{J}_s| \text{ and } \underline{J}_s \perp \underline{m} \right)$

$\Rightarrow \oint_S \underline{M} \times d\underline{s} + \int d\tau \underline{J}_m$ Now $\oint_S \underline{M} \times d\underline{s} = -\oint_S d\underline{s} \times \underline{M} = \int d\tau \nabla \times \underline{M}$ (Divergence theorem)

$\Rightarrow \underline{J}_m = \nabla \times \underline{M}$ So referring to Ampère: $\nabla \times \underline{B} = \mu_0(\underline{J}_f + \underline{J}_m)$ $\begin{cases} \underline{J}_{free} \\ \underline{J}_{magnetisation} \end{cases}$

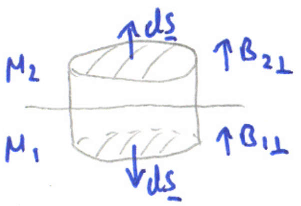
$\Rightarrow \nabla \times (\underline{B} - \mu_0 \underline{M}) = \mu_0 \underline{J}_f$ Define $\underline{H} = (\underline{B} - \mu_0 \underline{M})\frac{1}{\mu_0}$ $\Rightarrow \nabla \times \underline{H} = \underline{J}_f \Rightarrow \oint d\underline{l} \cdot \underline{H} = I$

Now for many isotropic media $\underline{M} \parallel \underline{H} \Rightarrow \underline{M} = \chi_m \underline{H}$ (χ_m = magnetic susceptibility)

$\Rightarrow \underline{B} = \mu_0(\underline{H} + \underline{M}) = \mu_0(1+\chi_m)\underline{H}$ so define μ_r = relative permeability s.t. $\mu_r = 1 + \chi_m$

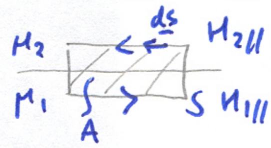
and $\underline{B} = \mu_r \mu_0 \underline{H}$ Now $\underline{B}$ and $\underline{H}$ have analogous properties to $\underline{E}$ and $\underline{D}$.

* $\underline{H}$ lines can end on 'magnetisation poles' i.e. N pole $\Leftrightarrow$ + charge
* b.cs of $\underline{B}$ and $\underline{H}$ are CONTINUITY of $B_\perp$ and $H_\parallel$.



$\oint \underline{B} \cdot d\underline{s} = 0$ $(\nabla \cdot \underline{B} = 0)$

$\Rightarrow B_{2\perp} = B_{1\perp}$



$\oint d\underline{s} \cdot \underline{H} = \int_A \nabla \times \underline{H} \cdot dA$

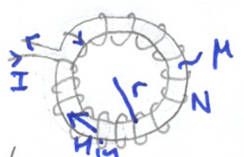
But if no free currents on boundary $\nabla \times \underline{H} = 0$

So $\oint d\underline{s} \cdot \underline{H} = 0 \Rightarrow H_{1\parallel} = H_{2\parallel}$. Note $\Rightarrow \phi_m$ is continuous.

Now like with $\underline{E}$ and $\underline{D}$ fields can use magnetostatic potential $\Leftrightarrow$ electrostatic potential to solve problems. (i.e. dielectric sphere = uniform field + dipole analogy can be applied to $\underline{H}$, $\underline{B}$ problems. ϕ_m instead of V.)

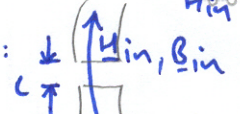
Note $\underline{B}_{in}$ = const. $\underline{B}_{out}$ for magnetisable material in uniform $\underline{B}$ fields. Cons. depends on shape as in electrostatics.

Now for a ring of $\mu_r = \mu$, N turns of current carrying wire



Ampère: $\oint \underline{H} \cdot d\underline{l} = 2\pi r H_{in} = NI \Rightarrow H_{in} = \frac{NI}{2\pi r}$

So $B_{in} = \mu_0 \mu NI/_{2\pi r}$ Now create small gap in ring:

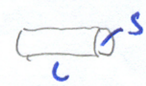


continuity of $B_\perp$ $\Rightarrow \underline{B}_{gap} = \underline{B}_{in}$

$\Rightarrow \mu_0 \mu H_{in} = \mu_0 H_{gap}$ Now Applying Ampère:

$\oint \underline{H} \cdot d\underline{l} = (2\pi r - l)H_{in} + lH_{gap} = NI \Rightarrow H_{gap} = \frac{\mu NI}{2\pi r + (\mu - 1)l}$ Now for most electromagnets $\mu \gg 1$, $\mu l \gg r$

$\Rightarrow B_{gap} \approx \mu_0 NI/_l$ ($\Rightarrow$ almost as if all the path is in the gap - rest has been 'shorted out').



