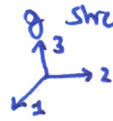
FLUID DYNAMICS

1) Fundamentals

* A true fluid is a material with no rigidity – i.e. subject to shear stress, no matter how small, it will FLOW.

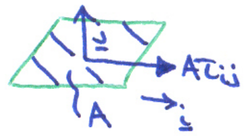
* A Newtonian fluid is one where stresses depend linearly on the gradient of velocity or, alternatively, rate of strain. $\underline{\underline{\tau}} = 2\eta \underline{\underline{\dot{e}}}$ or $\tau_{ij} = 2\eta \dot{e}_{ij}$ ($\underline{\underline{\tau}}, \underline{\underline{e}}$ are stress and strain tensors, η is the viscosity. Constant for Newtonian fluid).

In Cartesians (1, 2, 3)



τ_{ij} means ith component of force / unit area $\perp$ j direction

Displacement vector $\underline{x} = (x_1, x_2, x_3)$

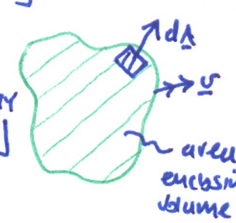


$$e_{ij} = \frac{1}{2}\left(\frac{\partial x_i}{\partial x_j} + \frac{\partial x_j}{\partial x_i}\right)$$

Now define velocity $\underline{v} = (v_1, v_2, v_3) \equiv \left(\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}, \frac{\partial x_3}{\partial t}\right)$

$$\therefore \dot{e}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \Rightarrow \tau_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

* Fluids, like most continuous media are continuous! i.e. any local outflow is balanced by a local change in density. i.e. $\int_A \rho \underline{v} \cdot d\underline{A}$ is mass outflow and this must equal $-\frac{\partial}{\partial t}\int_V \rho \, dV$.



Gauss theorem $\Rightarrow \int_A \rho \underline{v} \cdot d\underline{A} = \int_V \nabla \cdot (\rho \underline{v}) dV$ Hence $\nabla \cdot (\rho \underline{v}) = -\partial \rho / \partial t$ [CONTINUITY EQUATION]

Now for uniform fluids $\nabla \rho = 0$ and for incompressible fluids $\partial \rho / \partial t = 0$ $\therefore$ since $\nabla \cdot (\rho \underline{v}) = \rho(\nabla \cdot \underline{v}) + (\underline{v} \cdot \nabla)\rho$ CONTINUITY EQ. $\Rightarrow$ $\nabla \cdot \underline{v} = 0$

* Hydrostatic pressure $p = -\frac{1}{3}(\tau_{11} + \tau_{22} + \tau_{33})$ – independent of axis choice.

Now since $\tau_{ij} = 2\eta \dot{e}_{ij} \Rightarrow \tau_{ii} - \tau_{jj} = 2\eta(\dot{e}_{ii} - \dot{e}_{jj})$. $\therefore$ using above result for p

(i) $\tau_{11} = -3p - \tau_{22} - \tau_{33}$ (ii) $\tau_{11} = \tau_{33} + 2\eta(\dot{e}_{11} - \dot{e}_{33})$ (iii) $\tau_{11} = \tau_{22} + 2\eta(\dot{e}_{11} - \dot{e}_{22})$

$\rightarrow$ Add $\Rightarrow \tau_{11} = -p + \frac{2}{3}\eta(2\dot{e}_{11} - \dot{e}_{22} - \dot{e}_{33})$ Now $\dot{e}_{11} + \dot{e}_{22} + \dot{e}_{33} = \nabla \cdot \underline{v}$ so for uniform incompressible fluids

$$\tau_{11} = -p + \frac{2\eta}{3}(2\dot{e}_{11} + 2\dot{e}_{22} + 2\dot{e}_{33} - 3\dot{e}_{22} - 3\dot{e}_{33}) = -p + \frac{2\eta}{3}(2\nabla \cdot \underline{v} - 3\dot{e}_{22} - 3\dot{e}_{33})$$

$$= -p - 2\eta(\dot{e}_{22} + \dot{e}_{33}) \Rightarrow \tau_{11} = -p - 2\eta\left(\frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)$$ {And similarly for τ_{22}, τ_{33}}

* To include the effects of the uniform gravitational field present on earth can redefine p as the EXCESS PRESSURE $p^* = p + \rho g z - p_0$. p is the pressure due to the kinetic motion of the fluid molecules, $\rho g z$ is the weight pressure of the above fluid (z can be defined appropriately) p_0 is an optional constant which can allow us to be flexible with the 'direction' and 'starting point' of z. I will mostly ignore or absorb it into other constants (c.f. BERNOULLI'S THEOREM)

* The dynamics of a fluid element are described classically by Newton's law of motion (II) i.e. $\underline{f} = \rho \, d\underline{v}/dt$ where $\underline{f}$ is the force / unit volume acting on a volume element $dx_1 dx_2 dx_3$. Here we assume $\partial \rho / \partial t = 0$ i.e. fluid is incompressible to remove the $\underline{v}\,\partial \rho/\partial t$ term.

Now since $\underline{v} = \underline{v}(x_1, x_2, x_3, t)$, in general $d\underline{v}/dt \neq \partial \underline{v}/\partial t$. For a SCALAR function $A(x_1, x_2, x_3, t)$, $dA = \frac{\partial A}{\partial x_1}dx_1 + \frac{\partial A}{\partial x_2}dx_2 + \frac{\partial A}{\partial x_3}dx_3 + \frac{\partial A}{\partial t}dt \Rightarrow \frac{dA}{dt} = \frac{\partial A}{\partial x_1}v_1 + \frac{\partial A}{\partial x_2}v_2 + \frac{\partial A}{\partial x_3}v_3 + \frac{\partial A}{\partial t}$

$= (\underline{v} \cdot \nabla)A + \frac{\partial A}{\partial t}$. i.e. $\frac{dA}{dt} = \{(\underline{v} \cdot \nabla) + \partial/\partial t\}A$. So for vector field $\underline{v}(x_1, x_2, x_3, t)$ we can replace $\frac{d}{dt}$ by the same operator $\Rightarrow$ $d\underline{v}/dt = (\underline{v} \cdot \nabla)\underline{v} + \partial \underline{v}/\partial t$ CONVECTIVE DERIVATIVE

* Now $f_1 = \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3}$ $\{f_1 \partial x_1 \partial x_2 \partial x_3 = \partial \tau_{11} \partial x_2 \partial x_3 + \partial \tau_{12} \partial x_1 \partial x_3 + \partial \tau_{13} \partial x_1 \partial x_2\}$

Now using $\tau_{11} = -p^* - 2\eta\left(\frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)$ $\tau_{12} = \eta\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)$ $\tau_{13} = \eta\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right)$

$$\Rightarrow f_1 = -\frac{\partial p^*}{\partial x_1} - 2\eta\left(\frac{\partial^2 v_2}{\partial x_1 \partial x_2} + \frac{\partial^2 v_3}{\partial x_3 \partial x_1}\right) + \eta\left(\frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_1 \partial x_2}\right) + \eta\left(\frac{\partial^2 v_1}{\partial x_3^2} + \frac{\partial^2 v_3}{\partial x_3 \partial x_1}\right)$$

$$= -\frac{\partial p^*}{\partial x_1} - \eta\left[\frac{\partial}{\partial x_2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\right] = -\frac{\partial p^*}{\partial x_1} - \eta\left[\nabla \times \nabla \times \underline{v}\right]_1$$

$\Rightarrow \underline{f} = -\nabla p^* - \eta \nabla \times \nabla \times \underline{v}$. So from Newton's 2nd law we arrive at the NAVIER STOKES EQUATION describing isotropic, Newtonian, incompressible fluid flow

$$-\nabla p^* - \eta \nabla \times \nabla \times \underline{v} = \rho\left((\underline{v} \cdot \nabla)\underline{v} + \partial \underline{v}/\partial t\right)$$

2) Special cases of the N.S. equation * If viscosity is negligible compared to the other terms in the N.S. equation, get EULER'S EQUATION $-\nabla p^* = (\underline{v}\cdot\nabla)\underline{v}\rho + \rho\,\partial\underline{v}/\partial t$

For steady flow $\partial\underline{v}/\partial t = 0 \Rightarrow$ purely Non-linear inertial term $\rightarrow$ chaos, turbulence etc...

* If flow length scales are small or flow is slow s.t. terms $O(v^2)$ can be ignored get STOKES EQUATION $-\nabla p^* - \eta\nabla\times\nabla\times\underline{v} = \rho\,\partial\underline{v}/\partial t$. Now since $\nabla\times\nabla\times\underline{v} = \nabla(\nabla\cdot\underline{v}) - \nabla^2\underline{v}$ and for uniform incompressible flow $\nabla\cdot\underline{v} = 0 \Rightarrow -\nabla p^* + \eta\nabla^2\underline{v} = \partial\underline{v}/\partial t\,\rho$ [STOKES' EQUATION]

* Define VORTICITY $\underline{\Omega} = \nabla\times\underline{v}$. If $\underline{\Omega} = 0$ then regardless what η is, viscous effects will not play a part in fluid flow. Now if $\nabla\times\underline{v} = 0 \Rightarrow \underline{v}$ can be written as $\nabla\phi$ where ϕ is a scalar function $\phi(x_1, x_2, x_3, t)$ in general. $\Rightarrow$ POTENTIAL FLOW.

