Summary of Results from Previous Courses

Grad, Div, Curl and the Laplacian in Cartesian Coordinates

In Cartesian coordinates, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. For a scalar field $\Phi(\mathbf{x})$ and a vector field $\mathbf{F}(\mathbf{x}) = (F_1, F_2, F_3)$, we define:

Gradient	$\nabla\Phi = \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right)$	("grad Phi")
Divergence	$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$	("div F")
Curl	$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$	("curl F")
Laplacian	$\nabla^2\Phi = \nabla \cdot (\nabla\Phi) = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$	("del-squared Phi")

Grad, Div and the Laplacian in Polar Coordinates

Cylindrical Polars (r, θ, z)

When the components (F_1, F_2, F_3) of $\mathbf{F}$ are measured in cylindrical polar coordinates,

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial z}\right)$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_1) + \frac{1}{r}\frac{\partial F_2}{\partial\theta} + \frac{\partial F_3}{\partial z}$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Note: the formulae for plane polar coordinates (r, θ) are obtained by setting $\frac{\partial}{\partial z} = 0$.

Spherical Polars (r, θ, ϕ)

When the components (F_1, F_2, F_3) of $\mathbf{F}$ are measured in spherical polar coordinates,

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_2\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_3}{\partial\phi}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Divergence and Stokes' Theorems

Divergence Theorem in 3D: $\displaystyle\iiint_V \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$

where the surface S encloses a volume V and $\mathbf{n}$ is its outward-pointing normal.

Divergence Theorem in 2D: $\displaystyle\iint_S \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dx\, dy = \oint_C (f\, dy - g\, dx)$

where S is a plane region enclosed by a contour C traversed anti-clockwise. We can also write the right-hand side as $\oint_C \mathbf{F} \cdot \mathbf{n}\, dl$ where $\mathbf{F} = (f, g)$ and $\mathbf{n}$ is the outward-pointing normal on C.

Stokes' Theorem: $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{l}$

where the open surface S is bounded by a contour C, $\mathbf{n}$ is the normal to S and $d\mathbf{l}$ is a line element taken anti-clockwise around C.

Vectors and Matrices

Vector identities:

$$|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$$
$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$
$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$
$$\nabla(\Phi\Psi) = \Phi\nabla\Psi + \Psi\nabla\Phi$$
$$\nabla(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times (\nabla \times \mathbf{v}) + (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + (\mathbf{v} \cdot \nabla)\mathbf{u}$$
$$\nabla \cdot (\Phi\mathbf{u}) = \Phi\,\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla\Phi$$
$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$$
$$\nabla \times (\Phi\mathbf{u}) = \Phi\,\nabla \times \mathbf{u} + \nabla\Phi \times \mathbf{u}$$
$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\nabla \cdot \mathbf{v})\mathbf{u} - \mathbf{u} \cdot \nabla\mathbf{v} - (\nabla \cdot \mathbf{u})\mathbf{v} + \mathbf{v} \cdot \nabla\mathbf{u}$$
$$\nabla^2\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$$

A matrix A is orthogonal if $A^TA = AA^T = I$ where I is the identity matrix and A^T is the transpose of A. This is true if and only if the columns of A are mutually orthogonal unit vectors; similarly for the rows. Then $A^{-1} = A^T$. In 3D, an orthogonal matrix is either a rotation, a reflection, or a combination of the two.

$\mathbf{x}$ is an eigenvector of a symmetric matrix A with eigenvalue λ if $A\mathbf{x} = \lambda\mathbf{x}$. The eigenvalues can be found by solving the equation $\det(A - \lambda I) = 0$. The three unit eigenvectors are orthogonal (or in the case of repeated eigenvalues, can be chosen to be so).

The determinant of a matrix is unchanged by adding a multiple of one row to a different row, or by adding a multiple of one column to a different column. Swapping two rows changes the sign of the determinant, as does swapping two columns. Multiplying a row, or a column, by a constant factor α multiplies the determinant by α. If two rows, or columns, are the same, then the determinant is zero. For any square matrices A and B, $\det A^T = \det A$ and $\det AB = \det A \det B$.