Electric and magnetic circuits

(Noting $V = IR \Rightarrow \underline{J} = \sigma \underline{E}$ (power/unit volume = $\underline{J} \cdot \underline{E}$)

(σ = conductivity = $\frac{1}{\rho}$ resistivity)

Electric	$\int \underline{J} \cdot d\underline{s} = 0$ closed surface	$\oint \underline{E} \cdot d\underline{l} = EMF$	$\underline{J} = \sigma \underline{E}$	resistance $= L/_{\sigma S}$
Magnetic	$\int \underline{B} \cdot d\underline{s} = 0$ closed surface	$\oint \underline{H} \cdot d\underline{l} = $ 'MMF'	$\underline{B} = \mu \mu_0 \underline{H}$	'reluctance' $= \frac{L}{\mu \mu_0 S}$

Microscopic origin of magnetic behaviour

Simple model: 

electron orbits nucleus.

current $= \frac{e\omega_0}{2\pi}$

Area $= \pi r_0^2$ ∴ $|\underline{m}| = \frac{e\omega_0 r_0^2}{2}$

↑ dipole moment

1) Diamagnetism – all materials show magnetisation opposed to applied field → due to perturbation of 'electron orbits'

Now $|\underline{J}| = m_e\omega_0 r_0^2$ so $\underline{m}_e = \frac{e}{2m_e}\underline{J}$. In general for elementary particles: $\underline{m} = g\frac{q}{2m_q}\underline{J}$ (charge q, mass m_q, 'gyromagnetic factor g')

Now if applied $\underline{B}$ is 0, balance of forces gives $\frac{Ze^2}{4\pi\epsilon_0 r_0^2} = m_e\omega_0^2 r_0$ ($\underline{v} = \underline{\omega}\times\underline{r}$)

⇒ $\omega_0^2 = \frac{Ze^2}{4\pi\epsilon_0 m_e r_0^3}$

But if $\underline{B} \neq 0$ extra $-e\underline{v}\times\underline{B}$ force. ↑ inwards

⇒ New force balance at: $r = r_0+\Delta r$, $\omega = \omega_0+\Delta\omega$

i.e. $(\omega_0+\Delta\omega)^2 = \frac{Ze^2}{4\pi\epsilon_0 m_e}\frac{1}{(r_0+\Delta r)^3} + \frac{eB}{m_e}(\omega_0+\Delta\omega)$

⇒ expand to 1st order and note $\frac{\Delta r}{r_0} \ll \frac{\Delta\omega}{\omega_0}$ (J is QUANTISED – doesn't change when B is applied) ⇒ $\Delta\omega \simeq \frac{eB}{2m_e} = \frac{1}{2}\omega_L$

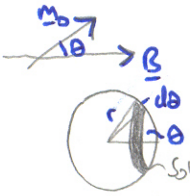
(ω_L = larmor frequency). So extra dipole moment opposed to field (Lenz's law)

$|\Delta\underline{m}| = \Delta I\cdot\pi r_0^2 = \frac{e}{2\pi}\cdot\Delta\omega\pi r_0^2 = \frac{e^2 r_0^2 B}{4m_e}$. Now average over orientations of orbit ↑ factor $\frac{2}{3}$

⇒ $\langle|\Delta\underline{m}|\rangle = \frac{e^2\langle r_0^2\rangle B}{6m_e}$

2) Paramagnetism – permanent dipoles in medium
$\underline{B} = 0$ – random dipole orientation. $\underline{B} \neq 0$ field aligns dipoles.

Assume atoms with permanent dipole moment m_0. So dipoles line up in the $\underline{B}$ field. Now probability of alignment in $\theta \to \theta+d\theta = P(\theta)d\theta$. ← NO COOPERATIVE EFFECTS OF DIPOL



$P(\theta)d\theta \propto \frac{1}{2}\sin\theta d\theta \cdot \exp\left(\frac{m_0B\cos\theta}{kT}\right)$ – $\underline{m}_0\cdot\underline{B}$ let $\mu = \cos\theta$, $x = m_0B/kT$

↑ probability of being at angle $\theta \to \theta+d\theta$

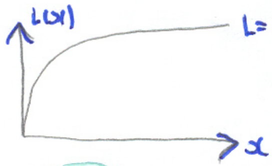
$\Omega = \frac{2\pi(r\sin\theta)\cdot r d\theta}{r^2}$ so $P(\Omega)d\Omega = \frac{\Omega}{4\pi} = \frac{1}{2}\sin\theta d\theta$

⇒ $P(\theta)d\theta = P(\mu)d\mu \propto e^{\mu x}d\mu$ ($= Ae^{\mu x}d\mu$)

Average alignment $\langle m_0\cos\theta\rangle = \langle m_0\mu\rangle = \langle m_\parallel\rangle$

$= m_0\int_{-1}^{1} d\mu\, \mu e^{\mu x} \Big/ \int_{-1}^{1} d\mu\, e^{\mu x}$ { limits $\cos\theta$ are $-1 \to 1$. $\int_{-1}^{1} e^{\mu x}d\mu = \frac{1}{A}$

Hence $\frac{\langle m_\parallel\rangle}{m_0} = \left(\coth(x) - \frac{1}{x}\right) \equiv L(x)$ (langevin function).