Since $\nabla\cdot\underline{v} = 0$ for uniform, incompressible flow $\Rightarrow \phi$ satisfies Laplace's equation $\nabla^2\phi = 0$.

If $\nabla\times\underline{v} = 0$ then N.S. $\rightarrow$ Euler equation. Now $(\underline{v}\cdot\nabla)v_1 = v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} + v_3\frac{\partial v_1}{\partial x_3}$

But since $\nabla\times\underline{v} = 0 \Rightarrow \frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i}$ $j \neq i$ $\Rightarrow (\underline{v}\cdot\nabla)v_1 = v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_2}{\partial x_1} + v_3\frac{\partial v_3}{\partial x_1} = \underline{v}\cdot\frac{\partial\underline{v}}{\partial x_1}$

Now $\frac{\partial}{\partial x_1}(\tfrac{1}{2}v^2) = \frac{\partial}{\partial x_1}(\tfrac{1}{2}\underline{v}\cdot\underline{v}) = \tfrac{1}{2}\underline{v}\cdot\frac{\partial\underline{v}}{\partial x_1} + \tfrac{1}{2}\underline{v}\cdot\frac{\partial\underline{v}}{\partial x_1} = \underline{v}\cdot\frac{\partial\underline{v}}{\partial x_1}$ Hence $(\underline{v}\cdot\nabla)v_1 = \frac{\partial}{\partial x_1}(\tfrac{1}{2}v^2)$

$\Rightarrow (\underline{v}\cdot\nabla)\underline{v} = \nabla(\tfrac{1}{2}v^2)$. Substitution of this result into Euler's equation yields

$-\nabla p^* = \rho\nabla(\tfrac{1}{2}v^2) + \rho\frac{\partial}{\partial t}\nabla\phi \Rightarrow \nabla\left(\frac{p}{\rho} + gz + \dot{\phi} + \tfrac{1}{2}v^2\right) = 0 \Rightarrow \frac{p}{\rho} + gz + \tfrac{1}{2}v^2 + \dot{\phi} = \text{constant}.$

This is BERNOULLI'S THEOREM.

3) Vortices and Vorticity * Taking the curl of both sides of the N.S. equation yields

$\nabla\times(-\nabla p^* - \eta\nabla\times\nabla\times\underline{v}) = \nabla\times(\rho(\underline{v}\cdot\nabla)\underline{v} + \partial\underline{v}/\partial t\,\rho) \Rightarrow \eta\nabla^2\underline{\Omega} = \rho\frac{\partial\underline{\Omega}}{\partial t} + \rho\nabla\times(\underline{v}\cdot\nabla)\underline{v}$

$(\nabla\times\nabla\times\nabla\times\underline{v} = \nabla\times\nabla\times\underline{\Omega} = \nabla(\nabla\cdot\underline{\Omega}) - \nabla^2\underline{\Omega}$. Since $\underline{\Omega}$ is $\nabla\times\underline{v} \Rightarrow \nabla\cdot\underline{\Omega} = 0)$

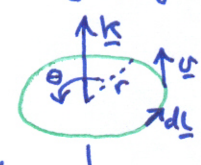
Now consider Stokes limit $\Rightarrow$ ignore term $\nabla\times(\underline{v}\cdot\nabla)\underline{v} \Rightarrow \nabla^2\underline{\Omega} = \frac{\rho}{\eta}\frac{\partial\underline{\Omega}}{\partial t}$ DIFFUSION EQUATION.

Now 1D diffusion equation takes the form $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ for some function $u(x,t)$.

If solution is required for $t>0$ and b.c's are $u(x,0) = 0$ $(x>0)$ $u(0,t) = u_0$, use method of Laplace transforms $\Rightarrow u(x,t) = u_0\left[1 - \mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right]$ i.e. $\tfrac{1}{2}$ r.m.s half width $\approx \sqrt{2kt}$

So for our equation in $\underline{\Omega}$, if $\underline{\Omega}$ were a scalar $\sqrt{2kt} = \sqrt{2\eta t/\rho}$. Define $\delta = \sqrt{2\eta t/\rho}$ as the BOUNDARY LAYER to give an idea to the extent of vorticity diffusion into (initially) vorticity free fluid.

* A vortex can be thought of as a core of vorticity of radius δ^* in a region of vorticity free fluid. A vortex has strength $\underline{K}$ where $K = \oint_{\text{vortex}} \underline{v}\cdot d\underline{l}$



Since $\underline{v}$ is continuous outside vortex core actual path around vortex chosen to evaluate $\oint_{\text{vortex}}\underline{v}\cdot d\underline{l}$ does not matter (c.f. Cauchy's theorem – can represent $\underline{v}$ in plane $\perp \underline{K}$ as complex variable) – choose circular path.

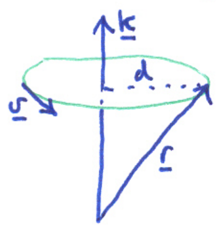
NOTE DEFINE $\underline{K} \perp \underline{v}$

$\therefore \underline{v} = v_\theta\hat{\underline{\theta}} = v_\theta\, d\underline{l}/dl$ $\therefore \underline{v}\cdot d\underline{l} = v_\theta\, dl \Rightarrow K = 2\pi r v_\theta \Rightarrow v_\theta = K/2\pi r$

$\Rightarrow \underline{v} = \frac{K}{2\pi r}\,d\underline{l}/dl$ Now $d\underline{l} = (\hat{\underline{K}}\times\hat{\underline{r}})dl$ (even if $\underline{r}$ is not $\perp \underline{K}$). $\therefore \underline{v} = \underline{K}\times\underline{r}/2\pi r^2$

↑ but for r in $\frac{K}{2\pi r}$ $\underline{r} \perp \underline{K}$

In general:



Now $d = \sqrt{r^2 - (\underline{r}\cdot\hat{\underline{K}})^2} = r\sqrt{1 - (\hat{\underline{r}}\cdot\hat{\underline{K}})^2}$

$\underline{v} = \frac{\underline{K}\times\hat{\underline{r}}}{2\pi d}$ [Not $\hat{\underline{K}}$]

$\Rightarrow \underline{v} = \frac{\underline{K}\times\underline{r}}{2\pi r^2\sqrt{1 - (\hat{\underline{r}}\cdot\hat{\underline{K}})^2}}$

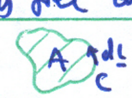
* Possible time dependent solutions for v_θ around a vortex core are

(1) $v = \frac{K}{2\pi r}(1 - \exp[-\rho r^2/4\eta t])$

(2) $v \propto \frac{\rho r}{16\pi\eta^2 t^2}\exp[-\rho r^2/4\eta t]$

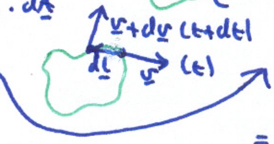
* If diffusion of vorticity into a fluid is ignored KELVIN'S CIRCULATION THEOREM HOLDS. i.e. "A vorticity free fluid at one instant of time remains vorticity free thereafter".

Proof: $K = \oint_C \underline{v}\cdot d\underline{l}$ By Stokes theorem $\oint\underline{v}\cdot d\underline{l} = \int_A \nabla\times\underline{v}\cdot d\underline{A} = \int_A \underline{\Omega}\cdot d\underline{A}$



Since $\underline{\Omega} = 0$ everywhere at $t=0 \Rightarrow K(t=0) = 0\ \forall$ loops.

Now $\frac{DK}{Dt} = \oint\left[\frac{D\underline{v}}{Dt}\cdot d\underline{l} + \underline{v}\frac{D(d\underline{l})}{Dt}\right]$. $\frac{D(d\underline{l})}{dt} = \frac{\partial\underline{v}}{\partial l}dl$ $(= d\underline{v}(l) = \frac{\partial\underline{v}}{\partial l}dl$ by chain rule)



$\therefore \underline{v}\cdot\frac{D(d\underline{l})}{Dt} = \underline{v}\cdot\frac{\partial\underline{v}}{\partial l}dl = \frac{\partial}{\partial l}(\tfrac{1}{2}\underline{v}\cdot\underline{v})dl = d(\tfrac{1}{2}v^2)$

Now from Euler's equation

$\frac{D\underline{v}}{Dt} = (\underline{v}\cdot\nabla)\underline{v} + \partial\underline{v}/\partial t = -\nabla p^*/\rho$ $\therefore$ since $-\frac{\nabla p^*}{\rho}\cdot d\underline{l} = -\frac{dp^*}{\rho}$

$\Rightarrow \frac{DK}{Dt} = \oint\left\{d(\tfrac{1}{2}v^2) - \frac{dp^*}{\rho}\right\}$. Since v^2 and p^* are not multivalued functions $\oint\ldots = 0 \Rightarrow \frac{DK}{Dt} = 0$

QED.