Mathematical Methods II

Fourier Series

Any (well-behaved) function $f(x)$ with period L may be represented as the infinite sum

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{2n\pi x}{L} + B_n \sin \frac{2n\pi x}{L} \right)$$

where

$$A_0 = \frac{1}{L} \int_0^L f(x)\, dx,$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi x}{L}\, dx,$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi x}{L}\, dx.$$

A function $f(x)$ which is defined only in the region $0 \leqslant x \leqslant L$ may be represented as a full Fourier Series as above by first making it into a periodic function with period L; or may alternatively be represented either by a Fourier cosine series, in which only the cosine terms appear, or by a Fourier sine series, in which only the sine terms appear. In the former case, f is first extended to $-L \leqslant x \leqslant 0$ by requiring it to be an even function, and then made into a periodic function with period $2L$; in the latter case, it is first extended into an odd function and then made periodic with period $2L$.

Legendre Polynomials

Legendre's equation for $P(x)$ is

$$\frac{d}{dx}\left((1-x^2) \frac{dP}{dx} \right) + \lambda P = 0.$$

The non-trivial solutions of this equation are ill-behaved at $x = \pm 1$ *except* when $\lambda = n(n+1)$ for some non-negative integer n. Then the solutions are the Legendre polynomials $P_n(x)$ of degree n. $P_n(x)$ is an even/odd function of x (i.e., contains only even/odd powers of x) when n is even/odd respectively. It is normalised so that $P_n(1) = 1$ (and therefore $P_n(-1) = (-1)^n$). Legendre polynomials are orthogonal:

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & m \neq n, \\ \frac{2}{2n+1} & m = n. \end{cases}$$

They can be found explicitly using Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \{(x^2-1)^n\}.$$

Taylor's Theorem (complex version)

Any smooth complex function can be expressed as a power series around $z = z_0$ in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n = f^{(n)}(z_0)/n!$.

Fourier Transforms

For suitable functions $f(x)$, the Fourier Transform is defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx,$$

and the inversion formula is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk.$$

The Fourier Transform of $f'(x)$ is $ik\tilde{f}(k)$. The Fourier Transform of $f(x-a)$ for constant a is $e^{-ika}\tilde{f}(k)$. The convolution $h = f * g$, defined by

$$h(y) = \int_{-\infty}^{\infty} f(x) g(y-x)\, dx,$$

satisfies $\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)$.

Laplace Transforms

For suitable functions $f(x)$ which satisfy $f(x) = 0$ for $x < 0$, the Laplace Transform is defined by

$$\bar{f}(p) = \int_0^{\infty} f(x) e^{-px} dx.$$

The Laplace Transform of $f'(x)$ is $p\bar{f}(p) - f(0)$. The Laplace Transform of $f(x-a)$ for constant a (remembering that $f(x-a) = 0$ for $x < a$) is $e^{-pa}\bar{f}(p)$. The convolution $h = f * g$ satisfies $\bar{h}(k) = \bar{f}(k)\bar{g}(k)$.

Summary of Results from Chapter 1: Poisson's Equation

Physical Origins of Poisson's Equation

Steady-state heat equation	$\nabla^2 T = -S(\mathbf{x})/k$.
Steady-state diffusion equation	$\nabla^2 \Phi = -S(\mathbf{x})/k$
Electrostatic potential	$\nabla^2 \Phi = -\rho(\mathbf{x})/\epsilon_0$
Gravitational potential	$\nabla^2 \Phi = 4\pi G \rho(\mathbf{x})$

Laplace's Equation in 2D Plane Polars

let $\Phi(r,\theta) = R(r)\Theta(\theta)$
rearrange after substitution s.t
LHS is $f(r)$, RHS is $f(\theta) \Rightarrow$
both must equal a constant