* High B or low T → saturates at $L(x) \to 1$.
* low field $L(x) \propto x$. Since $L(x) = \frac{x}{3} - \frac{x^3}{45} + O(x^5)$

Curie's law: low field strengths – no cooperative effects. $\frac{\langle m_\parallel\rangle}{m_0} \propto \frac{m_0B}{3kT}$ (i.e. $L(x) \approx \frac{x}{3}$).

So since $\chi_m = \frac{M}{H}$ for isotropic media ⇒ $\chi_m = \frac{C}{T}$

(C = curie constant $= \langle m_\parallel\rangle\cdot N / (B/\mu_r\mu_0)$ (N = dipoles / unit volume) $= m_0^2 N\mu_r\mu_0/3k$).

can be strong – Fe.

Weiss Theory – takes into account cooperative effects. $B_{local} = \mu_0(\underline{H}+\lambda\underline{M})$ λ = weiss constant (~10^3 for Fe)

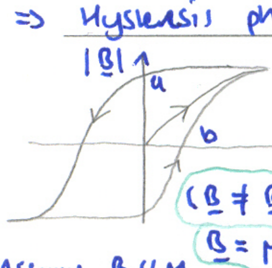
(strong cooperative paramagnetism is called Ferromagnetism since Fe is best at it).

Use B_{local} in langevin expression for $\langle m_\parallel\rangle$. In weak field limit $L(x) \propto \frac{x}{3}$

⇒ $\chi_m = \frac{C}{T-T_c}$, $C = Nm_0^2\mu_0/3k$, $T_c = \lambda C$

* $T > T_c$ ⇒ paramagnetism
* $T < T_c$ ⇒ spontaneous magnetisation

⇒ Hysteresis phenomenon with χ unstable below T_c.



a = remanence
b = coercivity

($\underline{B} \neq \underline{B}_{local}$)
$\underline{B} = \mu_0(\underline{M}+\underline{H})$

Assume $\underline{B} \parallel \underline{M}$.

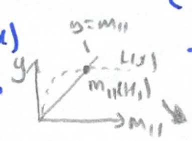
– Difficult to plot:

$B = \mu_0(M+H) = \mu_0(Nm_\parallel + H)$

But $m_\parallel = m_0 L\left(\frac{m_0\mu_0(H+\lambda M)}{kT}\right)$

i.e. $m_\parallel = m_0\left\{\coth\left(\frac{m_0\mu_0(H+\lambda N m_\parallel)}{kT}\right) - \frac{kT}{m_0\mu_0(H+\lambda N m_\parallel)}\right\}$

* Plot intersection of $y = m_\parallel$, $y = m_0L(x)$ for each value of H. Substitute into B expression and plot against H.



$B_1 = \mu_0(Nm_\parallel(H_1)+H_1)$ etc

Time dependent Electromagnetic fields

Electromagnetic induction

$\mathcal{E} = -\frac{\partial \Phi}{\partial t}$ (Voltage $\mathcal{E}$ (EMF) induced across circuit $= -\frac{\partial}{\partial t}$ electromagnetic flux 'cut' by circuit).

- sign is Lenz's law!

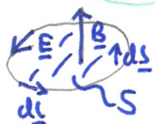
↓ quite general

So for a fixed loop: $\oint d\underline{l}\cdot\underline{E} = -\frac{\partial}{\partial t}\int_{area} d\underline{s}\cdot\underline{B} = -\int_{area} d\underline{s}\cdot\frac{\partial \underline{B}}{\partial t}$ ($d\underline{s}$ fixed) $\Rightarrow$ by Stokes theorem

$\underline{E} = -\nabla V = -\nabla\mathcal{E}$ Now $d\underline{l}\cdot\nabla\mathcal{E} = d\mathcal{E}$

so $\oint d\underline{l}\cdot\underline{E} = \oint d\mathcal{E} = \mathcal{E}$

(- sign absorbed by convention on $\mathcal{E}$?)



$\nabla\times\underline{E} = -\partial\underline{B}/\partial t$ (MAXWELL 2).

Now for a current loop I, flux linked by itself (current will create $\underline{B}$ field ~ Biot Savart etc...) $= \Phi$. Assume $\Phi_{self} \propto I$ $\Rightarrow$ Define Self inductance L as Φ_s/I

L is a geometrical factor which can be calculated by computing $L = \int \underline{B}\cdot d\underline{s}/I$.