4) Dynamical Similarity * The variation of fluid flow with geometrical scaling and change in fluid parameters like η, ρ and mean flow speed can be characterised by the Reynolds number associated with the flow. For a particular geometry, flows are identical when Re is the same, regardless of the scale of the geometry. (This is included in the definition of the Reynolds number). Given N.S. equation (Newtonian, uniform, incompressible flow) define 'characteristic velocity' v_0 and characteristic length L_0. Define $v = v_0 v'$, $x = L_0 x'$ where x', v' are dimensionless. Similarly (Noting Bernoulli) $P = \rho v_0^2 p'$, $\nabla = \nabla'/L$

$\Rightarrow$ N.S becomes $-\frac{\rho v_0^2}{L_0}\nabla' p^{*\prime} - \eta \frac{v_0}{L_0^2}\nabla'\times\nabla'\times\underline{v}' = \rho(\underline{v}'.\nabla)\underline{v}'\frac{v_0^2}{L_0} + v_0\frac{\partial \underline{v}'}{\partial t}\rho$

$\Rightarrow -\rho v_0 L_0 \nabla' p^{*\prime} - \eta\nabla'\times\nabla'\times\underline{v}' = (\underline{v}'.\nabla)\underline{v}' v_0 L_0 \rho + L_0^2 \frac{\partial \underline{v}'}{\partial t}\rho$ Also define $t = \frac{L_0}{v_0}t'$

$\Rightarrow \eta\nabla'\times\nabla'\times\underline{v}' = -\rho v_0 L_0\left((\underline{v}'.\nabla)\underline{v}' + \frac{\partial \underline{v}'}{\partial t'} + \nabla' p^{*\prime}\right) \Rightarrow \nabla'\times\nabla'\times\underline{v}' = -Re\left[(\underline{v}'.\nabla)\underline{v} + \frac{\partial \underline{v}'}{\partial t} + \nabla' p^{*\prime}\right]$

where $Re = \rho v_0 L_0/\eta$. Definition of Reynolds #. Clearly for same Re, N.S. yields same solutions. Re encompasses all flow parameters – the rest are dimensionless quantities.

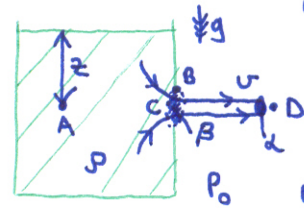
* other numbers are EULER # $E = \frac{v_0}{\sqrt{2p_0/\rho}}$ (Do above analysis with Euler equation)

MACH # $Ma = v_0/c_{sound}$ FROUDE # $F = v_0/c_{wave} = \frac{v_0}{\sqrt{g\lambda_0}}$ (λ_0 is characteristic wavelength, g is gravitational field strength)

↑ characteristic speed

For the results derived above $Ma \ll 1$. (i.e. flow pattern responds $\approx$ instantaneously to changes, info. transmitted at $c_{sound} \gg v_0$).

5) Examples of fluid flow * Potential flow: vena contracta and Borda's mouthpiece



Consider a water tank containing inviscid fluid (Potential flow)

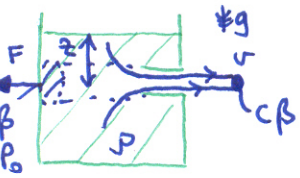
Pressure at A is $P_0 + \rho g z$, Pressure at D is P_0

↳ Apply Bernoulli assuming steady flow ($\dot\phi = 0$) $\Rightarrow P_0 + \rho g z = P_0 + \frac{1}{2}\rho v^2$

$\Rightarrow v = \sqrt{2gz}$ Now orifice has area β (assume circular but it need not be). Water jet emerging will have less cross section due to vena contracta. Why? velocity at C* will be > than velocity at orifice edge B. $\Rightarrow$ pressure gradient and $\therefore$ contracted flow. $\rightarrow$ eq as jet emerges.

$(P_B > P_C)$ * must outweigh loss of pressure head since $z_C > z_B$.

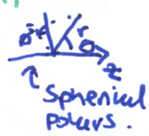
Now Borda's mouthpiece causes CONTRACTION COEFFICIANT $\alpha/\beta = C$ to be exactly $\frac{1}{2}$ (For circular holes $C \simeq 0.62$)

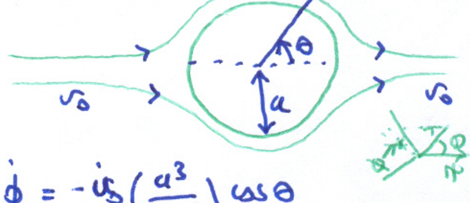


Force F on wall area β shown is $\rho g z \beta$. For reservoir to remain in e.q. – opposing force (which drives jet out) $= \beta\rho g z$ (well.... reservoir remains in eq because it is so heavy but reaction force (Newton III) on water drives it out because of hole, favourable pressure gradient. (And lack of opposing wall)). So rate of change of jet momentum is $\beta \rho g z = C\beta v \rho v$. Now $v^2 = 2gz$

$\Rightarrow \beta\rho g z = C\beta\rho . 2gz \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$ This analysis only works for Borda's mouthpiece because the flow convergence $\frac{1}{r}$ is so slow (since it is far from the orifice) and $\therefore$ we can ignore the force required to converge the liquid (Fluid) into the jet.

* Potential flow: flow round stationary sphere

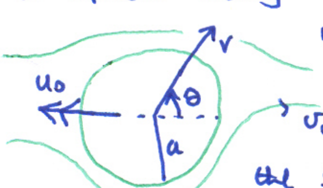




ϕ symmetric problem. Solve $\nabla^2\phi = 0$ with b.c's

$\underline{v}(r=\infty,\theta) = v_0\hat{\underline{z}} \Rightarrow v_0\cos\theta\,\hat{\underline{r}} = v_r\hat{\underline{r}}$ and $v_r(r=a) = 0$

and $v_\theta(r=a) = 0$ (No slip). $\Rightarrow \phi = v_0\cos\theta\left(r + \frac{a^3}{2r^2}\right)$. Now $\phi = -v_0\left(\frac{a^3}{2r^2}\right)\cos\theta$ for moving (well accelerating) sphere in fluid stream: why? more general ϕ is calculated for sphere moving with velocity u_0 in a fluid with opposing velocity v_0



New b.c $v_r(r=a) = -u_0\cos\theta$ instead of 0 $\Rightarrow \phi = v_0 r\cos\theta + \frac{1}{2}a^3\frac{\cos\theta}{r^2}(v_0 - u_0)$. For frame where sphere is stationary $u_0 = 0 \Rightarrow \phi = v_0\cos\theta\left(r + \frac{a^3}{2r^2}\right)$ For frame where fluid is stationary $\phi = -u_0\cos\theta\left(\frac{a^3}{2r^2}\right)$. Of course this is only valid for the snapshot in time when coordinate axis are aligned with centre of sphere.

$\rightarrow$ but clearly $\dot\phi$ can still be calculated. Now $\underline{v} = \nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, 0\right)$ $\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\phi}$

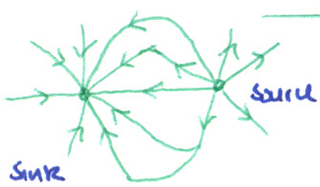
$= \left(\frac{a^3}{r^3}u_0\cos\theta, \frac{a^3}{2r^3}u_0\sin\theta, 0\right) \therefore$ by Bernoulli $\frac{p^*}{\rho} + \frac{1}{2}\left(\frac{a}{r}\right)^6\left(\cos^2\theta + \frac{1}{4}\sin^2\theta\right) - \dot v_0\left(\frac{a^3}{2r^2}\right)\cos\theta = 0$ (cancelling P_0 @ ∞ with P_0 at (r,θ)). Drag force is evaluate @ $r=a$

$F_D = -\int_0^\pi 2\pi a^2 p^*\cos\theta\sin\theta\, d\theta = \frac{1}{2}M_{sphere}\dot{v}_0$ $\rightarrow$ So when sphere accelerates in fluid appears like it has $\frac{3}{2}\times$ inertia at rest.

*** Potential flow: Sinks and sources** Unlike electric charges sources and sinks attract when alike and repel when not alike. (Source : Source, Sink : Sink attract ; Sink : Source repel)
Now continuity equation for $\nabla p = 0$ is $\nabla \cdot \underline{v} = -\frac{1}{\rho}\partial \rho/\partial t$. If $\nabla \times \underline{v} = 0$ in all space occupied by a fluid $\Rightarrow$ Potential flow and $\underline{v} = \nabla \phi$. Now $-\frac{1}{\rho}\partial\rho/\partial t = q$ = discharge rate / unit volume ($/s^{-1}$)
Hence $\nabla^2 \phi = q$ POISSON'S EQUATION. For point sources, sinks $q = Q\delta(\underline{x})$ { +ve source, -ve sink
overall flowrate from source $\int_{\text{all space}} Q\delta(\underline{x})d^3\underline{x} = Q$. Now consider sphere of radius R surrounding $\delta(\underline{x})$. Clearly $Q = \int_{V(R)} \nabla\cdot\nabla\phi\, d\tau = \int_S \nabla\phi \cdot d\underline{s}$ by divergence theorem. Now one expects ϕ to be radially symmetric (Hence spherical V) $\Rightarrow \nabla\phi\cdot d\underline{s} = \frac{d\phi}{dr}ds$

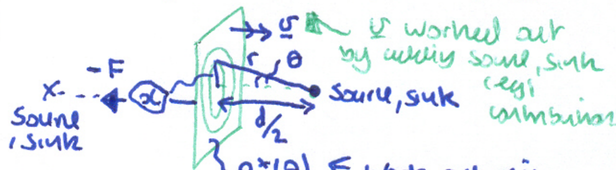
$\Rightarrow Q = 4\pi R^2 \phi'(R)$ $\therefore \phi(R) = -\frac{Q}{4\pi R} + C$ since R, r arbitrary. Now $\phi \to 0$ as $r \to \infty \Rightarrow C = 0$ $\Rightarrow \phi(r) = -\frac{Q}{4\pi r}$ (source) $\phi(r) = +\frac{Q}{4\pi r}$ (sink)
↑ i.e. ensure Q has +ve sign.

