RELATIVITY AND ELECTRODYNAMICS

1) Experimental basis of Special Relativity (SR) – i.e. observed problems with classical theory.

* Michelson–Morley (1887) – light is a wave and ∴ needs a medium in which to travel. MM demonstrated that it was not possible to measure the relative motion of the earth and the "ether" – the postulated medium which constitutes "free space". (Seems rather a contradiction now!)
  – corollary: speed of light was observed to have the same speed in different inertial frames.
* Fizeau (1860), Muscart (1872) and Rayleigh (1902) failed to find effects of the earth's motion on the refractive indices of certain dielectrics. * Trouton and Noble (1903) failed to detect the effects of the expected tendency of a charged plate capacitor to face the "ether drift".

..... All these null results prompted Einstein's Relativity principle

AXIOM (i) "The outcome of any physical experiment is the same when performed with identical initial conditions relative to any inertial frame"

["relative to any inertial frame" means at constant velocity w.r.t a non accelerating and gravity free frame of reference]

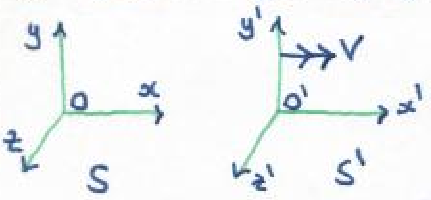
AXIOM (ii) "There exists an inertial frame in which light signals in vacuum always travel rectilinearly, at constant speed c, in all directions independently of the motion of the source"

Put (i), (ii) together and get Einstein's law of light propagation

"light signals in vacuum are propagated rectilinearly with the same speed c at all times, in all directions, in ALL INERTIAL FRAMES"

2) The structure of spacetime * Require four distinct coordinates to specify an EVENT in SPACETIME. * LORENTZ TRANSFORM relates these components between inertial frames. * We can group the 4 coordinates in objects called 4-vectors which are themselves independent of the frame chosen to evaluate their components. (i.e an extension of 3-vectors used to describe spatial position). ↙ i.e. cartesian. Rotating frames have acceleration! Polar frames are complicated. etc...

Define POSITION 4 vector $\underline{R} = (ct, x, y, z) \equiv (ct, \underline{r})$. Now define convention for cartesian geometry of inertial frames to align ∥ to $\hat{x}$ direction.



– let S, S' describe general pair of cartesian inertial frames.
[S' moves in +ve x direction with speed V relative to frame S]
– At t=0 [S frame time] origins O and O' coincide.

Using this geometry, can show S⇔S' transformations satisfy Einstein's law of light propagation when $\underline{R}' = \underline{\underline{L}}\,\underline{R}$. ($\underline{\underline{L}}$ = Lorentz transform matrix).

$$\underline{\underline{L}} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \beta = V/c \qquad \gamma = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}}$$

→ In fact define a 4-vector as an object that is frame independent and obeys the L.T. when transforming its components between frames. i.e $\underline{X}' = \underline{\underline{L}}\,\underline{X}$. $\underline{X}$ is any 4 vector. (′ ⇒ evaluate components in ′ frame).

* L.T. is linear so mathematical properties of 4 vectors are:

(i) Addition $\underline{A} = \underline{B} \pm \underline{C}$ [$\underline{A}$ is a 4 vector if $\underline{B}, \underline{C}$ are]
(ii) Commutation $\underline{A} + \underline{B} = \underline{B} + \underline{A}$ [$\underline{A}, \underline{B}$ 4 vectors]
(iii) Association $(\underline{A} + \underline{B}) + \underline{C} = \underline{A} + (\underline{B} + \underline{C})$
(iv) Scalar multiplication $\underline{B} = a\underline{A}$ ($\underline{B}$ is a 4 vector if a is a scalar, $\underline{A}$ is a 4 vector)

Define 4-vector scalar product between general 4-vectors $\underline{A}, \underline{B}$ as $\underline{A}\cdot\underline{B} = (\underline{\underline{g}}\,\underline{A})\cdot\underline{B}$

$\underline{\underline{g}}$ is the METRIC TENSOR. In Euclidean space where S.R. is valid $\underline{\underline{g}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

→ In General Relativity $\underline{\underline{g}}$ depends on mass distribution → "spacetime curvature". (In that case frames not inertial → acceleration, so L.T. not valid either...)

Now $\underline{A}\cdot\underline{B}$ is invariant under L.T. i.e $\underline{A}\cdot\underline{B} = \underline{A}'\cdot\underline{B}'$.

Proof: $\underline{A} = (A^0, A^1, A^2, A^3)$ $\underline{B} = (B^0, B^1, B^2, B^3)$ DEFINE

L.T. $\underline{A}' = (\gamma A^0 - \gamma\beta A^1, -\gamma\beta A^0 + \gamma A^1, A^2, A^3)$
$\underline{B}' = (\gamma B^0 - \gamma\beta B^1, -\gamma\beta B^0 + \gamma B^1, B^2, B^3)$ L.T.

∴ $\underline{A}\cdot\underline{B} = A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3$

and $\underline{A}'\cdot\underline{B}' = \gamma^2(A^0 - \beta A^1)(B^0 - \beta B^1) - \gamma^2(A^1 - \beta A^0)(B^1 - \beta B^0) - A^2B^2 - A^3B^3$
$= \gamma^2(1-\beta^2)(A^0B^0 - A^1B^1) - A^2B^2 - A^3B^3 = A^0B^0 - A^2B^2 - A^3B^3 - A^1B^1$

QED.

→ can also prove that if $\underline{A}\cdot\underline{B}$ is L.T. invariant and $\underline{A}$ is a 4 vector ⇒ $\underline{B}$ is a 4 vector.

3) other physically meaningful 4-vectors * 4-velocity. Start with 4-position $\underline{R} = (ct, \underline{r})$

consider element $d\underline{R}$ evaluated in frame where $\underline{r} = 0$. i.e. $d\underline{R}' = (c\,d\tau, 0,0,0)$ τ is "proper time"

Now since $d\underline{R}'$ is a 4 vector $d\underline{R}'.d\underline{R}' = c^2 d\tau^2$ is a L.T. invariant. i.e. can be used as a scalar. So if $c^2 d\tau^2$ is a scalar so is $d\tau$. $\therefore$ Form new 4-vector $\underline{U} = \frac{d\underline{R}}{d\tau}$ "4-velocity"

Now $\underline{\underline{L}}\, d\underline{R} = d\underline{R}'$ if S' frame is associated with rest frame of a particle moving with velocity $\underline{u}$ w.r.t S frame where $d\underline{R}$ evaluates its components in. Sensible to let $d\underline{R} = (c\,dt, u\,dt, 0, 0)$*

$\Rightarrow$ $d\tau = dt/\gamma_u$ (γ_u refers to particle speed u). Hence can define $\underline{U}$ with components from the same frame. $\underline{U} = \frac{d}{d\tau}\underline{R} = \gamma_u \frac{d}{dt}(ct, \underline{r}) = \gamma_u (c, \underline{u})$ 3 velocity.