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0$$

with general solution

For r equation substitute $\lambda = n^2$
and use $u = \ln r$ substitution

$$\Phi = A_0 + B_0\theta + C_0\ln r + \sum_{n=1}^{\infty}(A_n r^n + C_n r^{-n})\cos n\theta + \sum_{n=1}^{\infty}(B_n r^n + D_n r^{-n})\sin n\theta,$$

or, more compactly,

$r^2 R'' + rR' - n^2 R = 0$

$\Rightarrow R = \begin{cases} C + D\ln r & n=0 \\ Cr^n + Dr^{-n} & n \neq 0 \end{cases}$

$$\Phi = A_0 + B_0\theta + C_0\ln r + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} r^n(A_n\cos n\theta + B_n\sin n\theta)$$

Laplace's Equation in 3D Spherical Polars, Axisymmetric Case

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Phi}{\partial\theta}\right) = 0$$

$\Phi(r,\theta) = R(r)\Theta(\theta)$
use $z = \cos\theta$ for θ
equation gives
Legendre's equation:

$\frac{d}{dz}\left[(1-z^2)\frac{d\Theta}{dz}\right] + \lambda\Theta = 0$

with general solution

$R = Ar^n + Br^{-n-1}$ since $r^2R'' + 2rR' - n(n+1)R = 0$ ($u = \ln r$ sub again)

$$\Phi = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-n-1})P_n(\cos\theta)$$

For well behaved solutions at $z = \pm 1$
(i.e $\theta = 0, \pi$) need $\lambda = n(n+1)$ for $n \in \mathbb{Z}^+ \neq 0$
$\Rightarrow \Theta = CP_n(z) = CP_n(\cos\theta)$

Green's Identity

$$\iiint_V (\Phi\nabla^2\Psi - \Psi\nabla^2\Phi)\,dV = \iint_S (\Phi\nabla\Psi - \Psi\nabla\Phi)\,.\,\mathbf{n}\,dS$$

Green's Function & the Integral Solution of Poisson's Equation

If $G(\mathrm{x};\mathrm{x}_0)$ is Green's function satisfying

$$\begin{aligned} \nabla^2 G = \delta(\mathrm{x}-\mathrm{x}_0) \quad &\text{in } V \\ G = 0 \quad &\text{on } S \end{aligned}$$

then the solution to Poisson's equation with Dirichlet boundary conditions

$$\begin{aligned} \nabla^2 \Phi = \sigma \quad &\text{in } V \\ \Phi = f \quad &\text{on } S \end{aligned}$$

apply Green's Identity

is

$$\Phi(\mathrm{x}_0) = \iiint_V \sigma(\mathrm{x}) G(\mathrm{x};\mathrm{x}_0)\, dV + \iint_S f(\mathrm{x}) \frac{\partial G}{\partial n}\, dS$$

Start by saying $1 = \iiint_V \delta(\underline{x})\, dV$

Fundamental Solutions

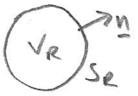
2D	$\dfrac{1}{2\pi} \ln\lvert\mathrm{x}-\mathrm{x}_0\rvert$
3D	$-\dfrac{1}{4\pi\lvert\mathrm{x}-\mathrm{x}_0\rvert}$

3D $1 = \iiint_V \delta(\underline{x}) = \iiint \nabla\cdot(\nabla G)\, dV$

$= \iint_{S_R} \nabla G\cdot\underline{n}\, dS = \iint_{S_R} g'(r)\, dS$

$= g'(R)\cdot 4\pi R^2$

$\Rightarrow g'(R) = \dfrac{1}{4\pi R^2}$



2D

$1 = \iint_{r\le R} \delta(\underline{x})\, dS = \iint_{r\le R} \nabla\cdot(\nabla G)\, dS$ ← 2D DIVERGENCE

$= \oint_{r=R} (\nabla G\cdot\underline{n})\, dl = \oint_{r=R} g'(r)\, dl = 2\pi R g'(R)$

$\Rightarrow g'(r) = \dfrac{1}{2\pi r}$ ↖ True for all R so $R\to r$.

$\Rightarrow g(r) = \dfrac{1}{2\pi} \ln r + C. \quad r = |\underline{x}-\underline{x}_0|$.