So $\underline{v}_{source} = \frac{Q}{4\pi r^2}\hat{\underline{r}}$ $\underline{v}_{sink} = \frac{-Q}{4\pi r^2}\hat{\underline{r}}$



So why do like attract and unlike repel? → Apply Bernoulli in plane separating sources, sinks. For like, like fluid velocity is high in this plane since fluid streams add and change direction to conserve fluid and momentum. ∴ pressure is lower ⇒ source, sinks attract. In source, sink case pressure is higher in intermediate region hence source, sinks repel. To prove in detail work out net force on infinite plate placed in intermediate region

$$F = \int_{\text{plate}} p^*(\theta) \cdot 2\pi x\, dx = \int_0^{\pi/2} p^*(\theta) \cdot \frac{\pi}{2} d^2 \frac{\sin\theta}{\cos^3\theta}\, d\theta$$

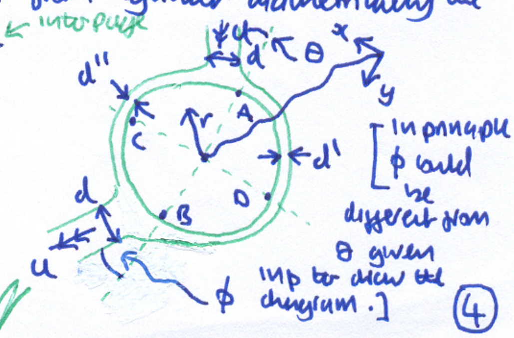


($x = r\sin\theta$, $\frac{d}{2} = r\cos\theta$ $\Rightarrow x = \frac{\sin\theta}{\cos\theta}\frac{d}{2}$, $dx = \frac{d}{2\cos^2\theta}d\theta$)
Now if F is -ve ⇒ "sources" repelled, F is +ve ⇒ "sources" attract.
Turns out $|F| = \rho\frac{Q^2}{4\pi d^2}$ for separation d. Note $|F| = \rho Q v(d)$
[Consider one side of the plate. Determine flow direction. Assume $\underline{F}$ points this way. Work out F. If -ve plate moves away from source, sink ⇒ repulsion. – Source, sink moves away from plate as if a force F was acting upon it].

*** Potential flow: Coanda Effect** – Manifests itself in the tendency of liquids to curl round the rims of vessels from which they are being poured and to drip where they are not wanted. To illustrate consider plane sheet of liquid (Euler) of thickness d striking a stationary cylinder of radius a (>>d) with velocity u. Let us ignore gravity (assume u is s.t. $\rho u^2 >> gz$ where $z \sim a$).

Now by Bernoulli, since P_A is the pressure at the fluid sheet surface, velocity at surface must be u. Now if the separated streams of thickness d' and d'' were to detach and carry straight on

to conserve flow, velocity profile must be uniform as before and $d = d' + d''$
→ but this cannot occur since if flow is vorticity free ⇒ $v_\theta \propto \frac{1}{r}$ round cylinder. i.e. velocity gradient which needs to be 'straightened out' by some pressure gradient which clearly does not exist beyond the cylinder ($p = P_A$). Only solution if flows carry on round cylinder until they meet. The simplest scenario is if cylinder is uniform and no 'delays' are placed in the flow (i.e. blotting paper etc) → in this case flow will detach from cylinder diametrically the point of impact. Now momentum change of fluid is (P_x, P_y) /s/m ← into page
$= (\rho u d u \sin\phi + \rho u^2 d \sin\theta,\ \rho u d u\cos\phi - \rho u d u\cos\theta)$ and rate of change of momentum / unit length of cylinder is – this. (≡ Force / unit length).
Now $P_D = P_A - d'\frac{dp}{dr}$, $P_C = P_A - d''\frac{dp}{dr}$. Considering initial impact
$du = d'u + d''u$, $\rho u^2 d\sin\theta = \rho u^2 d' - \rho u^2 d''$ $\Rightarrow d' = \frac{1}{2}d(1+\sin\theta)$, $d'' = \frac{1}{2}d(1-\sin\theta)$
↑ continuity ↑ x momentum.



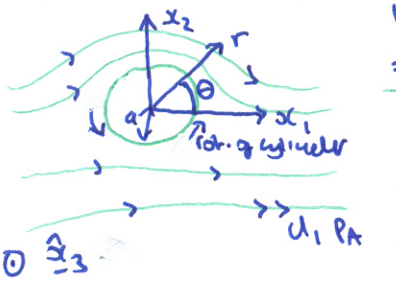
[In principle φ could be different from θ given imp to draw the diagram.]

Now Bernoulli ⇒ $P_A = P(r) + \frac{1}{2}\rho v^2$ and $v = A/r$, $u = \frac{A}{a+d'} \approx \frac{A}{a}$ } $\Rightarrow v = \frac{ua}{r}$ $\therefore P(r) = P_A - \frac{1}{2}\rho\frac{u^2a^2}{r^2}$ $\Rightarrow \frac{dp}{dr} = +\rho\frac{u^2a^2}{r^3}$ $\Rightarrow \left.\frac{dp}{dr}\right|_{r=a} = +\frac{\rho u^2}{a}$

Now $P_D \simeq P_A - d' \frac{dp}{dr}\Big|_{r=a}$, $P_C \simeq P_A - d'' \frac{dp}{dr}\Big|_{r=a}$ ($d'', d' \ll a$ crucial here).

$\therefore$ Net force in x direction on cylinder / unit length $= (P_D - P_C) \cdot 2a$
(could prove by integration but easy to justify). $= (d'' - d')(+\rho u^2/a) \cdot 2a$
$= +\frac{2a}{2}\rho \frac{u^2}{a}(-\sin\theta - \sin\theta) = -2\rho d u^2 \sin\theta$ so to get this result $\phi = \theta$. QED.

* Potential flow with circulation - Vortex images, magnus effect, smoke rings, submerged jets
With the analogy of a current carrying wire in a magnetic field (see section on magnetic analogy) force on vortex line $\underline{k}$ in a uniform velocity field $\underline{v}$ is $\underline{f} = \rho \underline{v} \times \underline{k}$ [MAGNUS FORCE]
Justification: consider potential flow round a cylinder of radius a. / unit length

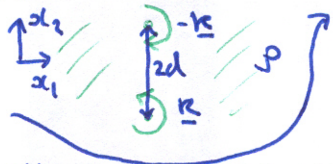


Let $\underline{v}(r\to\infty)$ be $u_1\hat{x}_1$. Solution of Laplace's equation for $r > a$
$\Rightarrow \phi = u_1\cos\theta\left(r + \frac{a^2}{r}\right)$ Now for a vortex $\parallel \hat{x}_3$ of strength k, $v_\theta = \frac{k}{2\pi r}$ $\therefore \phi_{vortex} = \frac{k\theta}{2\pi}$. $\left[\left(\frac{\partial\phi_{vortex}}{\partial r}, \frac{1}{r}\frac{\partial\phi_{vortex}}{\partial\theta}, \frac{\partial\phi_{vortex}}{\partial z}\right) = \underline{v} = \frac{k}{2\pi r}\hat{\theta}\right.$ as required.$\left.\right]$
$\therefore$ for rotating cylinder
$\Rightarrow \phi = u_1\cos\theta\left(r + \frac{a^2}{r}\right) + \frac{k\theta}{2\pi}$

Now pressure at surface of cylinder (well excess pressure really) $p^*(r=a) = \frac{1}{2}\rho(u_1^2 - v_\theta^2) = \frac{1}{2}\rho\left\{u_1^2 - \left(-2u_1\sin\theta + \frac{k}{2\pi a}\right)^2\right\}$
$\uparrow$ in this case $P(r=a) - P_A = p^*$
$= \frac{1}{2}\rho\left(\frac{k}{2\pi a}\right)^2 + \frac{1}{2}\rho u_1^2(1 - 4\sin^2\theta) + \frac{\rho u_1 k}{\pi a}\sin\theta$
symmetric over θ so no net force

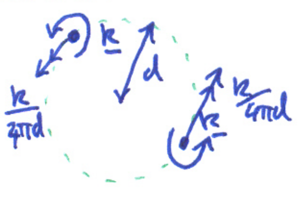
$\Rightarrow$ noting symmetry of 1st two terms - only $\frac{\rho u_1 k}{\pi a}\sin\theta$ will contribute to net force $\parallel x_2$ - clearly no net force $\parallel x_1$ $\therefore \underline{F} = F\hat{x}_2$ ↙ /unit length
$= -\int_0^{2\pi} p^* \sin\theta \, a \, d\theta = -\frac{\rho u_1 k}{\pi a}\int_0^{2\pi} a\sin^2\theta \, d\theta = -\rho u_1 k = (\rho u_1 \hat{x} \times \underline{k}) \cdot \hat{x}_2$ So $\underline{F} = \rho \underline{v} \times \underline{k}$

Now vortex lines with opposite circulation will propel each other along with speed $u = \frac{k}{4\pi d}$ if $2d$ is their separation.