*can be more elegant and general: $d\underline{R} = (c\,dt, \underline{u}\,dt)$ $d\underline{R}.d\underline{R}$ = invariant = $c^2 d\tau^2$

$\Rightarrow$ $d\tau^2 = dt^2 (1 - u^2/c^2)$ $\Rightarrow$ $d\tau = dt/\gamma_u$. Now $\underline{U}.\underline{U} = \gamma_u^2 (c^2 - u^2) = c^2$ [Proof that $\underline{U}$ is a 4 vector]

and since $\underline{U}' = \underline{\underline{L}}\,\underline{U}$ $\Rightarrow$

$$\left.\begin{aligned} \gamma_u' c &= \gamma_v(\gamma_u c - \beta \gamma_u u_x) \\ \gamma_u' u_x' &= \gamma_v(\gamma_u u_x - \beta\gamma_u c) \\ \gamma_u' u_y' &= \gamma_u u_y \\ \gamma_u' u_z' &= \gamma_u u_z \end{aligned}\right\}$$

$$u_x' = \frac{c u_x - \beta c^2}{c - \beta u_x} = \frac{u_x - V}{1 - u_x V/c^2}$$

$$u_y' = \frac{u_y}{\gamma_V (1 - u_x V/c^2)}$$

$$u_z' = \frac{u_z}{\gamma_V (1 - u_x V/c^2)}$$

[$\beta = V/c$ γ_V refers to speed V of S' frame w.r.t S frame. In general $\gamma_V \neq \gamma_u$].

Also: $\gamma_u' = \gamma_u \gamma_V \left(1 - \frac{V u_x}{c^2}\right)$

* 4-Acceleration Since $\underline{U}$ is a 4 vector, $d\tau$ is a scalar $\Rightarrow$ $\frac{d}{d\tau}\underline{U}$ is a 4-vector.

Define 4-acceleration $\underline{A} = \gamma_u \frac{d}{dt}(\gamma_u c, \gamma_u \underline{u}) = (\gamma_u c \dot\gamma_u, \gamma_u^2 \dot{\underline{u}} + \gamma_u \dot\gamma_u \underline{u})$ $\dot{\underline{u}}$ is 3-acceleration $\underline{a} = d\underline{u}/dt$

$\Rightarrow$ $\underline{A} = \gamma_u \dot\gamma_u \left(c, \frac{\gamma_u}{\dot\gamma_u}\underline{a} + \underline{u}\right)$ where $\dot\gamma_u = \frac{u\dot u/c^2}{(1 - u^2/c^2)^{3/2}}$. Now in "instantaneous rest frame" IRF, $\underline{u} = 0$ but $\underline{a}$ may not $= 0$.

$\therefore$ in IRF $\underline{A} = (0, \underline{\alpha})$ since $\gamma_u = 1, \dot\gamma_u = 0$.

$\underline{\alpha}$ is the "Proper acceleration". Note $\underline{A}.\underline{A} = -\alpha^2$ – invariant. $\alpha^2 = \gamma_u^4 a^2 + (\underline{a}.\underline{u})^2 \gamma^6/c^2$

$\Rightarrow$ $\alpha^2 = \gamma_u^4 a^2 + \gamma_u^2 \dot\gamma_u^2 u^2 + 2\gamma_u^3 \dot\gamma_u \underline{a}.\underline{u} - \gamma_u^2 \dot\gamma_u^2 c^2$ (useful identity) Note $\underline{a}.\underline{u} = u\dot u$

Note also $\underline{A}.\underline{U}$ evaluated in IRF $= (0, \underline{\alpha}).(c, \underline{0}) = 0$. So $\underline{A}.\underline{U} = 0$ in all frames (inertial).

* Frequency 4-vector

Consider plane wave of amplitude A, phase ϕ observed in some inertial frame S.

$A = |A| e^{i\phi}$ where $\phi = \omega t - \underline{k}.\underline{r}$ {frequency $\omega/2\pi$ wave-vector $\underline{k} = \frac{2\pi}{\lambda}\hat{\underline{k}}$}

Now phase is a Lorentz invariant - i.e. all observers will agree whether A is a max or min. Consider possible 4-vector $\underline{K} = (\omega/c, \underline{k})$. Now $\underline{K}.\underline{R} = \omega t - \underline{k}.\underline{r} = \phi$

Since $\underline{R}$ is a 4-vector and ϕ is invariant $\Rightarrow$ $\underline{K}$ is a 4-vector.

* 4-momentum - classical momentum $\underline{p}_{classical} = m\underline{u}$ is NOT conserved when one transforms between inertial frames using the L.T. $\therefore$ Need relativistic 4-momentum that has same low speed conservation properties as $\underline{p}_{classical}$. Now since mass is a scalar and $\underline{U}$ is a 4-vector $\Rightarrow$ $\underline{P} = m\underline{U}$ is also a 4-vector $\Rightarrow$ DEFINE 4-momentum $\underline{P} = (mc, m\underline{u})\gamma_u$

Now in limit $u \ll c$ $\gamma_u = (1 - u^2/c^2)^{-\frac{1}{2}} \simeq 1 + \frac{1}{2}u^2/c^2$ $\Rightarrow$ $\lim_{u \ll c} \underline{P} \simeq (mc + \frac{1}{2}mu^2/c, m\underline{u})$ ignore $u^2\underline{u}, O(u^3)$

$= \left(\frac{1}{c}(mc^2 + T), \underline{p}_{classical}\right)$ [T is classical kinetic energy]. Temporal part of $\lim_{u \ll c} \underline{P}$ is total energy which is conserved. (including new mc^2 term which must be conserved since mass must be for non gauge boson interactions). Spatial part is $\underline{p}_{classical}$ which is conserved. → So if E conserved ⇒ mass, energy relationship. We can infer from this that $\underline{P}$ is, in general, a conserved quantity if no external forces act on a system of particles. Write $\underline{P} = (E/c, \underline{p})$

where $E = \gamma_u mc^2$ (Total energy) $\underline{p} = \gamma_u m\underline{u}$ (3-momentum) Note presence of fields has been ignored. → see VECTOR POTENTIAL

- noting invariance of $\underline{P}.\underline{P}$, evaluating $\underline{P}.\underline{P}$ in zero 3-momentum frame, $\underline{P} = (mc, 0)$
$\Rightarrow$ Energy, momentum invariant. $E^2/c^2 - p^2 = m^2c^2 \Rightarrow E^2 - p^2c^2 = m^2c^4$