$\Rightarrow g(r) = -\dfrac{1}{4\pi r} + \text{const}$ since true for all R.

$G\to 0$ as $r\to\infty \Rightarrow A = 0$

$\Rightarrow G(\underline{x},\underline{x}_0) = \dfrac{-1}{4\pi|\underline{x}-\underline{x}_0|}$

Images in Circles and Spheres

In either case, for Dirichlet boundary conditions the image point is at

$$\mathrm{x}_1 = \frac{a^2}{\lvert\mathrm{x}_0\rvert^2}\mathrm{x}_0,$$

with strength -1 in the 2D case (circle) and strength $-a/\lvert\mathrm{x}_0\rvert$ in the 3D case (sphere)

Divergence theorem in 3D $\quad \iint_S (\underline{F}\cdot\underline{n})\, dS = \iiint_V (\nabla\cdot\underline{F})\, dV$

2D $\quad \oint (\underline{F}\cdot\underline{n})\, dl = \iint_S \nabla\cdot\underline{F}\, dS$

Summary of Results from Chapter 2: Cartesian Tensors

Transformation Law

If a tensor of rank n has components $T_{ijk\ldots}$ measured in a frame with unit Cartesian axes $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ then its components in a frame with axes $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are given by

$$T'_{ijk\ldots} = l_{ip} l_{jq} l_{kr} \ldots T_{pqr\ldots}$$

where the rotation matrix L is defined by

$$l_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j .$$

The components of L satisfy $l_{ik} l_{jk} = \delta_{ij}$ and $l_{ki} l_{kj} = \delta_{ij}$.

The Kronecker Delta and the Alternating Tensor

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is a cyclic permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an anticyclic permutation of } (1,2,3) \\ 0 & \text{if any two of } (i,j,k) \text{ are equal} \end{cases}$$

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$[\mathbf{x} \times \mathbf{y}]_i = \varepsilon_{ijk} x_j y_k$$

$$\det A = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$$

Symmetric and Antisymmetric Tensors

A tensor $T_{ijk\ldots}$ is symmetric in i, j if $T_{ijk\ldots} = T_{jik\ldots}$ and antisymmetric in i, j if $T_{ijk\ldots} = -T_{jik\ldots}$.

Any second rank tensor T can be decomposed into a symmetric part S and an antisymmetric part A where

$$S_{ij} = \tfrac{1}{2}(T_{ij} + T_{ji}),$$

$$A_{ij} = \tfrac{1}{2}(T_{ij} - T_{ji})$$

and

$$T_{ij} = S_{ij} + A_{ij}.$$

Any antisymmetric second rank tensor A can be expressed in terms of a suitable vector $\boldsymbol{\omega}$ such that $A_{ij} = \varepsilon_{ijk}\omega_k$. (In fact, $\omega_k = \frac{1}{2}\varepsilon_{klm}A_{lm}$.)

Diagonalisation of Symmetric Second Rank Tensors

If T is a symmetric second rank tensor with eigenvalues (principal values) λ_1, λ_2 and λ_3 and corresponding unit eigenvectors (principal axes) $\mathbf{e}_1'$, $\mathbf{e}_2'$ and $\mathbf{e}_3'$, then the components of T in a frame whose axes coincide with the principal axes are

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Isotropic Tensors

The most general isotropic tensors are:

Rank 0: Any scalar

Rank 1: Only the zero vector

Rank 2: $\lambda\delta_{ij}$

Rank 3: $\lambda\varepsilon_{ijk}$

Rank 4: $\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}$

Differential Operators

$\partial_i = \partial/\partial x_i$ is a tensor differential operator of rank 1.

$$[\nabla\Phi]_i = \frac{\partial\Phi}{\partial x_i}$$

$$\nabla\,.\,\mathbf{F} = \frac{\partial F_i}{\partial x_i}$$

$$[\nabla\times\mathbf{F}]_i = \varepsilon_{ijk}\frac{\partial F_k}{\partial x_j}$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x_i \partial x_i}$$

$$[\nabla^2\mathbf{F}]_i = \nabla^2(F_i) = \frac{\partial^2 F_i}{\partial x_j \partial x_j}$$

Summary of Results from Chapter 3: Complex Analysis

Analyticity

A function $f(z)$ is differentiable at z if the limit

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

exists and is independent of the direction taken by δz in the limiting process.