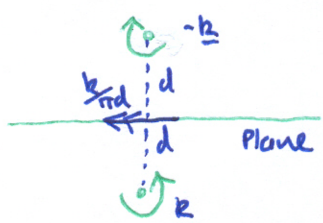


generated by the other ⇐ Note self $\underline{v}$ of each $\underline{k}$ will be zero on the streamline
$\underline{v} = \underline{k} \times \underline{r} / 2\pi r^2$ ($\underline{r} \perp \underline{k}$)
$\to$ if $\underline{r} = 2d\hat{x}_2$ $\therefore \underline{v} = \frac{-k}{4\pi d}\hat{x}_1$ (2) for v_{vortex} p2.
Now Magnus force between the vortices is $\rho \underline{v} \times \underline{k}$. Each $\pm\underline{k}$ is in the $\underline{v}$ field

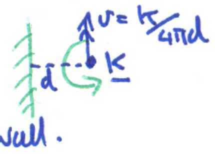
result of Magnus force is that vortices attract each other with a force $= \frac{\rho k^2}{4\pi d}$. i.e. vortex
BUT THEY DON'T ATTRACT because the vortices are embedded in the fluid. must move with a velocity precisely equal to the 'applied field' relative to an outside observer in order to remain stationary relative to the fluid. Hence the $\underline{v}$ in the magnus force is zero, the presence of the other vortex simply propels the other along at velocity $\frac{k}{4\pi d}$.
Like vortices also have the same effect (propulsion speed $\frac{k}{4\pi d}$) but $\underline{v}$ points in alternate directions $\rightarrow$ mutual rotation about common centre.
(Angular speed $\frac{v}{r} = \frac{k}{4\pi d^2}$).



Now for the case of opposite vortices if one inserted a plane between them (providing boundary layer $\ll d$ VELOCITY FIELD WOULD NOT BE AFFECTED. $\therefore$ can



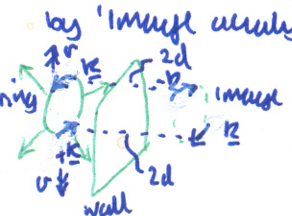
replace a real plane and a vortex above it by a vortex, VORTEX IMAGE (opposite k sign) pair. $\therefore$ A vortex 'placed' next to a wall will tend to drift along it.



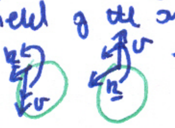
A SMOKE ring is an example of a vortex ring
since each $\delta\underline{k}$ vector round the ring has a diametric opposite $\Rightarrow$ ring propels itself along (into the paper) with a certain velocity $v_{ring} \simeq \frac{k}{4\pi d}\left\{\ln\left(\frac{8d}{a}\right) - \frac{1}{4}\right\}$

a = core radius $\simeq \sqrt{2\nu t/\rho}$ for viscous medium.

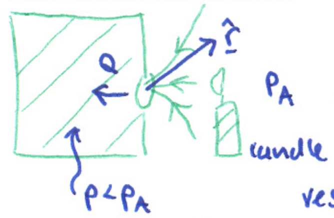
Note smoke rings approaching a wall will radially expand by 'image analysis' above. Note this increases d which slows ring down.



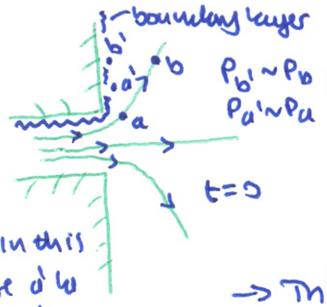
If two rings are released one after another, the velocity field of the one behind causes the front one to enlarge, the behind one to contract - the increase of d slows the first one down and vice versa for the one behind $\rightarrow$ result is a leapfrogging of smoke rings.



a) Submerged jets – why it is easier to blow out a candle than to suck it out.

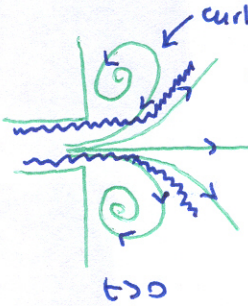


If one tries to suck a candle out one effectively creates a sink with velocity distribution $\underline{v} = -\frac{Q}{4\pi r^2}\hat{\underline{r}}$. If one tries to blow it out, if fluid inside tubing is the air or otherwise (it makes sense for it to be air – a rather strange jet of other liquid will surely result otherwise) we get a SUBMERGED JET rather than a source. When the fluid leaves the orifice at sufficiently high speeds, adverse pressure gradient due to diverging flow lines (Bernoulli) causes eddy formation, overall this results in a "solenoid" of vortex rings which surround the emerging fluid in a tube – submerged jet. Although this vortex sheet is unstable → turbulence not far from orifice it is clearly localising the fluid momentum more than a source ⇒ hence easier to blow out candle.



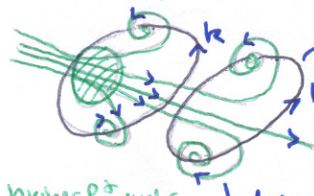
$P_{b'} \sim P_b$ BUT $P_b > P_a$ since $v \propto \frac{1}{r^2}$
$P_{a'} \sim P_a$

⇒ Pressure gradient b' → a'
⇒ resultant flow b' → a'
⇒ EDDIES



curl round since $P_{b'}$ drops as fluid increases speed. ∴ Force b → b' .....
⇒ result is B.L detaches from walls.

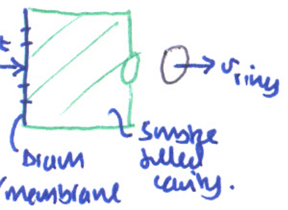
Eventually get submerged jet.



diverging streamlines from eddies which confine flow.

In this case d ... cracker's mouthpiece

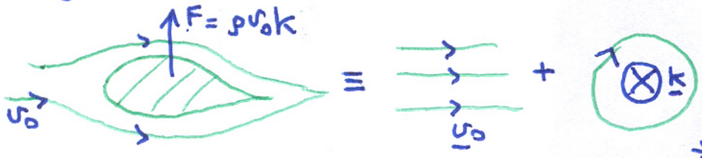
→ This is a good way to produce smoke rings – use a 'pulse' of flow out of the orifice instead of a continuous jet.



higher P under asymmetric aerofoil α>0.

* Aerofoils and wings (+ trailing vortices and stalling)

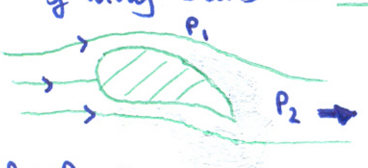
To create lift one must create a bound vortex around the wing of a plane. If the plane is moving at speed v_0 (velocity) w.r.t. air ⇒ air moves at velocity $-\underline{v}_0$ w.r.t. plane. ⇒ Magnus lift

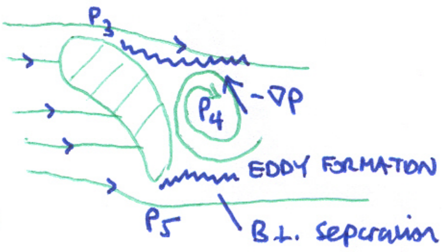


Now for special types of aerofoils i.e. Zhukovskii aerofoils $\frac{k}{v_0} = \pi \sin\alpha(D + 0.77d)$

(Note: Force on wing ↑ = $\rho v_0 k$ × length of wing – so make wings long!)

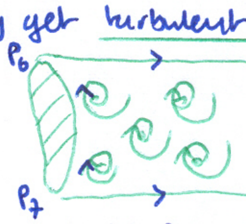
Now if $\alpha \gtrsim 8°$ then B.L at rear of wing starts to separate.





$P_1 \sim P_2$ ⇒ Bound vortex
stagnations increase P
$P_4 \sim P_5$ BUT $P_3 << P_4$ (slow air behind aerofoil)

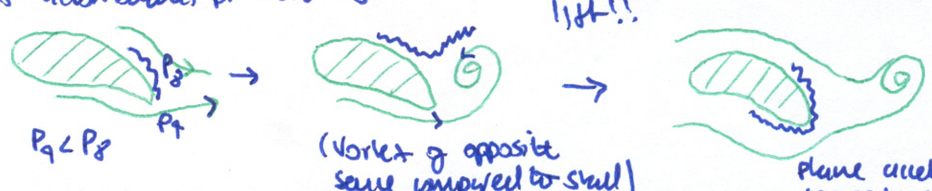
Eventually get turbulent wake.



In this case when B.L separates – no bound vortex ⇒ no lift. STALLING.

If $P_6 > P_7$ then start to form reverse bound vortex ⇒ opposite of lift!!

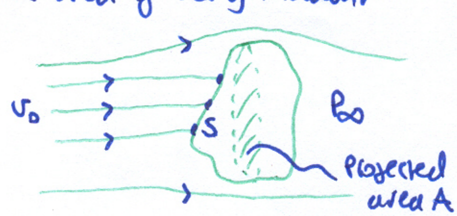
Now when plane accelerates one observes vortex shedding from TRAILING EDGE as bound vortex accomodates for new v_0 ↑(Not leading edge as for stall)



Plane accelerates, leaves trailing vortex behind.
– New bound vortex forms.

b) Drag * maximum rate of momentum transfer from fluid to a body moving within it results if there is stagnation everywhere over the area of body incident to the fluid. If fluid obeys potential flow conditions → Bernoulli: $P_S = P_\infty + \frac{1}{2}\rho v_0^2$

∴ $F_{Drag} = \frac{1}{2}\rho v_0^2 A$ [If body was moving through stationary fluid at speed v_0]
'ideal'



* extending this expression for ideal drag – define DRAG COEFFICIENT to model non ideal situations.