* The 4-Force. By analogy to 3-momentum classically we defined the 4-momentum. By considering $\lim_{u \ll c} \underline{P}$ we inferred $\underline{P}$ is conserved in absence of external forces. Now, classically, presence of 3-forces result in changes in $\underline{p}_{classical}$ via Newton's 2nd law. $\underline{f}_{classical} = \frac{d}{dt}\underline{p}_{classical}$.
$\rightarrow$ By analogy DEFINE 4-Force $\underline{F} = \frac{d}{d\tau}\underline{P}$ (which must be a 4-vector since $\underline{P}$ is and $d\tau$ is a scalar).
Now $\underline{F} = \gamma_u \frac{d}{dt}(E/c, \underline{p}) = (\gamma_u W/c, \gamma_u \underline{f})$ where W = power input dE/dt, $\underline{f}$ = 3-force $d\underline{p}/dt$
(Note in limit $u \ll c \Rightarrow \gamma_u \sim 1$ spatial part of $\underline{F}$ is indeed the 3-force $\underline{f}$. Now if $\gamma_u \sim 1$ and $\underline{p} = \gamma_u m \underline{u} \Rightarrow \underline{f} = m\dot{\underline{u}}$ i.e. Newton's 2nd law - if $\dot{m} = 0$). Now consider $\underline{F}.\underline{U}$ evaluated in IRF; $\underline{F} = m\underline{A}$ if $\dot{m} = 0$ $= (0, m\underline{\alpha})$ in IRF, $\underline{U} = (c, \underline{0})$ in IRF
assume this is correct in SR.
$\Rightarrow \underline{F}.\underline{U} = 0$ (i.e. like $\underline{A}.\underline{U} = 0$). $\therefore$ Since $\underline{F}.\underline{U}$ is invariant $\Rightarrow 0 = (\gamma_u W/c, \gamma_u \underline{f}).(\gamma_u c, \gamma_u \underline{u})$
$\Rightarrow W = \underline{f}.\underline{u}$ so power $= \underline{f}.\underline{u}$ in SR as in Newtonian mechanics.
(difference is $\underline{f} = \frac{d}{dt}(\gamma_u m \underline{u})$). $\Rightarrow$ we can $\therefore$ write $\underline{F}$ as $\underline{F} = (\frac{\gamma_u}{c}\underline{f}.\underline{u}, \gamma_u \underline{f})$
$\underline{F}$ boosts transforms as follows: $f_x' = \dfrac{f_x - \frac{\beta}{c}\underline{f}.\underline{u}}{1 - \beta u_x/c}$ $\quad f_y' = \dfrac{f_y}{\gamma_v(1 - \frac{\beta u_x}{c})}$ $\quad f_z' = \dfrac{f_z}{\gamma_v(1 - \frac{\beta u_x}{c})}$
(using result from $\underline{U}$ transform
$\gamma_v \gamma_u / \gamma_u' = (1 - \frac{\beta u_x}{c})^{-1}$). $\quad \therefore$ if $\underline{f} = (f_x, 0, 0) \Rightarrow f_x' = f_x$ i.e. force same in all inertial frames.
In IRF, $\underline{u} = \underline{0} \Rightarrow f_x' = f_x$; $f_y' = f_y/\gamma_v$; $f_z/\gamma_v = f_z'$ [useful - transform from IRF get simple results].
Now if in a frame S, $\underline{f}_1 = \underline{f}$; $\underline{f}_2 = -\underline{f}$ where
3 forces $\underline{f}_1, \underline{f}_2$ act on a body at points moving at velocities $\underline{u}_1, \underline{u}_2 \Rightarrow \underline{f}_1' = \underline{f}'$; $\underline{f}_2' = -\underline{f}'$ only if $\underline{u}_1 = \underline{u}_2$. Now definition of $\underline{F}$ is not that useful in itself since one needs to relate the position and time of a particle to the 3-forces acting on it. In the absence of external fields $\underline{f} = \frac{d\underline{p}}{dt} = \frac{d}{dt}(\gamma_u m \underline{u})$. $\Rightarrow \underline{f} = \gamma_u m \dot{\underline{u}} + \underline{u}\frac{d}{dt}(\gamma_u m)$
(in fact this result is general.)
$= \gamma_u m \underline{a} + \frac{\underline{u}}{c^2}\frac{dE}{dt}$ $(E = \gamma_u mc^2)$ $= \gamma m \underline{a} + \frac{\underline{f}.\underline{u}}{c^2}\underline{u}$. So if $\underline{f} \parallel \underline{u} \Rightarrow (\underline{f}.\underline{u})\underline{u}$
$= (\underline{f}.u\hat{\underline{f}})u\hat{\underline{f}} = fu^2\hat{\underline{f}} = u^2\underline{f}$
$\Rightarrow \underline{f} = \gamma^3 m \underline{a}$ $\quad$ if $\underline{f} \perp \underline{u} \Rightarrow \underline{f}.\underline{u} = 0 \Rightarrow \underline{f} = \gamma m \underline{a}$

4) Useful deductions from the above results * The status of IRF is deemed resistant to infinitessimal changes in 3-velocity. (So $\underline{A} = (0, \underline{\alpha}) \Rightarrow \underline{f} = m\underline{\alpha}$ since $\gamma_u = 1$ and $f_x' = f_x$, $f_y' = f_y/\gamma_v$, $f_z' = f_z/\gamma_v$ when transforming to another frame (inertial)).
Now since in IRF $u' : 0 \rightarrow du'$; some inertial frame $u : u \rightarrow u + du$
$\rightarrow$ turns out $du = du'/\gamma_u^2$ *. So work out dynamics in IRF (consider momentum, forces etc) - transform to some inertial frame using simple transforms above then integrate w.r.t. that frame. $\rightarrow$ Helps solve problems with rockets etc...
* Proof: use velocity transform (Assume u above $\equiv u_x$) $u_x' = \dfrac{u_x - V}{1 - u_x V/c^2}$
Now $u_x' = du'$ $\quad u_x = u + du$, $V = u$ (IRF still ok)
$\Rightarrow du' = \dfrac{du}{1 - \frac{u^2 + udu}{c^2}}$ $\quad$ Now $u^2 \gg udu \Rightarrow du' \propto \dfrac{du}{1 - u^2/c^2} \Rightarrow du' = \gamma_u^2 du$ QED.

* Time Dilation. Consider two inertial frames S and S' and two events which occur at the same place in S' separated by time interval $\Delta t'$. i.e. $\Delta \underline{R}' = (c\Delta t', 0, 0, 0)$
Now $\Delta \underline{R} = (c\Delta t, +V\Delta t, 0, 0)$ using L.T. $c\Delta t' = \gamma_v(c\Delta t - \beta V \Delta t) \Rightarrow \Delta t' = \gamma_v(1 - \frac{V^2}{c^2})\Delta t$
$\Rightarrow \Delta t' = \Delta t/\gamma_v$ so "MOVING CLOCKS RUN SLOW".

* **Length contraction** Consider a rod of length L_0 in its rest frame, moving at velocity $\beta c \parallel \hat{x}$ axis of frame S. In S let the measurement of the rod length be described by two temporally simultaneous events. $\underline{R}_{front} = (0, x_f, 0, 0)$ $\underline{R}_{back} = (0, x_b, 0, 0)$. Length of rod in S is clearly the $\hat{x}$ component of $\underline{R}_{front} - \underline{R}_{back} = x_f - x_b \equiv L$.

Now in S' $\underline{R}'_{front} - \underline{R}'_{back} = \begin{pmatrix} \gamma_v & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ L \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\gamma\beta L \\ \gamma L \\ 0 \\ 0 \end{pmatrix}$ Now $x'_f - x'_b = \gamma_v L = L_0$ $\Rightarrow L = L_0/\gamma_v$

$\Rightarrow$ In S frame we observe rod to be contracted by factor $\frac{1}{\gamma_v}$.

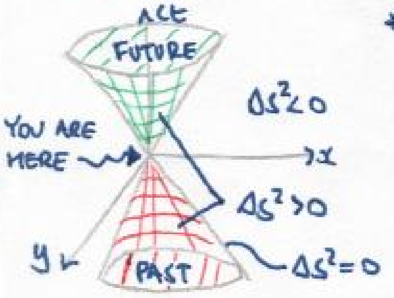
Note in S' frame though observe measurement events to differ by time $t'_f - t'_b = -\gamma\beta L/c = -L_0 v/c^2$ ($v = \beta c$).

* **Intervals and light cones** Consider two position 4 vectors (or events) $\underline{R}_1$ and $\underline{R}_2$. Define interval $\Delta \underline{S} = \underline{R}_2 - \underline{R}_1$. Clearly this must be a 4 vector. $\Rightarrow \Delta S^2 = \Delta \underline{S} \cdot \Delta \underline{S}$ is a Lorentz invariant. i.e. $\Delta S'^2 = \Delta S^2 \Rightarrow c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2) = c^2\Delta t'^2 - (\Delta x'^2 + \Delta y'^2 + \Delta z'^2)$

- For light $\Delta S^2 = 0$. If $\Delta S^2 = 0$ we say events are in "light cone" of each other.