Now for a LR circuit:

L = constant for rigid circuits

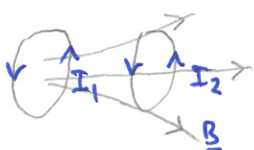
Now back EMF $= -\partial\Phi/\partial t = -L\partial I/\partial t$

so $V = RI + L\partial I/\partial t$ Now rate of power loss

$= VI = I^2R + LI\partial I/\partial t = I^2 R + \frac{\partial}{\partial t}(\frac{1}{2}LI^2)$

Now I^2R is dissipation in resistor and $\therefore \frac{\partial}{\partial t}(\frac{1}{2}LI^2)$ must be rate of magnetic field energy gain. So $U_{LR} = \frac{1}{2}LI^2$ (c.f. $U_C = \frac{1}{2}CV^2$).

For flux sources near to each other we have mutual inductance. (M_{ij})



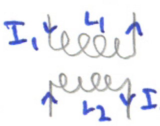
$\Phi_2 = M_{21}I_1$; $\Phi_1 = M_{12}I_2$ Now by reciprocity theorem $M_{ij} = M_{ji} = M$

Note total energy in such binary system is $\frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + MI_1I_2$

(For coupled LR circuits: $V_1 = L_1\partial I_1/\partial t + M\partial I_2/\partial t$ $V_2 = L_2\partial I_2/\partial t + M\partial I_1/\partial t$

Net energy loss rate: $V_1I_1 + V_2I_2 = \frac{\partial}{\partial t}(\frac{1}{2}L_1I_1^2) + \frac{\partial}{\partial t}(\frac{1}{2}L_2I_2^2) + \frac{\partial}{\partial t}(MI_1I_2)$ .... use argument as above for U_{LR})

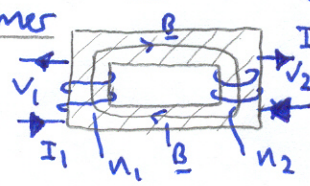
Now for coupled inductors (e.g. A TRANSFORMER) $\frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + MI_1I_2 \geq 0$.



(Total energy +ve). $\Rightarrow \frac{1}{2}\begin{pmatrix}I_1\\I_2\end{pmatrix}^T\begin{pmatrix}L_1 & M\\M & L_2\end{pmatrix}\begin{pmatrix}I_1\\I_2\end{pmatrix} \geq 0 \Rightarrow \begin{vmatrix}L_1 & M\\M & L_2\end{vmatrix} \geq 0$

i.e. $M^2 \leq L_1L_2$. Define $M = k(L_1L_2)^{1/2}$ $(0 \leq k \leq 1)$ k = coupling constant.

So for perfect coupling $k=1$. Example: Ideal transformer



Assume $k=1$. + No losses in wire, hysteresis in coupling medium. (Don't use iron!) So $\Phi_1 = n_1\Phi$ $\Phi_2 = n_2\Phi$

(Φ = flux linkage / turn). So $V_1 = -\frac{\partial\Phi_1}{\partial t} = -n_1\frac{\partial\phi}{\partial t}$ $V_2 = -n_2\frac{\partial\phi}{\partial t}$

$\Rightarrow \frac{V_1}{V_2} = \frac{n_1}{n_2}$ (self inductance of solenoid $\propto n^2$ so $\frac{L_1}{L_2} = \left(\frac{n_1}{n_2}\right)^2$).

Magnetic energy

– current flowing in inductance L – energy $U = \frac{1}{2}LI^2$, $L = \Phi/I$

$\Rightarrow W = \frac{1}{2}\Phi I$ for many circuits $W = \sum\frac{1}{2}\Phi_iI_i$ Now $\Phi = \int\underline{B}\cdot d\underline{s}$, $\underline{B} = \nabla\times\underline{A}$

$\Rightarrow \Phi = \int d\underline{s}\cdot\nabla\times\underline{A} = \oint\underline{A}\cdot d\underline{l}$ (Stokes). $\Rightarrow W = \frac{1}{2}\sum_i(\oint\underline{A}\cdot I d\underline{l})_i$ Now in DISTRIBUTED LIMIT

$I d\underline{l} \to \underline{J}\,d\tau$ $\Rightarrow W = \frac{1}{2}\int d\tau\,\underline{A}\cdot\underline{J}$ Now by Ampere's theorem $\nabla\times\underline{H} = \underline{J}$ (IF $\frac{\partial\underline{D}}{\partial t} = 0$ – see later)

$\Rightarrow W = \frac{1}{2}\int d\tau\,\underline{A}\cdot\nabla\times\underline{H}$ Now $\nabla\cdot(\underline{A}\times\underline{H}) = \underline{H}\cdot(\nabla\times\underline{A}) - \underline{A}\cdot(\nabla\times\underline{H}) \Rightarrow W = -\frac{1}{2}\int d\tau\,\nabla\cdot(\underline{A}\times\underline{H}) + \frac{1}{2}\int d\tau\,\underline{H}\cdot\nabla\times\underline{A}$

Now $ds \propto R^2$, $A \propto R^{-1}$, $H \propto R^{-2}$ so surface integral $\to 0$ as $R \to \infty$.