A function $f(z)$ is analytic (or regular) in a region $R \subseteq \mathbb{C}$ if $f'(z)$ exists and is continuous for all $z \in R$. It is analytic at a point z_0 if it is analytic in some neighbourhood of z_0, i.e., in some region enclosing z_0.

$f(z)$ is analytic in R if and only if the Cauchy–Riemann equations

Take: $f(z) = u(x,y) + iv(x,y)$ $z = x+iy$ let $\delta z = \delta x$ or $i\delta y$ in each case.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold in R, where

$$u(x,y) = \operatorname{Re} f(x+iy) \qquad \text{and} \qquad v(x,y) = \operatorname{Im} f(x+iy).$$

The functions u and v are harmonic, i.e., satisfy Laplace's equation in two dimensions.

$\nabla^2 u = \nabla^2 v = 0.$

Laurent Expansions

If $f(z)$ is analytic in some annulus centred at z_0 then there exist complex constants a_n such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

within the annulus.

An isolated singularity of a function $f(z)$ is a point z_0 at which f is singular, but where otherwise f is analytic within some circle centred at z_0. A Laurent Expansion of f always exists about an isolated singularity.

If there are non-zero a_n for arbitrarily large negative n then f has an essential isolated singularity at z_0.

If $a_n = 0$ for all $n < -N$, but $a_{-N} \neq 0$, where N is a positive integer, then f has a pole of order N at z_0.

If $a_n = 0$ for all $n < 0$ then f has a removable singularity at z_0.

Residues

If $f(z)$ has an isolated singularity at z_0, then its residue there is given by the coefficient a_{-1} in the Laurent Expansion.

If $f(z)$ has a pole of order N at z_0, then

$$\operatorname*{res}_{z=z_0} f(z) = \lim_{z\to z_0} \left\{ \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} \left((z-z_0)^N f(z) \right) \right\}.$$

If $f(z)$ has a simple pole at z_0 then

$$\operatorname*{res}_{z=z_0} f(z) = \lim_{z\to z_0} \{(z-z_0) f(z)\}.$$

The residue of $f(z)$ at infinity is defined to be equal to the residue of $\zeta^{-2} f(1/\zeta)$ at $\zeta = 0$.

Branch Cuts

The canonical branch cut for both $\log z$ and z^α, where α is not an integer, is along the negative real axis from 0 to $-\infty$. With this cut,

$$\log z = \log r + i\theta$$

and

$$z^\alpha = r^\alpha e^{i\alpha\theta}$$

where $z = re^{i\theta}$ and $-\pi < \theta \leqslant \pi$.

No curve may cross a branch cut.

Summary of Results from Chapter 4: Contour Integration

Cauchy's Theorem

$f(z) = u + iv$
$dz = dx + i\,dy$

If $f(z)$ is analytic in a simply-connected domain R, then for any simple closed curve C in R,

use: * Cauchy Riemann
* 2D divergence

C Analytic, simply connected domain.

$$\oint_C f(z)\,dz = 0.$$

$\oint_C f(z)dz = \oint_C (u+iv)(dx+idy)$
$= \oint_C (udx - vdy) + i\oint_C (vdx + udy)$
let $F_1 = (u,-v)$, $F_2 = (v,-u) \to$ apply 2D Div, Note CR and QED.

$dl = (dx, dy)$
$n \cdot dl = 0$
$dl = (dx, dy)$
∴ let $n\,dl = (-dy, dx)$

(We deduce that for any integral between two points z_0 and z_1, or round a closed curve, we may deform one contour of integration into another without affecting the value of the integral so long as we do not cross any singularities of the integrand during the deformation.)

C_1 = C_2 Proof consider, use usual result. middle bits cancel.