* If $\Delta S^2 > 0 \Rightarrow$ interval is "time like" - inertial frame exists in which the two events 1,2 occur at the same place.

* If $\Delta S^2 < 0 \Rightarrow$ interval is "space like" - inertial frame exists where 1,2 events occur at the same place. $\rightarrow$ consider a slice through spacetime ($z=0$) - light cone illustrates above.



* **Aberration of light and the Relativistic Doppler Effect.**

Consider a frame S where a plane wave propagates with wave vector $\underline{k} = (k\cos\theta, k\sin\theta, 0)$ $\omega = v_{wave} k$

$\rightarrow$ Frequency 4-vector in S is $\therefore$

$\underline{K} = (\omega/c, k\cos\theta, k\sin\theta, 0) \therefore \underline{K}' = (\omega'/c, k'\cos\theta', k'\sin\theta', 0)$

for S' frame moving at speed $\beta c \parallel \hat{x}$ axis of S frame. Applying L.T. we find:

$\omega'/c = \gamma_v(\frac{\omega}{c} - \beta k\cos\theta)$; $k'\cos\theta' = \gamma_v(k\cos\theta - \beta\omega/c)$; $k'\sin\theta' = k\sin\theta$

Now for light $\omega/k = \omega'/k' = c \Rightarrow \sin\theta' = \frac{\sin\theta}{\gamma_v(1-\beta\cos\theta)}$; $\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}$;

$\tan\theta'/2 = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \tan\theta/2$ and $\omega'/\omega = \gamma(1-\beta\cos\theta)$ $\Leftarrow$ relativistic Doppler formula.

In classical limit $\gamma \to 1 \Rightarrow \omega'/\omega = 1 - \beta\cos\theta$ i.e. not much less/greater than 1 as expected.

[For other waves $v_{wave} \ll c$ this method is an elegant way of deriving Doppler formula. In this case $k \neq \omega/c$ but $\omega/v \Rightarrow \omega' = \gamma_v(\omega - V\omega/v \cos\theta) \Rightarrow \frac{\omega'}{\omega} = (1 - \frac{V}{v}\cos\theta)\gamma_v$].

* **4-Angular momentum** has no direct 3-vector analogy since $\underline{L} = \underline{r} \times \underline{p}$ - no cross product operation for 4-vectors. It can be formed by a combination of $\underline{R}$ and $\underline{P}$ 4-vectors though this relies on particular symmetry arguments.

* **Compton Scattering** is the elastic scattering of a photon from an initially stationary electron. Note 4-momentum of a photon $= (p, p, 0, 0)$ - since mass $= 0 \Rightarrow p = E/c$. $E = \hbar\omega \Rightarrow p = \hbar\omega/c$

Before: p, $\underline{P}_2 = (m_e c, 0, 0, 0)$, $\underline{P}_1 = (p, p, 0, 0)$

Want to find $p_3 - p = f(\theta)$

After: $\underline{P}_3 = (p_3, p_3\cos\theta, p_3\sin\theta, 0)$, θ, $\underline{P}_4$

Now $\underline{P}$ conservation $\Rightarrow \underline{P}_1 + \underline{P}_2 = \underline{P}_3 + \underline{P}_4$

Now $\underline{P}_2 \cdot \underline{P}_2 = \underline{P}_4 \cdot \underline{P}_4$ since collision elastic

$\rightarrow \underline{P}_2 \cdot \underline{P}_2 = m_e^2 c^2$

$\therefore m_e^2 c^2 = \underline{P}_4 \cdot \underline{P}_4$

$= (\underline{P}_1 + \underline{P}_2 - \underline{P}_3) \cdot (\underline{P}_1 + \underline{P}_2 - \underline{P}_3)$

Now $\underline{P}_1 \cdot \underline{P}_1 = \underline{P}_3 \cdot \underline{P}_3 = 0$

$\rightarrow m_e^2 c^2 = 2\underline{P}_1 \cdot \underline{P}_2 - 2\underline{P}_1 \cdot \underline{P}_3 - 2\underline{P}_2 \cdot \underline{P}_3 + m_e^2 c^2$

$\Rightarrow \underline{P}_1 \cdot \underline{P}_2 = \underline{P}_1 \cdot \underline{P}_3 + \underline{P}_2 \cdot \underline{P}_3$

$\therefore m_e c p = p p_3 - p p_3 \cos\theta + m_e c p_3$

$\Rightarrow (\div p p_3)$ $\frac{1}{p_3} - \frac{1}{p} = \frac{1}{m_e c}(1 - \cos\theta)$

$\Rightarrow \Delta\lambda = \lambda_3 - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$

5) Electrodynamics * The theory of electromagnetism describes the interaction of charged objects. It can be summarised by the following equations.

MAXWELL'S EQUATIONS

$\nabla \cdot \underline{D} = \rho_{free}$ — free charge

$\nabla \cdot \underline{B} = 0$

$\nabla \times \underline{E} = -\partial \underline{B}/\partial t$

$\nabla \times \underline{H} = \underline{J}_{free} + \partial \underline{D}/\partial t$ — free current density

CONSTITUTIVE RELATIONS

$\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$ — polarisation

$\underline{B} = \mu_0 \underline{H} + \mu_0 \underline{M}$ — magnetisation

$\underline{P} = \varepsilon_0 \underline{\chi_e} \underline{E}$

$\underline{M} = \underline{\chi_M} \underline{H}$ — electric and magnetic susceptibilities

$\underline{J}_{free} = \underline{\sigma} \underline{E}$ — conductivity

$\rho = \rho_{free} - \nabla \cdot \underline{P}$

$\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$ (Lorentz force on charge q moving in $\underline{E}, \underline{B}$ with velocity $\underline{v}$)

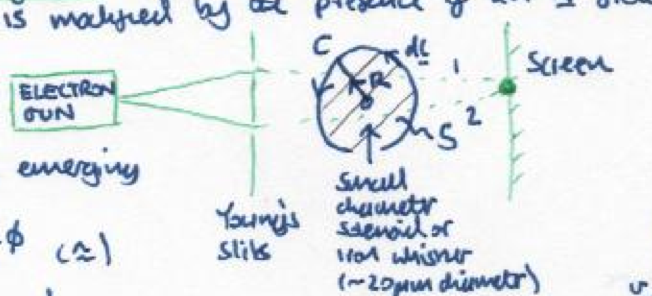
$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0$ (continuity of charge).

So effectively can write a set of equations involving only $\underline{E}, \underline{B}$ in terms of constants ε_0, μ_0 and problem specific parameters $\rho_{free}, \underline{\chi_e}, \underline{\chi_M}, \underline{\sigma}$ (could in general be functions of space and time). → can be more fundamental though i.e. like introducing 4-momentum ⇒ conservation of energy and 3-momentum. Need $\underline{E}, \underline{B}$ as functions of known things!

* Vector potential $\underline{A}$ Since $\nabla \cdot \underline{B} = 0$ if $\underline{B} = \nabla \times \underline{A}$, $\underline{A}$ will automatically satisfy $\nabla \cdot \underline{B} = 0$ ∴ take this as the definition of $\underline{A}$. Now consider $\underline{A}' = \underline{A} + \nabla \psi$ (ψ some scalar function of position). $\nabla \times \underline{A}' = \nabla \times \underline{A}$ ∴ $\underline{A} \to \underline{A}'$ does not alter physical meaning of $\underline{A}$. This is called a GAUGE TRANSFORM. i.e. $\nabla \psi$ is arbitrary. Now $\nabla \cdot \underline{A}' = \nabla \cdot \underline{A} + \nabla^2 \psi$. Since $\nabla^2 \psi$ is arbitrary ⇒ free choice of $\nabla \cdot \underline{A}$

Real existence of $\underline{A}$ is demonstrated by the Aharonov-Bohm effect. i.e. quantum mechanical phase of a particle is modified by the presence of an $\underline{A}$ field.