$\Rightarrow W = \int d\tau\,\frac{1}{2}\underline{B}\cdot\underline{H}$ i.e. magnetic energy density $U_M = \frac{1}{2}\underline{B}\cdot\underline{H}$

Now all electromagnetic field equations must be consistent with each other and the conservation laws which apply to many (all?) physical systems. i.e like charge conservation.

i.e $\oint d\underline{S}\cdot\underline{J} + \int d\tau \frac{\partial\rho}{\partial t} = 0$ (Flux of current density leaving a volume's surface = rate of change of charge density integrated over volume).

$\Rightarrow \nabla\cdot\underline{J} + \frac{\partial\rho}{\partial t} = 0$ (Continuity equation).

Now Ampere $\Rightarrow \nabla\times\underline{H} = \underline{J} \Rightarrow \nabla\cdot\underline{J} = 0$ (Rubbish!) Modify $\nabla\times\underline{H}$ by Maxwell's Displacement current. i.e

$\nabla\times\underline{H} = \underline{J} + \partial\underline{D}/\partial t$ ↑ MAXWELL 4.

(Taking divergence yields $\nabla\cdot\underline{J} + \frac{\partial}{\partial t}\nabla\cdot\underline{D} = 0$

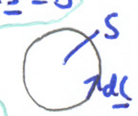
$\Rightarrow \nabla\cdot\underline{J} + \partial\rho/\partial t \Rightarrow$ by MAXWELL 1. $\nabla\cdot\underline{D} = \rho_{free}$)

So MAXWELL 4 $\Rightarrow$ continuity so is justified.

We can now summarise the key results in EM by Maxwells 4 equations, their definitions, integrated forms + Lorentz force equation and equation of EM energy density. ($U_m + U_E$).

Equation

MAXWELL 1	$\nabla\cdot\underline{D} = \rho_{free}$	or	$\int_S \underline{D}\cdot d\underline{S} = \int_V d\tau\rho_{free}$
MAXWELL 2	$\nabla\cdot\underline{B} = 0$	or	No magnetic poles or $\oint_S \underline{B}\cdot d\underline{S} = 0$
MAXWELL 3	$\nabla\times\underline{E} = -\partial\underline{B}/\partial t$	or	$\mathcal{E} = -\frac{\partial}{\partial t}\int_S d\underline{S}\cdot\underline{B}$
MAXWELL 4	$\nabla\times\underline{H} = \underline{J}_{free} + \partial\underline{D}/\partial t$		$\oint \underline{H}\cdot d\underline{l} = \int_S \underline{J}\cdot d\underline{S} + \frac{\partial}{\partial t}\int_S \underline{D}\cdot d\underline{S}$
TOTAL ENERGY DENSITY	$U = \frac{1}{2}\underline{E}\cdot\underline{D} + \frac{1}{2}\underline{B}\cdot\underline{H}$		Energy $= \int_V d\tau\, U$
LORENTZ FORCE	$\underline{F} = q(\underline{E} + \underline{v}\times\underline{B})$		$\underline{v}$ = velocity of charge q.



Definitions: $\underline{E}$ and $\underline{B}$ fundamental - present in force law. (Though use above to work out $\underline{E}$ and $\underline{B}$ from charge distributions - otherwise Coulombs and Biot-Savart are fundamental generators of $\underline{E}$ and $\underline{B}$ from charged particles.

$\underline{D} = \varepsilon_0\underline{E} + \underline{P}$ ($\underline{D} = \varepsilon_0\varepsilon\underline{E}$ for isotropic material) $\underline{B} = \mu_0(\underline{H} + \underline{M})$ ($\underline{B} = \mu_0\mu\underline{H}$ for isotropic material)

$\underline{P}$: polarisation / unit volume $\underline{P} = \varepsilon_0\chi\underline{E}$ $\underline{M} = \chi_M\underline{H}$ $\underline{M}$: magnetic dipole moment / unit volume

Electromagnetic waves in free space ($\Rightarrow \rho = \underline{J} = 0$, $\underline{D} = \varepsilon_0\underline{E}$ $\underline{B} = \mu_0\underline{H}$)

Maxwells equations (start with $\nabla\times\nabla\times\underline{H}$, $\underline{E}$) $\Rightarrow$ wave equations.

$$\nabla^2\underline{E} = \varepsilon_0\mu_0\frac{\partial^2\underline{E}}{\partial t^2}$$
$$\nabla^2\underline{H} = \varepsilon_0\mu_0\frac{\partial^2\underline{H}}{\partial t^2}$$

$\nabla^2\underline{E} = (\nabla^2 E_x, \nabla^2 E_y, \nabla^2 E_z)$ IN CARTESIAN i.e basis vectors position invariant. Not true for polars.

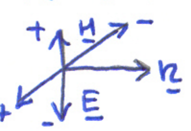
i.e EM waves propagate at speed $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$ in free space REGARDLESS of frame of reference. (So EM is a relativistic theory).