The Integral of f′(z)

If $f(z)$ is analytic in a simply-connected domain R and the contour of integration lies entirely in R, then

$$\int_{z_0}^{z_1} f'(z)\,dz = f(z_1) - f(z_0).$$

The Residue Theorem

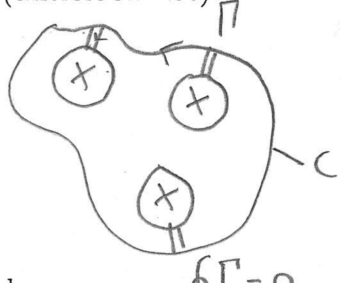
If $f(z)$ is analytic in a simply-connected domain R except for a finite number of poles at $z = z_1, z_2, \ldots, z_n$, and C is a simple closed anticlockwise contour in R encircling the poles, then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{res}_{z=z_k} f(z).$$

z_0 $C \to C = \varepsilon$ take circular contour.
let $z = z_0 + \varepsilon e^{i\theta}$.

In particular, if C is a simple closed curve encircling z_0 in a positive (anticlockwise) sense and n is an integer,

$$\oint_C (z-z_0)^n\,dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$



$\oint \Gamma = 0$
(total closed contour)
$= \int$ joins $+ \int_C + \int$ circles round poles
joins $= 0$

Cauchy's Formula

If $f(z)$ is analytic in a simply-connected domain R and z_0 lies in R, then

prove by shrinking and use residue theorem with integrand.

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz$$

for any simple closed anticlockwise contour C in R encircling z_0.

If instead $f(z)$ has a singularity at z_0, and has Laurent Expansion $\sum\limits_{n=-\infty}^{\infty} a_n(z-z_0)^n$ there, then the coefficients of the expansion are given by

consider $I = \oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$ let $f(z) = \sum_{-\infty}^{\infty} a_m (z-z_0)^m$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

$\Rightarrow \oint \sum_{-\infty}^{\infty} a_m (z-z_0)^{m-n-1} dz$ Now

$\oint_C (z-z_0)^k dz = \begin{cases} 0 & k \neq -1 \\ 2\pi i & k = -1 \end{cases} \Rightarrow I = 2\pi i a_n$ (when $n = m$ integral is non zero)

So $a_n = \frac{I}{2\pi i}$ QED.

Jordan's Lemma

Let C_R be the semicircular contour of radius R in the upper half plane with centre at the origin, traversed from $+R$ on the real axis to $-R$; let C'_R be the semicircular contour of radius R in the lower half plane with centre at the origin, traversed from $+R$ to $-R$; and let $f(z)$ be an analytic function (except possibly for a finite number of poles) which satisfies $f(z) \to 0$ as $|z| \to \infty$. Then for any real constant $\lambda > 0$,

$$\int_{C_R} f(z) e^{i\lambda z} dz \to 0$$

as $R \to \infty$; while for $\lambda < 0$,

$$\int_{C'_R} f(z) e^{i\lambda z} dz \to 0$$

as $R \to \infty$.

The Inverse Laplace Transform

If $f(t)$ (which vanishes for $t < 0$) has Laplace Transform $\bar{f}(p)$ then the Bromwich Inversion Formula is

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(p) e^{pt} dp$$

where the Bromwich inversion contour runs along the line $\mathrm{Re}\, p = \gamma$, where γ is a real constant which lies to the right of all the singularities of $\bar{f}(p)$.

If $\bar{f}(p) \to 0$ as $|p| \to \infty$, and if $\bar{f}(p)$ has poles at $p = p_1, p_2, \ldots, p_n$ (but no other singularities, e.g. branch cuts), then

$$f(t) = \begin{cases} 0 & t < 0 \\ \displaystyle\sum_{k=1}^{n} \operatorname*{res}_{p=p_k} \left(\bar{f}(p) e^{pt}\right) & t > 0 \end{cases}$$

For Normal Modes:

(Methods III) Note $\underline{q} = \underline{x}^T \underline{\underline{I}}\, \underline{x}$

are orthogonal for eigenvectors of