Experimental setup



$\underline{B}$ field inside solenoid points out of the page. No $\underline{B}$ outside solenoid (i.e. where path C is drawn intersecting with the electron trajectories 1, 2).

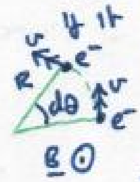
W.F. of electron emerging from gun is

$\psi(t) = \psi(0) e^{i\phi}$ (≈)

$\phi = -\frac{Et}{\hbar}$ (phase)

$\therefore d\phi = -\frac{E\,dt}{\hbar}$

Now electron moves in circular trajectories if it were in the presence of a $\underline{B}$ field.



centripetal acceleration (or centrifugal force in frame of stationary electron) = Lorentz force to maintain eq.

$\Rightarrow m_e a = evB \Rightarrow \frac{m_e v^2}{R} = evB$ ← circular motion.

$\therefore m_e v = ReB = p$ (momentum).

Now $dt = \frac{R\,d\theta}{v} = \frac{mR\,d\theta}{p}$

and $E = \frac{p^2}{2m}$ classically ($v \ll c$)

$\therefore d\phi = -\frac{eBR^2\,d\theta}{2\hbar}$ Now $\frac{1}{2}R^2 d\theta = dS$

so $d\phi = -\frac{e}{\hbar} B\,dS \Rightarrow \phi = -\frac{e}{\hbar}\iint_S \underline{B} \cdot d\underline{S} \xrightarrow{\text{Stokes}} \Rightarrow \phi = -\frac{e}{\hbar}\oint_C \underline{A} \cdot d\underline{l}$

{ we will gloss over fact that $\underline{B} = 0$ outside solenoid ...... }

- infact the actual result for ϕ is $\phi = -\frac{e}{\hbar}\int \underline{A} \cdot d\underline{l}$ since the electron does not in this case undergo a loop. [Correct way of showing this (Gasiorowicz pp 220-232) uses time independent Schrodinger equation with $\underline{A}$ included in Hamiltonian].

∴ Phase shift between beams 1, 2 is $\Delta\phi = -\frac{e}{\hbar}\int_1 \underline{A} \cdot d\underline{l} + \frac{e}{\hbar}\int_2 \underline{A} \cdot d\underline{l} = \frac{e}{\hbar}\oint \underline{A} \cdot d\underline{l} = \frac{e}{\hbar}\iint_S \underline{B} \cdot d\underline{S}$

$= \frac{e}{\hbar} \times$ magnetic flux through solenoid. So electrons interact with $\underline{A}$ set up by $\underline{B}$ within solenoid - observable result is change of interference pattern on screen.

* Maxwell's equations in terms of $\underline{A}, \phi$ Define $\underline{A}$ by $\underline{B} = \nabla \times \underline{A}$ and $\nabla \cdot \underline{A}$ = gauge choice

Now $\nabla \times \underline{E} = -\partial \underline{B}/\partial t = -\frac{\partial}{\partial t}\nabla \times \underline{A} = -\nabla \times \dot{\underline{A}} \Rightarrow \underline{E} = -\dot{\underline{A}} +$ "constant" ("constant" means - time independent - or should vanish when we tune out)

⇒ let "constant" $= -\nabla\phi$ where ϕ is a scalar function $\phi(\underline{r}, t)$. → scalar potential in electrostatics.

$\therefore \underline{E} = -\frac{\partial}{\partial t}\underline{A} - \nabla\phi$. Now $\nabla \times \underline{H} = \underline{J}_{free} + \partial \underline{D}/\partial t \Rightarrow \frac{1}{\mu_0}\nabla \times [(1 + \underline{\chi}_M)^{-1}\underline{B}] = \underline{\sigma}\underline{E} + \varepsilon_0 \frac{\partial}{\partial t}(1 + \underline{\chi}_e)\underline{E}$

Now let's consider a uniform medium with $\mu = 1 + \underline{\chi}_M$, $\varepsilon = 1 + \underline{\chi}_e$ and free current density $\underline{j} = \underline{\sigma}\underline{E} \Rightarrow \nabla \times \underline{B} = \mu\mu_0 \underline{j} + \varepsilon\varepsilon_0\mu\mu_0 \dot{\underline{E}}$. Replace $\underline{B}, \underline{E}$ by our new definitions in terms of $\underline{A}, \phi$: $\Rightarrow \nabla \times \nabla \times \underline{A} = \mu\mu_0 \underline{j} - \varepsilon\varepsilon_0\mu\mu_0(\ddot{\underline{A}} + \nabla\dot{\phi})$. Now $\nabla \times \nabla \times \underline{A} = \nabla(\nabla \cdot \underline{A}) - \nabla^2 \underline{A}$

$\Rightarrow \nabla(\nabla \cdot \underline{A} + \varepsilon\varepsilon_0\mu\mu_0\dot{\phi}) - \nabla^2\underline{A} = \mu\mu_0\underline{j} - \varepsilon\varepsilon_0\mu\mu_0\ddot{\underline{A}}$. Define gauge for $\underline{A}$ as $\nabla \cdot \underline{A} = -\varepsilon\varepsilon_0\mu\mu_0\dot{\phi}$

$\equiv -\dot{\phi}/c^2$ where c = speed of light in medium. $\Rightarrow \nabla^2\underline{A} - \frac{\ddot{\underline{A}}}{c^2} = -\mu\mu_0\underline{j}$ WAVE EQUATION + SOURCE TERM

Now since $\nabla\cdot\underline{D} = \rho_{free} \Rightarrow \nabla\cdot\underline{E} = \rho_{free}/\epsilon\epsilon_0 \Rightarrow -\frac{\partial}{\partial t}\nabla\cdot\underline{A} - \nabla^2\phi = \rho_{free}/\epsilon\epsilon_0$. Now since $\nabla\cdot\underline{A}$ is chosen to be $-\dot{\phi}/c^2$

$\Rightarrow \nabla^2\phi - \ddot{\phi}/c^2 = -\rho_{free}/\epsilon\epsilon_0$ WAVE EQUATION FOR ϕ. Now wave equation has general solution in terms of functions of the form $f(t - r/c)$ for outgoing waves.

Volume / element

$\therefore$ since ϕ for uniform medium will be spherically symmetric

$\Rightarrow \phi$ will take form $\phi(\underline{r},t) = \frac{1}{r} g(t - r/c)$. Now consider point charge $\rho_f(\underline{r},t)dV$ at location $\underline{r}_0$. i.e. $\rho_f(\underline{r},t)dV = \rho_f(t, \underline{r}-\underline{r}_0)\delta(\underline{r}-\underline{r}_0)dV$ (Note convenient variable change for ρ_f).

Now near $\underline{r} = \underline{r}_0$ $\ddot{\phi} \approx 0$ since $|\underline{r}-\underline{r}_0| \ll \lambda$. $\therefore \nabla^2\phi \approx -\frac{\rho_f(\underline{r}-\underline{r}_0,t)\delta(\underline{r}-\underline{r}_0)dV}{\epsilon\epsilon_0}$. Integrating over all space we find

$\iiint_{\text{all space}} \nabla\cdot\nabla\phi \, d\tau = -\frac{\rho_f(\underline{r}_0,t)dV}{\epsilon\epsilon_0}$. Now $\iiint \nabla\cdot\nabla\phi \, d\tau = \iint_S \nabla\phi\cdot d\underline{S}$ where, for convenience we take S to be the surface of a sphere of radius $|\underline{r}-\underline{r}_0|$ centred on $\underline{r}-\underline{r}_0$

let $R = |\underline{r}-\underline{r}_0|$. Define variable x to measure radial length from $\underline{r}-\underline{r}_0$ to S.