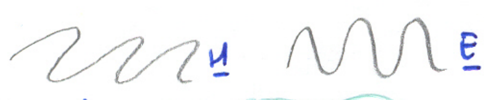
* $\underline{E}$ and $\underline{H}$ ($\underline{B}$) fields are in phase

* Impedance of free space $\frac{E_x}{H_y} = \frac{E_y}{-H_x} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\Omega \Rightarrow$ energy density has equal electric and magnetic contributions $\frac{1}{2}\varepsilon_0|\underline{E}|^2 = \frac{1}{2}\mu_0|\underline{H}|^2$

Now if $\underline{E}$ and $\underline{H}$ are of form $\underline{E}(\underline{x},t) = \underline{E}_0 e^{i(\underline{k}\cdot\underline{x}-\omega t)}$ $\underline{H}(\underline{x},t) = \underline{H}_0 e^{i(\underline{k}\cdot\underline{x}-\omega t)}$ – Maxwell's equations $\Rightarrow \{\underline{E}_0, \underline{H}_0, \underline{k}\}$ form a R.H. set.

i.e EM waves are Transverse and components $\underline{E}, \underline{H}$ are mutually orthogonal.





Now work done on fields is:

$\oint d\underline{S}\cdot\underline{E}\times\underline{H} + \frac{\partial}{\partial t}\int d\tau\frac{1}{2}(\underline{E}\cdot\underline{D} + \underline{B}\cdot\underline{H})$ ↑ U. (stored).

$dW = -q\underline{E}\cdot d\underline{l}$ (Lorentz) rate $= -q\underline{E}\cdot\underline{v}$ $-\int d\tau\underline{E}\rho\underline{v} = \int d\tau\underline{E}\cdot\underline{J}$ then apply M4.

Poynting vector $\underline{N} = \underline{E}\times\underline{H}$ = rate of energy flow (Power / unit area). Now $E = pc$ for photons $\Rightarrow$ momentum density $\underline{g} = \underline{N}/c^2$ and Radiation Pressure $= \underline{N}/c$

(p: momentum)

Plane EM waves in isotropic insulating media i.e. ε_r, μ_r real and constant.

In these cases replace $\varepsilon_0 \to \varepsilon_r \varepsilon_0$, $\mu_0 \to \mu_r \mu_0$ in solutions above.

So $c' (= c_{\text{medium}}) = \dfrac{1}{\sqrt{\varepsilon\varepsilon_0 \mu\mu_0}} \Rightarrow \dfrac{c}{c'} = \sqrt{\varepsilon\mu} = n$ REFRACTIVE INDEX of material. ($\mu \simeq 1$ for optical materials).

Medium impedance $= Z = Z_0 \sqrt{\dfrac{\mu}{\varepsilon}}\ \Omega$. Energy densities in E and H waves are still equal.

Waves in plasmas – 'quasineutral' mixture of free electrons and ions. More mobile electrons dominate electromagnetic properties. For electron in plasma: $m_e \ddot{\underline{r}} = -e(\underline{E} + \underline{v} \times \underline{B})$

If $|\underline{v}| \ll$, $\underline{v} \times \underline{B}$ is negligible so consider $\underset{E\text{ wave}}{\text{'solution'}}$ $\underline{E} = \underline{E}_0 e^{i(kz - \omega t)}$ (propagation in z direction)

At $z = 0$: $m_e \ddot{\underline{r}} = -e \underline{E}_0 e^{-i\omega t} \Rightarrow$ steady state solution $\underline{r} = \dfrac{e}{m_e\omega^2} \underline{E}_0 e^{-i\omega t}$. i.e. plasma electrons oscillate in E field of wave. Now dipole moment $\underline{p} = -e\underline{r} = -\dfrac{e^2}{m_e \omega^2} \underline{E}$

If N electrons / unit volume $\underline{P} = N\underline{p} = -\dfrac{Ne^2}{m_e\omega^2}\underline{E}$ Now $\chi = \dfrac{|\underline{P}|}{\varepsilon_0 |\underline{E}|}$, $\varepsilon = 1 + \chi$

$\Rightarrow \varepsilon = 1 - \dfrac{Ne^2}{m_e \varepsilon_0 \omega^2} = 1 - \dfrac{\omega_p^2}{\omega^2}$ Plasma frequency $\omega_p^2 = \dfrac{Ne^2}{m_e \varepsilon_0}$ $n = \sqrt{\varepsilon} = \left(1 - \dfrac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}$

* $\omega > \omega_p$ – dispersive. Phase velocity $c' = \dfrac{c}{n}$, group velocity $= \dfrac{d\omega}{dk} = v_g$.

($\omega = v_p k = c'k$ so $k = \omega / c'$ work out $\dfrac{dk}{d\omega}$ then reciprocate).