$F_{Drag} = \frac{1}{2}\rho v_0^2 A C_D$ where $C_D = f(Re)$. (since C_D is dimensionless and we assume very much subsonic speeds ($M \ll 1$) this is plausible). The form of f purely depends on the geometry of the body since we make the assumption of uniform flow (v_0) in the absence of the body feeling the drag force.

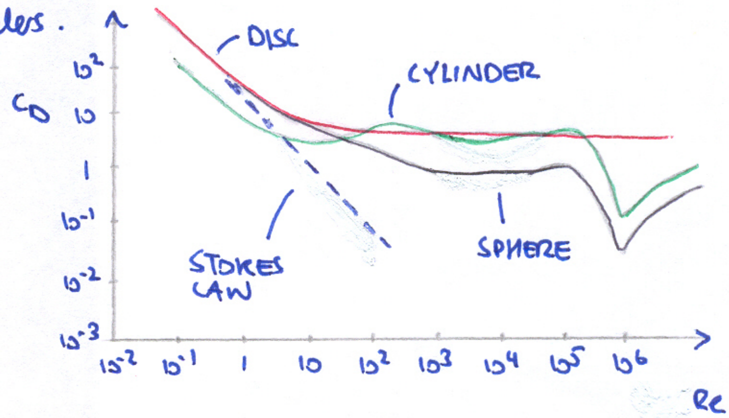
* Stokes Drag – If inertial terms in the N.S equation are negligible compared to viscous terms ⇒ Stokes equation. For creeping flow i.e. $\rho|\partial \underline{v}/\partial t| \ll |\nabla p^*|, |\eta \nabla^2 \underline{v}| \Rightarrow \nabla^2 \underline{v} = \frac{1}{\eta}\nabla p^*$

Now for a sphere (stationary) in moving Stokes fluid (or vice versa) with sphere/far field fluid velocity v_0 – drag force $F_{Drag} = 6\pi\eta a v_0$ where a = sphere radius.

$\therefore C_D = \dfrac{6\pi\eta a v_0}{\frac{1}{2}\rho v_0^2 \pi a^2} = 12\eta/\rho v_0 a$. Define $Re_{sphere} = 2a v_0 \rho/\eta \Rightarrow C_D = 24/Re$

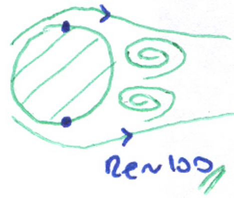
Now one would expect from 'ideal' case, $C_D \leq 1$. Clearly when $Re \ll 1$, $C_D \gg 1$ for Stokes drag and Stokes law is known to hold experimentally for $Re < 0.5$ so this is a correct result. Explanation is C_D is defined via inertial effects rather than viscous.

* Drag coefficient $C_D(Re)$ for $M \ll 1$ varies with Re in the following way. Note similarity in trends between discs, spheres and cylinders.

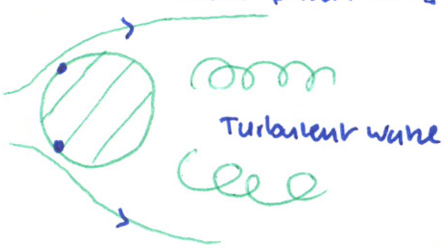


For a sphere this trend is illustrated by the following flow description:

$Re \sim 30$, $\delta \sim 0.4a$

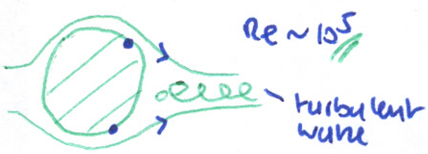


[B.L. separation points move forward....]



Re ~ 1000 – 10,000

[Then as B.L becomes turbulent separation point shifts back. (greater ∇p needed to separate B.L due to higher sideways $\underline{v}$)]



This is called the "Drag crisis" – dip in graph above.

– Exact 'crisis point' depends on surface geometry of body moving in fluid. If B.L. turbulence can be stimulated at lower Re – good way of drag reduction. c.f dimples on golf balls.

($M \ll 1$. close to sonic speeds start becomes important too)

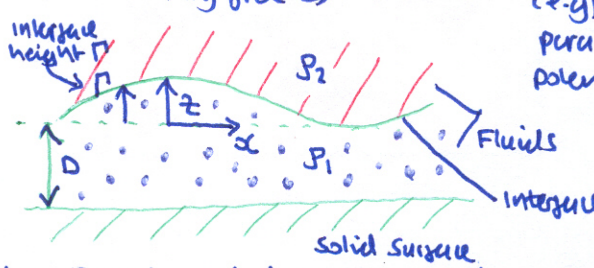
– cylinders behave in similar ways to spheres except at $Re \sim 100$ – alternate eddy detachment (Kármán vortex sheet) → oscillatory Magnus force on cylinder – Explains "swish of a cane, helical fins on tall chimneys (reduce oscillation of chimney).

TO REDUCE DRAG one needs to streamline back of object to prevent eddy formation (c.f aerofoil) – because of symmetry of potential flow round a sphere (e.g) – no drag. (D'Alembert's paradox) So want to minimise non potential flow round back!

7) Waves * consider an interface between two Newtonian, incompressible fluids. Flow of fluid is vorticity free ⇒ potential flow. i.e. $\underline{v} = \nabla\phi$.

* let fluid extend infinitely in y direction ⇒ no y dependence of $\underline{v}$

* k, ω of waves on interface must be same for 1, 2



(It can be shown attenuation constant ($\phi \propto e^{-\alpha t}$) due to viscosity is $\alpha \simeq 16\pi^2\eta/3\rho c_s\lambda^2$, c_s is speed of sound)

to preserve velocity continuity at interface. * Assume no attenuation (or viscous dissipation).

$\therefore$ let $\phi_1 = f_1(z)e^{i(kx-\omega t)}$ and $\phi_2 = f_2(z)e^{i(kx-\omega t)}$. Now since fluid is incompressible and uniform ⇒ $\nabla \cdot \underline{v} = 0$.

$\therefore$ Since $\partial^2\phi_1/\partial x_1^2 = -k^2\phi_1 \Rightarrow \partial^2\phi_1/\partial z^2 = k^2\phi_1$ similarly for ϕ_2........ (see next page!)

So, if $\partial^2\phi_{1,2}/\partial z^2 = k^2\phi_{1,2} \Rightarrow \phi_{1,2}$ must take form $\phi_{1,2} = (A_{1,2}e^{kz} + B_{1,2}e^{-kz})e^{i(kx-\omega t)}$

Now $v_z = \partial\phi/\partial z = 0$ at $z=-D$ and $z=\infty \Rightarrow A_1e^{-kD} - B_1e^{kD} = 0$ and $A_2 = 0$. $\Rightarrow B_1 = +A_1e^{-2kD}$

$\therefore \phi_1 = 2A_1e^{-kD}\cosh(kz+kD)e^{i(kx-\omega t)}$; $\phi_2 = B_2e^{i(kx-\omega t)}e^{-kz}$. Now $\Pi(x,t)$ is interface height.

clearly $\left.\partial\phi_1/\partial z\right|_{z=\Pi} = \left.\partial\phi_2/\partial z\right|_{z=\Pi} = \frac{D\Pi}{Dt} = \frac{\partial\Pi}{\partial t} + (\underline{v}.\nabla)\Pi \approx \frac{\partial\Pi}{\partial t}$ ignoring terms $O(A_1^2, B_2^2)$. [we will use this later]

* In general there will be a pressure difference across the surface of the interface $= \sigma/R$ (surface tension/radius of curvature) $= -\sigma\,\partial^2\Pi/\partial x^2$ (-ve sign: $P_1 > P_2 \Rightarrow$ +ve curvature $\Rightarrow \partial^2\Pi/\partial x^2$ -ve.)

$\therefore P_1 - P_2 = -\sigma\,\partial^2\Pi/\partial x^2$. Now since $\left.\partial\phi_2/\partial z\right|_{z=\Pi} = \partial\Pi/\partial t$

$\Rightarrow +\frac{1}{i\omega}B_2ke^{-k\Pi}e^{i(kx-\omega t)} = \Pi \Rightarrow -\sigma\,\partial^2\Pi/\partial x^2 = \sigma k^2\Pi$. Now applying Bernoulli in (neglect $v^2 - O(A_1^2)$ term).