$\therefore \nabla\phi\cdot d\underline{S} = \frac{\partial\phi}{\partial x}\Big|_{x=R} dS$ clearly $\phi = \phi(x) \Rightarrow \iint_S \nabla\phi\cdot d\underline{S} = \frac{\partial\phi}{\partial x}\Big|_{x=R} 4\pi R^2$. Now since (ignore time dependence of ϕ here)

R is clearly a variable also $\frac{\partial\phi}{\partial x}\Big|_{x=R} = \frac{\partial\phi}{\partial R}$, writing $\phi = \phi(R)$ now

$\Rightarrow \frac{\partial\phi(R)}{\partial R} = -\frac{\rho_f(\underline{r}_0,t)dV}{\epsilon\epsilon_0}\frac{1}{4\pi R^2} \Rightarrow \phi(R) = \frac{\rho_f(\underline{r}_0,t)dV}{4\pi\epsilon\epsilon_0 R}$ (Assume $\phi \to 0$ as $R \to \infty$ so no integration constant).

Now noting general form of $\phi(\underline{r},t) \Rightarrow$ replace t by $t - |\underline{r}-\underline{r}_0|/c$ to get result (distance "r" $\equiv R$ from source $\underline{r}_0$)

for $\phi(R) \Rightarrow \phi(\underline{r},t) = \frac{\rho_f(\underline{r}_0, t - |\underline{r}-\underline{r}_0|/c)dV}{4\pi\epsilon\epsilon_0|\underline{r}-\underline{r}_0|}$

I can manipulate ϕ functional dependence since $\underline{r}_0$ is a constant at this stage, $\underline{r}$ is the variable

Now since the wave equations are linear one can find a general $\phi(\underline{r},t)$ by integrating (linear superposition) over all sources. Let $dV = d^3\underline{r}_0$ for clarity of integration variable.

$$\Rightarrow \phi(\underline{r},t) = \frac{1}{4\pi\epsilon\epsilon_0}\int_{\text{all space}} \frac{\rho_f(\underline{r}_0, t - |\underline{r}-\underline{r}_0|/c)}{|\underline{r}-\underline{r}_0|} d^3\underline{r}_0 \quad \text{"Retarded potential"}$$

Now noting very similar wave equation for $\underline{A} \Rightarrow \underline{A}(\underline{r},t) = \frac{\mu\mu_0}{4\pi}\int_{\text{all space}} \frac{\underline{j}(\underline{r}_0, t - |\underline{r}-\underline{r}_0|/c)\, d^3\underline{r}_0}{|\underline{r}-\underline{r}_0|}$

Note convention to write $\underline{j}$ or $\rho_f(\underline{r}_0, t - |\underline{r}-\underline{r}_0|/c)$ as $[\underline{j}]$ or $[\rho_f]$.

* 4-vector formulation of electromagnetism. Postulate 4-vectors for 4-potential and 4-current density ${}^4\underline{A} = (\phi/c, \underline{A})$ $\underline{S} = (c\rho_f, \underline{j})$. ($\underline{j}$ = free current density.) Consider operator $\underline{\Box} = (\frac{1}{c}\frac{\partial}{\partial t}, -\nabla)$ (frame S)

Now in frame S' $\frac{1}{c}\frac{\partial}{\partial t'} = \frac{1}{c}\left(\frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial t'}\frac{\partial}{\partial z}\right)$ by chain rule.

Now noting L.T. of $\underline{R}$ $\frac{\partial t}{\partial t'} = \gamma_v$ $\frac{\partial x}{\partial t'} = \gamma_v V$ $\frac{\partial y}{\partial t'} = \frac{\partial z}{\partial t'} = 0 \Rightarrow \frac{1}{c}\frac{\partial}{\partial t'} = \gamma_v\left(\frac{1}{c}\frac{\partial}{\partial t}\right) - \gamma_v\beta\left(-\frac{\partial}{\partial x}\right)$

By similar means can show $-\frac{\partial}{\partial x'} = \gamma_v\left(-\frac{\partial}{\partial x}\right) - \gamma_v\beta\left(\frac{1}{c}\frac{\partial}{\partial t}\right)$ and $-\frac{\partial}{\partial y'} = -\frac{\partial}{\partial y}$ $-\frac{\partial}{\partial z'} = -\frac{\partial}{\partial z}$

$\therefore \underline{\Box}\cdot\underline{\Box} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ (D'Alambertian) and $\underline{\Box}'\cdot\underline{\Box}' = \left(\gamma_v\frac{1}{c}\frac{\partial}{\partial t} + \gamma_v\beta\frac{\partial}{\partial x}\right)^2 - \left(\gamma_v\frac{\partial}{\partial x} + \gamma_v\beta/c\frac{\partial}{\partial t}\right)^2 - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

$= \frac{\gamma_v^2}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\gamma_v^2\beta^2}{c^2}\frac{\partial^2}{\partial t^2} + \gamma_v^2\beta^2\frac{\partial^2}{\partial x^2} - \gamma_v^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2}(1-\beta^2)\gamma_v^2\frac{\partial^2}{\partial t^2} - \gamma_v^2(1-\beta^2)\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$. So under LT $\Rightarrow \underline{\Box}\cdot\underline{\Box} = \underline{\Box}'\cdot\underline{\Box}'$ i.e. $\underline{\Box}$ must be a 4-vector since $\underline{\Box}\cdot\underline{\Box}$ is invariant.

Now consider $\underline{\Box}\cdot\underline{S} = \frac{\partial\rho_f}{\partial t} + \nabla\cdot\underline{j}$ if this is a Lorentz invariant then $\underline{S}$ is a 4-vector.
NOTE CHARGE IS LORENTZ INVARIANT – EXPERIMENTAL FACT. Consider microscopic charge Q

in rest frame of charge $\frac{\partial\rho_f}{\partial t} + \nabla\cdot\underline{j} = 0$ CONTINUITY EQUATION. occupying volume V_0 in IRF of charge. In this frame $\rho_f' = Q/V_0$. In general frame charge moves at speed $u \parallel \underline{S}$ axis of frame and has volume Lorentz contracted to $V = V_0/\gamma_u$ $\therefore \rho_f = \gamma_u\rho_f'$

Now $\underline{j} = \rho_f\underline{u} = \gamma_u\rho_f' \, d\underline{r}/dt$ Now $d\tau = dt/\gamma_u \Rightarrow \underline{j} = \rho_f'\frac{d\underline{r}}{d\tau}$ $\therefore \underline{S} = (c\gamma_u\rho_f', \rho_f'\frac{d\underline{r}}{d\tau})$

$\Rightarrow \underline{J} = \left(c\gamma_u \rho_f' \frac{dt}{d\tau}, \rho_f' \frac{d\underline{r}}{d\tau}\right) = \rho_f' \frac{d}{d\tau}(ct, \underline{r}) = \rho_f' \frac{d\underline{R}}{d\tau} = \rho_f' \underline{U}$. Now ρ_f' is defined only in IRF of charge so invariant. $\underline{U}$ is a 4 vector $\Rightarrow \underline{J}$ is a 4 vector.