Note $v_p > c$, $v_g < c$ and $v_p v_g = c^2$. * Below ω_p waves evanescent as n is imaginary. All points oscillate in phase. $\underline{E} = \underline{E}_0 e^{i(kz - \omega t)} = \underline{E}_0 e^{i\omega(\frac{nz}{c} - t)}$

$= \underline{E}_0 e^{-\omega\sqrt{|\varepsilon|} z / c} e^{-i\omega t}$ ← decaying wave. Now $Z^{-1} = i\sqrt{|\varepsilon|}/Z_0 \Rightarrow \underline{H}$ is $\frac{\pi}{2}$ behind $\underline{E}$.

* No net $\underline{N}$. (Note $\underline{N} = \text{Re}\,\underline{E} \times \text{Re}\,\underline{H}$ – similar for energies take real part first).

Waves in conducting media – i.e. metals

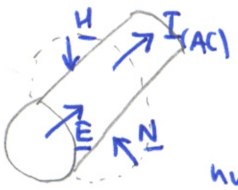
$\underline{D} = \varepsilon\varepsilon_0 \underline{E}$ $\underline{B} = \mu\mu_0 \underline{H}$ $\underline{J} = \sigma \underline{E}$ M4: $\nabla \times \underline{H} = \sigma \underline{E} + \varepsilon\varepsilon_0 \partial \underline{E}/\partial t$ look for oscillating solutions $\underline{E}, \underline{H} \propto e^{-i\omega t}$ $\Rightarrow \nabla \times \underline{H} = -\left(\varepsilon + \dfrac{i\sigma}{\omega\varepsilon_0}\right) i\omega\varepsilon_0 \underline{E}$ Define 'effective ε', ε'

as $\varepsilon' = \varepsilon + \dfrac{i\sigma}{\omega\varepsilon_0} \Rightarrow \nabla \times \underline{H} = -i\omega\varepsilon'\varepsilon_0 \underline{E} \Rightarrow \nabla \times \underline{H} = \varepsilon'\varepsilon_0 \partial\underline{E}/\partial t$ i.e. same

solutions for EM waves in insulating media now (other Maxwell equations are unchanged) but ε is ε'. Now $n = \dfrac{c}{c'} = \sqrt{\varepsilon'\mu} = \pm\dfrac{(1+i)}{\sqrt{2}}\sqrt{\dfrac{\sigma\mu}{\omega\varepsilon_0}}$ $\therefore k = \dfrac{\omega}{(c/n)}$

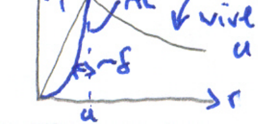
$= (1+i)\dfrac{\omega}{c}\sqrt{\dfrac{\sigma\mu}{2\omega\varepsilon_0}}$ (+ve k). $= (1+i)\sqrt{\dfrac{\sigma\mu\mu_0}{2}}$ define $\delta = \sqrt{\dfrac{2}{\sigma\mu\mu_0\omega}}$ as SKIN DEPTH

Since as $\underline{E} = \underline{E}_0 e^{i(kz - \omega t)} \Rightarrow \underline{E} = \underline{E}_0 e^{i(\frac{z}{\delta} - \omega t)} e^{-z/\delta}$ i.e. decaying travelling wave with decay constant δ. (take $-k$ for opposite wave so decay is always in the direction of motion) Now for wire carrying current I, $\underline{E} \parallel \underline{J}$ so: (at frequency ω)



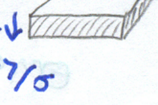
$\underline{N} = \underline{E} \times \underline{H}$

$\underline{N}$ and $\underline{H}$ are $\perp$ but both point INTO the wire. (well $\underline{H}$ oscillates but $\underline{N}$ doesn't). Flow of energy into wire. Now 'skin depth' $\Rightarrow$ current is actually a surface phenomenon. Since $\delta \propto \omega^{-\frac{1}{2}}$ higher $\omega \Rightarrow$ smaller δ and hence lower field strength into wire. (Reduced inductance since flux decreases).



cylinder wire radius a.

Now at high ω, wire $a \gg \delta$ – unwrap wire to plane surface width $2\pi a$ ($\underline{J} = \sigma \underline{E}$)



Total current $= I = \int_{2\pi a} dz\, J_x(z) \simeq 2\pi a J_0 e^{-i\omega t} \int_0^{\infty} dz \exp\left(\frac{z}{\delta}(-1+i)\right)$

$= \pi a J_0 e^{-i\omega t} \delta(1+i)$ Now $\langle I^2 \rangle = \langle \text{Re}(I)^2 \rangle = \pi^2 a^2 J_0^2 \delta^2$ Now $\underline{J} \cdot \underline{E} = \langle J^2 \rangle / \sigma$

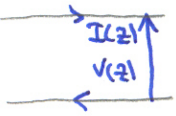
And $\langle J^2/\sigma \rangle = \dfrac{J_0^2}{2\sigma} e^{-2z/\delta}$ → cont on next page...