- regions 1, 2 at interface Π: $\frac{P_1}{\rho_1} + g\Pi + \left.\partial\phi_1/\partial t\right|_{z=\Pi} = \frac{P_0}{\rho_1}$; $\frac{P_2}{\rho_2} + g\Pi + \frac{\partial\phi_2}{\partial t} = \frac{P_0}{\rho_2}$

(P_0 = far field pressure - no disturbance). Now $P_1 - P_2 = (P_1 - P_0) - (P_2 - P_0) = \sigma k^2\Pi$

$\Rightarrow \sigma k^2\Pi = g\Pi(\rho_2 - \rho_1) + \rho_2\left.\partial\phi_2/\partial t\right|_{z=\Pi} - \rho_1\left.\partial\phi_1/\partial t\right|_{z=\Pi}$ Now $\left.\partial\phi_1/\partial t\right|_{z=\Pi} = -i\omega.2A_1e^{-kD}\cosh(k\Pi+kD)\cdot\exp(ikx - i\omega t)$

and $\left.\partial\phi_2/\partial t\right|_{z=\Pi} = -i\omega B_2e^{-k\Pi}e^{i(kx-\omega t)}$. $\Rightarrow \left.\partial\phi_1/\partial t\right|_{z=\Pi} = -\frac{\omega^2}{k}\coth(k\Pi+kD)\Pi$

(using $\frac{\partial\Pi}{\partial t} \propto \left.\partial\phi_1/\partial z\right|_{z=\Pi} = \frac{+i}{\omega}.2A_1ke^{-kD}\sinh(k\Pi+kD)e^{i(kx-\omega t)}$), and $\left.\partial\phi_2/\partial t\right|_{z=\Pi} = \frac{\omega^2}{k}\Pi$

$\therefore \sigma k^2\cancel{\Pi} = g\cancel{\Pi}(\rho_2-\rho_1) + \rho_2\frac{\omega^2}{k}\cancel{\Pi} + \rho_1\frac{\omega^2}{k}\coth(k\Pi+kD)\cancel{\Pi}$ (Now assume $k\Pi \ll kD$)

$$\Rightarrow \omega^2 = \frac{\sigma k^3 + g(\rho_1-\rho_2)k}{\rho_2 + \rho_1\coth(kD)}$$

DISPERSION RELATION. (phase velocity of waves is ω/k, group velocity is $d\omega/dk$). ← can compute

← in general since $\omega/k = f(k)$ as is $d\omega/dk$. → dispersive waves.

* Special cases: (set $\rho_1 \gg \rho_2$)

i) σ is negligible

$\Rightarrow \omega^2 = gk\tanh(kD)$ if $D \ll 1$ (shallow water)

$\Rightarrow \omega^2 = gDk^2$ if $D\to\infty \Rightarrow \omega^2 = gk$ [$D \ll 1 \Rightarrow \tanh(kD) \propto kD$, $D\to\infty$ $\tanh(kD) \to 1$]

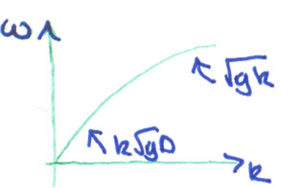
ii) σ not negligible, shallow water

$\omega^2 = \frac{\sigma k^4D}{\rho_1} + gk^2D$; and deep water: $\omega^2 = \frac{\sigma k^3}{\rho_1} + gk$

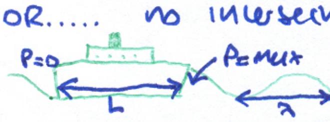
iii) σ not negligible, $\rho_1 \sim \rho_2$ shallow water $\omega^2 = \frac{\sigma k^4D + g(\rho_1-\rho_2)k^2D}{\rho_2kD + \rho_1}$

and for deep water $\omega^2 = \frac{\sigma k^3}{\rho_1+\rho_2} + \frac{gk(\rho_1-\rho_2)}{\rho_1+\rho_2}$

* Most practical purposes (i.e ships in sea, canals, lakes etc) - case (i).



* If ship is moving at speeds $> \sqrt{gD}$, no waves can be formed since steepest gradient of $\omega(k)$ is $\sqrt{gD}$. (corresponding to a group velocity or maximum phase velocity). → OR..... no intersection of line $\omega = v_{ship}k$ with curve on the left!



scenario for max drag occurs when $L = \frac{2m+1}{2}\lambda$

* Wave resistance occurs for $v_{ship} < \sqrt{gD}$

define $F_{drag} = C_W\frac{1}{2}\rho v^2A$ w/ C_W the drag coefficient. In this case A is the "wetted area". Now $C_W = f(F)$ where F = Froude #. $F = v/c_{wave}$ ← characteristic speed $= \frac{v}{\sqrt{gL}}$ where L is ship length.

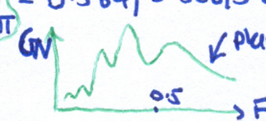
[use $\omega = \sqrt{gk} \Rightarrow c_{wave} \sim \frac{\omega}{k} = \sqrt{g/k}$. $k \sim \frac{1}{L}$]

$\Rightarrow L = (2m+1)\pi/k$ $(m = 0,1,2\ldots)$

Now if $v_{ship} = \sqrt{g}/\sqrt{k}$ (i.e creates deep water waves)

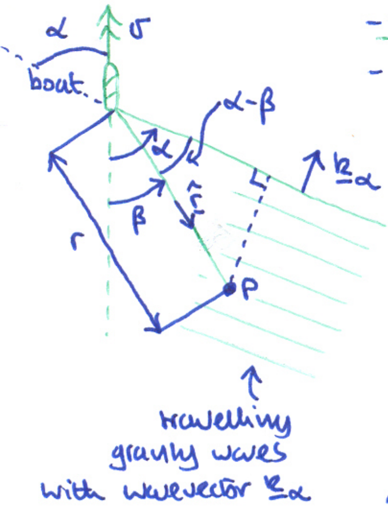
$\Rightarrow F = \frac{1}{\sqrt{(2m+1)\pi}} = 0.564, 0.326, 0.252\ldots$

- Explains peaks in C_W, F curve.



FLUIDS

* **Kelvin wedge** Consider a boat moving at uniform velocity v over the surface of a deep fluid obeying potential flow conditions. Waves generated by the movement of the ship will interfere with gravity waves already travelling along the fluid interface. Assume these waves obey the dispersion relation $\omega^2 = gk \Rightarrow$ phase velocity $c_p = \frac{\omega}{k} = \sqrt{g/k}$



- phase at P is $\phi_p = |\underline{r}\cdot\underline{k}_\alpha| = rk_\alpha \sin(\alpha-\beta)$
- Now since boat must also contribute to the k_α waves (can only feed energy into a wave fourier component to replace viscous losses if boat travels with crest / stern with trough – "surf riding condition")

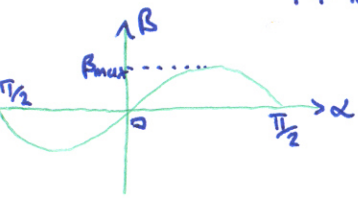
$$\Rightarrow \quad v\sin\alpha = c_p(k_\alpha) = \sqrt{\frac{g}{k_\alpha}} \quad \Rightarrow \quad k_\alpha = \frac{g}{v^2\sin^2\alpha}$$

Hence $\phi_p = \frac{rg}{v^2}\cdot\frac{\sin(\alpha-\beta)}{\sin^2\alpha}$. Now P represents a maximum disturbance (constructive interference) if $\partial\phi_p/\partial\alpha = 0$

$$\partial\phi_p/\partial\alpha = \left[\frac{\cos(\alpha-\beta)}{\sin^2\alpha} + \frac{\sin(\alpha-\beta)}{\sin^4\alpha}(-2\sin\alpha\cos\alpha)\right]\frac{rg}{v^2}$$

$$\therefore \partial\phi_p/\partial\alpha = 0 \Rightarrow \frac{2\sin(\alpha-\beta)\cos\alpha}{\sin\alpha} = \cos(\alpha-\beta) \Rightarrow 2\tan(\alpha-\beta) = \tan\alpha$$

Now $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$ $\Rightarrow 2\tan\alpha - 2\tan\beta = \tan\alpha + \tan^2\alpha\tan\beta \Rightarrow \tan\beta = \frac{\tan\alpha}{2+\tan^2\alpha}$



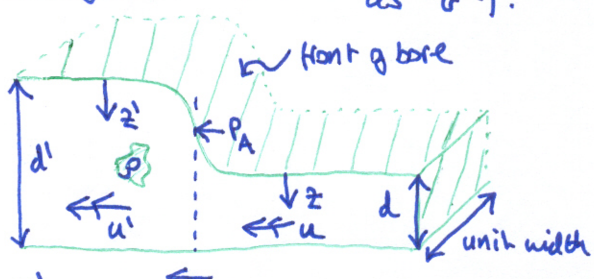
Now $\frac{1}{\cos^2\beta}\frac{\partial\beta}{\partial\alpha} = \frac{(2+\tan^2\alpha)\sec^2\alpha - 2\tan^2\alpha\sec^2\alpha}{(2+\tan^2\alpha)^2}$ $\therefore \beta_{max}$ when $\partial\beta/\partial\alpha = 0$

$\Rightarrow 2+\tan^2\alpha = 2\tan^2\alpha \Rightarrow \alpha = \tan^{-1}(\sqrt{2}) \approx \pm 54°$ (54.7 actually)

$\Rightarrow \tan\beta_{max} = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}} \Rightarrow \beta_{max} = \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) = \pm 19.5°$

Note wedge angle is independent of g, v .... (Though k_α scales as v^{-2}).

* **Bores and hydraulic jumps**
- Consider reference frame where front of bore (travelling 'step' disturbance as shown) is stationary.