Now consider $\Box^2\,{}^4\underline{A} = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\begin{pmatrix}\phi/c \\ \underline{A}\end{pmatrix} = \left(\ddot{\phi}/c^3 - \frac{\nabla^2\phi}{c}, \frac{1}{c^2}\ddot{\underline{A}} - \nabla^2\underline{A}\right)$

Now Maxwell's equations $\Rightarrow \nabla^2\phi - \ddot{\phi}/c^2 = -\rho_f/\varepsilon\varepsilon_0$ and $\nabla^2\underline{A} - \frac{1}{c^2}\ddot{\underline{A}} = -\mu\mu_0\underline{j}$

$\therefore -\frac{\nabla^2\phi}{c} + \ddot{\phi}/c^3 = \frac{\rho_f}{c\varepsilon\varepsilon_0} = \mu\mu_0 c\rho_f$ and $\frac{1}{c^2}\ddot{\underline{A}} - \nabla^2\underline{A} = \mu\mu_0\underline{j}$

$\Rightarrow \Box^2\,{}^4\underline{A} = \mu\mu_0(c\rho_f, \underline{j}) = \mu\mu_0\underline{J}$. This proves existence of ${}^4\underline{A}$ and describes EM in 1 equation!

Now $\Box\cdot\underline{A}$ = invariant since ${}^4\underline{A}$ is a 4 vector. $\Box\cdot{}^4\underline{A} = \dot{\phi}/c + \nabla\cdot\underline{A} \Rightarrow \nabla\cdot\underline{A} = -\dot{\phi}/c$ if this invariant $= 0$. i.e. gauge choice.

$\Box = \left(\frac{1}{c}\partial/\partial t, -\nabla\right)$

So EM summarised by $\Box^2\underline{A} = \mu\mu_0\underline{J}$ with ${}^4\underline{A} = (\phi/c, \underline{A})$ and $\underline{J} = (c\rho_f, \underline{j})$

* $\underline{E}, \underline{B}$ Lorentz transforms. By considering L.T. of ${}^4\underline{A}$ we can arrive at the following results

$E_x' = E_x$ $\quad B_x' = B_x$

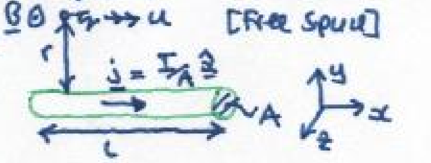
$E_y' = \gamma_v(E_y - vB_z)$ $\quad B_y' = \gamma_v\left(B_y + \frac{v}{c^2}E_z\right)$

$E_z' = \gamma_v(E_z + vB_y)$ $\quad B_z' = \gamma_v\left(B_z - \frac{v}{c^2}E_y\right)$

$\Rightarrow \underline{E}, \underline{B}$ fields are manifestations of the same relativistic effect - they often interchange with inertial frame choice.

Example ① Frame S. [Free space]

charge q at velocity $\underline{u} = u\hat{\underline{x}}$ r away from current carrying wire



in S $\underline{J} = (0, I/A, 0, 0)$ [No free charge seen by q].

Ampère $\Rightarrow B(r) = \frac{\mu_0 I}{2\pi r} \Rightarrow \underline{B}_q = \frac{\mu_0 I}{2\pi y}\hat{\underline{z}}$

$\therefore$ Lorentz force felt by q is $\underline{f} = q\underline{u}\times\underline{B} = -quB\hat{\underline{y}}$. Now in rest frame of charge S' $\underline{J}' = (-\gamma_u\beta I/A, \gamma_u I/A, 0, 0)$

$\Rightarrow c\rho_f' = -\gamma_u\beta I/A$ $\therefore$ charge in wire $Q_w = -\gamma_u u I L/c^2$. Now by Gauss

$\Rightarrow \int_{\text{cylinder radius } r} \nabla\cdot\underline{E}\, d\tau = Q_w/\varepsilon_0$. Clearly $\underline{E} = E(r)$

So $Q_w/\varepsilon_0 = \int \frac{dE}{dr} r\,d\theta\,dr\,L = 2\pi r L\int dE = E\cdot 2\pi r L$

[r turns out to be a constant so no need for dummy variable] $\Rightarrow \underline{E}(r=y) = \frac{Q_w}{2\pi y L\varepsilon_0} = \frac{-\gamma_u u I L}{2\pi y L c^2\varepsilon_0}$

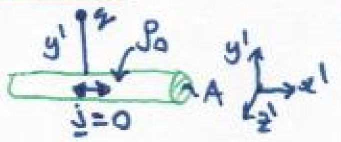
$= \frac{-\mu_0\gamma_u u I}{2\pi y} = -\gamma_u u B$

$\therefore$ Lorentz force $\underline{f}' = -\gamma_u u q B\hat{\underline{y}}$. So "$\underline{B}$ force" $\leftrightarrow$ "$\underline{E}$ force"

Now since $\underline{u} = 0$ (IRF) $\Rightarrow \underline{f} = \underline{f}'/\gamma_u = -uqB\hat{\underline{y}}$ and $\underline{f} \parallel \hat{\underline{y}}$ as calculated above.

Example ②

S'



$\underline{J}' = (c\rho_0, 0, 0, 0)$ i.e. $\rho_f' = \rho_0$, $\underline{j}' = 0$

Let S be frame moving at $-v$ in $\hat{\underline{x}}$ direction $\therefore \underline{J} = \begin{pmatrix}\gamma_v & \gamma_v v/c & 0 & 0 \\ \gamma_v v/c & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}c\rho_0 \\ 0 \\ 0 \\ 0\end{pmatrix}$

$= (\gamma_v c\rho_0, \gamma_v v\rho_0, 0, 0) \Rightarrow \rho_f = \gamma_v\rho_0$, $\underline{j} = \gamma_v v\rho_0\hat{\underline{x}}$

So $\underline{B}(r=y) = \frac{\mu_0}{2\pi y}\gamma_v v\rho_0 A\hat{\underline{z}}$ Ampère

$\underline{E}(r=y) = \frac{Q_w}{2\pi y L\varepsilon_0} = \frac{\gamma_v\rho_0 A}{2\pi y\varepsilon_0}$ Gauss

$\therefore \frac{|qv\hat{\underline{x}}\times\underline{B}|}{|q\underline{E}|}$ (= ratio of electric and magnetic forces)

$= \frac{vB_z}{E_y} = v^2\varepsilon_0\mu_0 = \frac{v^2}{c^2} = \beta^2$. Now typical drift velocity $\sim 10^{-3}\,\text{ms}^{-1}$

$\Rightarrow vB_z/E_y \sim 10^{-23}$ i.e. $\beta \sim 10^{-11}$. Now conductors are neutral to one part in 10^{23} (hence $\rho_f = 0$ in example 1 S frame) $\Rightarrow$ why we see magnetic effects.

* Energy density and Poynting vector. A region of free space containing ISOTROPIC EM radiation of mean energy density U has no net Poynting flux $\underline{N} = \underline{E}\times\underline{B}/\mu_0$ i.e. $B_iE_j = B_jE_i$ $i \neq j$

Now $U = \frac{1}{2}\underline{E}\cdot\underline{D} + \frac{1}{2}\underline{B}\cdot\underline{H} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}B^2/\mu_0$. Using $\underline{E}, \underline{B}$ transforms and $B_iE_j = B_jE_i$ where ' frame is one moving with velocity $v\hat{\underline{x}}$ relative to original.