Now total power dissipated / unit length of wire $= \frac{2\pi a J_0^2}{2\sigma}\int_0^\infty dz\, e^{-2z/\delta} = \frac{\pi a J_0^2 \delta}{2\sigma}$

So resistance / unit length $= \frac{\langle Power \rangle}{\langle I^2 \rangle} = \frac{1}{2\pi a \delta \sigma}$ i.e. effective wire cross section

$= 2\pi a \delta$ – just as if current were flowing uniformly in skin depth δ

Now pairs of conductors can be used to carry EM waves. – Efficient energy transport.



For lossless line: Transmission line equations. (1) $\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$ (Faraday + $L = \Phi/I$)

(2) $\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$ ($Q = CV$, $I = dQ/dt$) ⇒ wave equation for V, I.

i.e. $\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}$ so V, I waves travel in z direction at speed $\frac{1}{\sqrt{LC}}$. For

lines of resistance R / unit length (1) → $\frac{\partial V}{\partial z} = -RI - L\frac{\partial I}{\partial t}$ ⇒ $\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2} + RC\frac{\partial V}{\partial t}$

(Damped waves). → i.e. some resistive loss. ($\underline{N}$ not entirely ∥ to z ?).

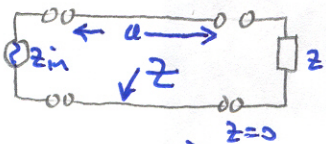
For lossless lines $Z = \frac{V}{I} = \frac{\omega L}{k} = \sqrt{\frac{L}{C}}$ – long line looks like resistance $R = \sqrt{\frac{L}{C}}$.

Power flow is VI between two ends of a line. { use $U = \int_a^b dz\left(\frac{1}{2}LI^2 + \frac{1}{2}CV^2\right)$ and (1) and (2) }

If line is terminated by load resistance Z_L – do (waves) reflected, transmitted, incident analysis to derive reflection coefficients $r = \frac{Z_L - Z}{Z_L + Z}$, $t = \frac{2Z_T}{Z_T + Z}$

(Note slight differences for waves incident on a boundary due to sign conventions of what V is measured between). b.c.s in this case are provided by $Z_L = \frac{V_t}{I_t}$ and $V_t = V_i + V_r$, $I_t = I_i + I_r$ – I, V having $e^{i(kz - \omega t)}$ dependence.

Now for finite line a:



$Z_{in} = \left.\frac{V_i + V_r}{I_i + I_r}\right|_{z=-a}$ ⇒ $\frac{Z_{in}}{Z} = \frac{Z_L\cos ka - iZ\sin ka}{Z\cos ka - Z_L\sin ka}$

Note if $a = \frac{\lambda}{4}$ (use $k = \frac{2\pi}{\lambda}$)

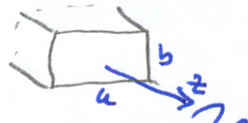
⇒ $Z_{in}Z_L = Z^2$. i.e. can match load to impedance of a transmission line.

($\omega = vk$ ⇒ $k = \omega\sqrt{LC}$)

Now hollow conductor TUBES act as WAVEGUIDES. Constraints on EM waves are provided by conductor solutions of maxwell equations. i.e. $\rho_f = 0$ inside conductor, no $\underline{E}$ fields ∥ to conductor surface ⇒ no $\underline{H}$ fields ⊥ to surface ($\underline{E}, \underline{H}, \underline{N}$) are R.H. set. Need to satisfy ($E_\parallel, D_\perp$) ($H_\parallel, B_\perp$) b.c's as well. ⇒ constraint of wave-fields for $\underline{E}, \underline{H}$ quantise $\underline{k}$, ω and hence get TE_{mn} modes ('Transverse, electric') m, n correspond to various quantisation numbers. * rectangular waveguides.

or TM_{mn} (if conductor magnetisable)

i.e. $\frac{\omega^2}{c^2} = k_z^2 + k_x^2 + k_y^2 = k_z^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$



k_z not fixed as EM wave is free to propagate in this direction.

($\underline{E} = \underline{E}_0 e^{i(k_z z - \omega t)}$ (standing wave x, y)). Note $k_z^2 = \frac{\omega^2}{c^2} - (k_x^2 + k_y^2)$ ⇒ if $k_z^2 < 0$

⇒ evanescent wave. Leads to 'cutoff frequency' $\nu_c = \frac{ck_c}{2\pi} = c\left(\frac{m^2}{4a^2} + \frac{n^2}{4b^2}\right)^{\frac{1}{2}}$ for rectangular waveguides. * Note conducting guide cannot support TM_{m0} or TM_{0n} modes – no 'magnetic charge' on walls. * Resonant cavities are fully constrained waveguides, i.e. $k_z^2 = \frac{l^2\pi^2}{d^2}$ ~> c.f. Black body radiation.