→ by continuity: $d'u' = du$

(assume $u' < u$) → force $\parallel \hat{x}$: (/unit width of bore)

$\rightarrow P_A(d'-d) - \int_0^{d'} dz'(P_A + \rho g z') + \int_0^{d} dz(P_A+\rho g z) = F_x$ = rate of change of momentum / unit width ($=\dot{p}_x$)

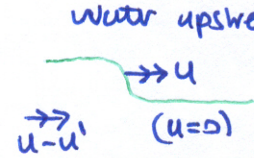
- momentum flux $\parallel \hat{x} = \dot{p} = \rho u'^2 d' - \rho u^2 d$

$\therefore \dot{p} = F_x \Rightarrow \rho u'^2 d' - \rho u^2 d = -\frac{1}{2}\rho g(d'^2 - d^2)$. From continuity $u' = du/d'$

$$\Rightarrow u^2 d = \frac{d' d^2 u^2}{d'^2} + \frac{g}{2}(d'^2 - d^2) \Rightarrow u^2\left(d - \frac{d^2}{d'}\right) = \frac{g}{2}(d'^2-d^2)$$

$$\Rightarrow u^2 = \frac{g}{2}\frac{(d'^2-d^2)}{\left(d-\frac{d^2}{d'}\right)} = \frac{gd'}{2d}\frac{(d'^2-d^2)}{(d'-d)} \Rightarrow u = \sqrt{\frac{gd'}{2d}(d+d')}$$

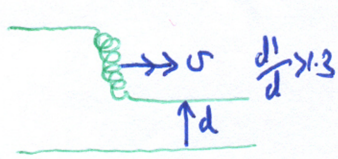
Now in frame with stationary water upstream



$\Rightarrow$ Bore speed $= u$. Now when $\frac{d'}{d} < 1.3$ [if water is shallow (i.e. $\omega^2 = gDk^2$ – $D \equiv d$ in this notation) $\Rightarrow c_w = \sqrt{gd}$

$\Rightarrow u = c_w\sqrt{\frac{d'}{d^2}\frac{(d+d')}{2}}$ so $u > c_w$] – get smooth profile with undulations

If $\frac{d'}{d} > 1.3$, bore sharpens then breaks. → turbulent foaming front.



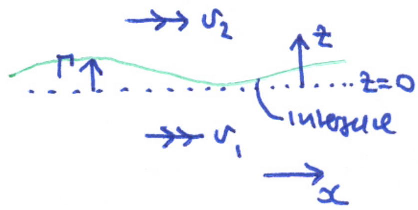
Note: "Stationary bores" (i.e. original ref. frame) are called "hydraulic jumps". c.f. step in flow i.e. tap meets sink, spill way at weir.

(Note if $d' \sim d$ then $u \sim \sqrt{gd}$ i.e. shallow water wave speed. Works nicely!)

8) Kelvin-Helmholtz instability

* consider two layers of Newtonian, uniform, incompressible and non viscous (⇒ potential flow) fluid moving at velocities v_1 and v_2 ∥ to the interface between them. (The fluids are the same).

* from previous analysis of waves in (potential) fluid – in deep water limit (fluid) – no surface tension



Potential flow everywhere apart from boundary/interface.

$$\phi_1 = v_1 x + A_1 e^{kz} e^{i(kx-\omega t)}$$
$$\phi_2 = v_2 x + A_2 e^{-kz} e^{i(kx-\omega t)}$$

[Here we add $v_{1,2}x$ term to potential]

Now as before $\partial\phi_1/\partial z|_{z=\Gamma} = \partial\phi_2/\partial z|_{z=\Gamma} \approx \frac{\partial\Gamma}{\partial t} + v_1\frac{\partial\Gamma}{\partial x}$ or $\frac{\partial\Gamma}{\partial t} + v_2\frac{\partial\Gamma}{\partial x}$ [$\partial\phi_{1,2}/\partial z|_{z=\Gamma} = \frac{D\Gamma}{Dt} = \frac{\partial\Gamma}{\partial t} + (v\cdot\nabla)\Gamma$ – ignore $O(A^2)$ terms]

Now before $\partial\phi_{1,2}/\partial z|_{z=\Gamma} = \pm kA_{1,2}e^{\pm k\Gamma}e^{i(kx-\omega t)} \Rightarrow \Gamma = \frac{\pm i}{\omega}kA_{1,2}e^{\pm k\Gamma}e^{i(kx-\omega t)}$

⇒ $\partial\phi_{1,2}/\partial z|_{z=\Gamma} = -i\omega\Gamma$. So in this case we expect $\partial\phi_{1,2}/\partial z|_{z=\Gamma} = -i\omega\Gamma + ik\Gamma v_{1,2}$

$\partial\phi_{1,2}/\partial z|_{z=\Gamma} = \partial\Gamma/\partial t$ only

Now $\partial\phi_{1,2}/\partial z|_{z=\Gamma} \approx \pm kA_{12}$ so, approximately: $\begin{cases} A_1 k = i\Gamma(v_1k-\omega) \\ -A_2k = i\Gamma(v_2k-\omega)\end{cases}$

$$\Rightarrow A_1/A_2 = -\frac{v_1k-\omega}{v_2k-\omega} \quad (1)$$

Now apply Bernoulli in regions 1, 2: (at interface)

$$\frac{P_0}{\rho} = \frac{P_1}{\rho} + g\Gamma + \frac{\partial\phi_1}{\partial t}\Big|_{z=\Gamma} + \frac12 v^2\Big|_{z=\Gamma} \; ; \; \frac{P_0}{\rho} = \frac{P_2}{\rho} + g\Gamma + \frac{\partial\phi_2}{\partial t}\Big|_{z=\Gamma} + \frac12 v^2\Big|_{z=\Gamma}$$

Now $v^2 = (v_{1,2} + \partial\phi_{1,2}/\partial x)^2$
$\approx v_{1,2}^2 + 2v_{1,2}\partial\phi_{1,2}/\partial x$
$\approx 2v_{1,2}\partial\phi_{1,2}/\partial x$
(neglect $O(A_{1,2}^2)$ terms).

Now $P_1 \approx P_2$ so $\frac{P_0-P_1}{\rho} - g\Gamma = \begin{cases} -i\omega A_1 e^{k\Gamma}e^{i(kx-\omega t)} + iv_1kA_1e^{k\Gamma}e^{i(kx-\omega t)} \\ -i\omega A_2 e^{-k\Gamma}e^{i(kx-\omega t)} + iv_2kA_2e^{k\Gamma}e^{i(kx-\omega t)}\end{cases}$

$\Rightarrow 1 = e^{2k\Gamma}\frac{A_1}{A_2}\left(\frac{v_1k-\omega}{v_2k-\omega}\right)$. Now $e^{2k\Gamma} \approx 1$

$$\Rightarrow A_1/A_2 = \frac{v_2k-\omega}{v_1k-\omega} \quad (2)$$

So, if (1) = (2) ⇒ $(v_1k-\omega)^2 = -(v_2k-\omega)^2$

$$\Rightarrow \frac{\omega}{k} = \frac12(v_1+v_2) \pm \frac{i}{2}(v_1-v_2)$$

Now since Γ varies as $e^{i(kx-\omega t)}$, $e^{-i\omega t} = \exp\left[-ik\frac{t}{2}(v_1+v_2) \mp i^2k\frac{t}{2}(v_1-v_2)\right]$

$= e^{\pm k\frac{t}{2}(v_1-v_2)}e^{-ik\frac{t}{2}(v_1+v_2)}$ → i.e. oscillatory term with frequency $\frac{k}{4\pi}(v_1+v_2)$ + exponential term. +ve "root" of this ⇒ increasing amplitude exponentially with time. i.e. Γ UNSTABLE.

9) Atmospheric lapse rate, Föhn (+ Bernoulli's theorem for compressible flow of gases)

* Assume atmosphere is – Isotropic, Newtonian fluid AND Adiabatic gas (⇒ $P \propto \rho^\gamma$; $\gamma = c_p/c_v$)
* consists of dry air * convects at rates s.t. inertial effects negligible * non viscous

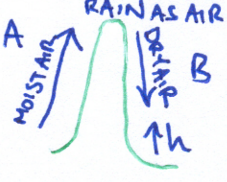
⇒ Bernoulli for compressible flow (steady state) $\frac{\gamma}{\gamma-1}\frac{P}{\rho} + gh + \frac12 v^2 = $ constant

$\frac{\gamma}{\gamma-1} = \frac{c_p}{c_p-c_v}$ $\quad c_p - c_v = R$ (ideal gas) $\quad P = \frac{\rho RT}{M_{mol}} \Rightarrow R = M_{mol}P/\rho T \quad \therefore \frac{\gamma}{\gamma-1} = c_p\rho T/M_{mol}P$

⇒ $\frac{c_pT}{M_{mol}} + gh = $ const. if ignore v^2. Now from Equipartition $c_p = \frac72 R$ ($c_v \approx \frac52 R$)

⇒ $\frac72\frac{RT}{M_{mol}} + gh = $ const ⇒ $\frac72\frac{R}{M_{mol}}\frac{dT}{dh} + g = 0 \Rightarrow \frac{dT}{dh} = \frac{-gM_{mol}}{R}\frac27$ LAPSE RATE (~10K/km)

Real result is ~6.5 K km^{-1}. – Due to water vapour in atmosphere. – Tends to condense as the air rises and cools. releases latent as it does so which reduces cooling effect. → Warm Föhn winds from Alps explained by this difference.



$\frac{dT}{dh}\big|_A \sim -6.5$ K/km

$\frac{dT}{dh}\big|_B \sim -10$ K/km