$\Rightarrow U' = \frac{4\gamma_v^2 - 1}{3}U$

$\Rightarrow$ Net Poynting flux in $-\hat{\underline{x}}'$ direction. $N_x' = \frac{1}{\mu_0}(E_y'B_z' - E_z'B_y') \xrightarrow{\underline{E}\,\underline{B}\text{ transforms}} = -\frac{4\gamma_v^2 vU}{3}$

$\Rightarrow \underline{N}' = \left(-\frac{4\gamma_v^2 vU}{3}, 0, 0\right)$

6) Radiation. For a system to radiate electromagnetic energy we expect a non-zero Poynting Flux. Since static charges do not manifest $\underline{B}$ fields and uniformly moving charges, though exhibit $\underline{E}$ and $\underline{B}$ fields, cannot radiate since we can always find a inertial frame where the charge is stationary, (Note – analysis on E, B fields due to a moving charge $\Rightarrow \underline{N}$ points in direction of motion i.e. can be associated with energy of the flow of charge!) $\Rightarrow$ accelerated charges only can radiate.

* Hertzian Dipole consider small oscillating dipole $\underline{p} = q\underline{d}$ where $|\underline{d}| \ll \lambda_{oscillation} = \frac{2\pi c}{\omega}$ (if oscillation generates EM radiation which it does).

Vector potential $\underline{A}(\underline{r},t) = \frac{\mu_0}{4\pi}\int_{dipole} \frac{[\underline{j}]dV}{r}$

Magnitude of A is

$$\frac{\mu_0}{4\pi r}\int_{dipole}[j]dV = \frac{\mu_0}{4\pi r}[I]d = \frac{\mu_0}{4\pi r}[\dot{p}]$$

Note $[\underline{j}] = (\underline{j} - \frac{|\underline{r}|}{c})$ $\rightarrow$ No $\underline{r}_0$ since one dipole $\Rightarrow r$ is a constant.

Note $\dot{q}d = Id = \dot{p}$

So $A = \frac{\mu_0}{4\pi r}[\dot{p}]$ and $A_r = A\cos\theta$, $A_\theta = -A\sin\theta$, $A_\phi = 0$ (ϕ symmetry in s. polars).

Now $\underline{B} = \nabla\times\underline{A}$ $\rightarrow$ spherical polars $\Rightarrow$ $\underline{B} = (0,0,-\sin\theta\frac{\partial A}{\partial r}) = (0,0,-\frac{\mu_0}{4\pi}\sin\theta\frac{\partial}{\partial r}(\frac{[\dot{p}]}{r}))$

Now $\frac{\partial}{\partial r}[X] = -\frac{1}{c}[\dot{X}]$ and $\frac{\partial}{\partial r}[x] = [\dot{x}]$ where $[X] = X(t - r/c)$ for any $X(\ ,t)$

Proof: let $u = t - r/c$. $\therefore \frac{\partial}{\partial r}[X] = \frac{\partial}{\partial r}X(t - r/c) = \frac{\partial X(u)}{\partial u}\frac{\partial u}{\partial r} = -\frac{1}{c}\frac{\partial X(u)}{\partial u} = -\frac{1}{c}\dot{X}(t - r/c) = -\frac{1}{c}[\dot{X}]$

$\Rightarrow$ for $\frac{\partial}{\partial t}[X]$ the result is clearly $[\dot{X}]$.

$\therefore \frac{\partial}{\partial r}\frac{[\dot{p}]}{r} = -\frac{[\ddot{p}]}{rc} - \frac{[\dot{p}]}{r^2}$ $\Rightarrow$ $\underline{B} = (0,0,\frac{\mu_0}{4\pi}\sin\theta(\frac{[\ddot{p}]}{rc} + \frac{[\dot{p}]}{r^2}))$ $\int[\dot{X}]dt = [X]$ etc.

Now $\underline{E} = -\dot{\underline{A}} - \nabla\phi$ Get ϕ from Lorentz Gauge i.e. $\nabla\cdot\underline{A} = -\frac{1}{c^2}\dot{\phi}$ $\Rightarrow$ $\phi = \frac{\cos\theta}{4\pi\epsilon_0}\left\{\frac{[p]}{r^2} + \frac{[\dot{p}]}{rc}\right\}$

$$\Rightarrow \underline{E} = \left(\frac{2\cos\theta}{4\pi\epsilon_0}\left\{\frac{[p]}{r^3} + \frac{[\dot{p}]}{r^2c}\right\}, \frac{\sin\theta}{4\pi\epsilon_0}\left\{\frac{[p]}{r^3} + \frac{[\dot{p}]}{r^2c} + \frac{[\ddot{p}]}{rc^2}\right\}, 0\right)$$

assume $E\times B \perp \hat{r}$ – it does for $\frac{1}{r}$ terms.

Now only $\frac{1}{r}$ terms $\rightarrow$ radiation since if $|\underline{E}\times\underline{B}| \propto \frac{1}{r^2}$, net energy flow from dipole $= 4\pi r^2|\underline{E}\times\underline{B}|/\mu_0$ = constant if $|\underline{E}\times\underline{B}| \propto \frac{1}{r^2}$. Other terms die away with r. So for large r only need consider these terms.

Radiation field: $\underline{E} = (0, \frac{\sin\theta}{4\pi\epsilon_0}\frac{[\ddot{p}]}{rc^2}, 0)$ $\underline{B} = (0,0,\frac{\mu_0}{4\pi}\sin\theta\frac{[\ddot{p}]}{rc})$ Note orthogonal $\underline{e}_\theta\times\underline{e}_\phi = \underline{e}_r$

$\Rightarrow$ purely radial $\underline{N} = \underline{E}\times\underline{B}/\mu_0 = E_\theta B_\phi/\mu_0\,\underline{e}_r \Rightarrow \underline{N} = \frac{\mu_0}{16\pi^2 c}\sin^2\theta\frac{[\ddot{p}]^2}{r^2}\hat{\underline{r}}$ ($\underline{e}_r \equiv \hat{\underline{r}}$)

Total radiated power $= P = \int_{\substack{sphere\\radius\ r}}\underline{N}\cdot d\underline{S} = \frac{\mu_0}{16\pi^2 c}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\sin^2\theta\frac{[\ddot{p}]^2}{r^2}\cdot r^2\sin\theta d\theta d\phi$

$= \frac{\mu_0}{6\pi c}[\ddot{p}]^2 = \frac{\mu_0}{6\pi c}[\dot{I}]^2 d^2$

Now if $I = I_0\cos\omega t \Rightarrow \langle P\rangle = \frac{\mu_0\omega^2}{12\pi c}I_0^2 d^2$ or $P = P_0\cos\omega t \Rightarrow \langle P\rangle = \frac{\mu_0\omega^4}{12\pi c}P_0^2$

Define Power Gain or Directivity $G(\theta,\phi) = \frac{r^2 N(\theta,\phi)}{\frac{1}{4\pi}\int r^2 N(\theta,\phi)d\Omega}$ using solid angle $d\Omega = \frac{dA}{r^2} = \frac{r^2\sin\theta d\phi d\theta}{r^2}$ $\Rightarrow d\Omega = \sin\theta d\phi d\theta$.

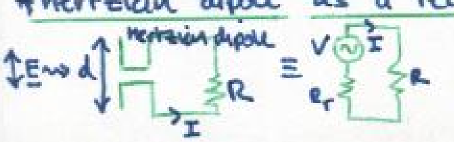
For Hertzian dipole $G = \frac{4\pi\sin^2\theta}{\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\sin^3\theta d\theta d\phi} = \frac{3}{2}\sin^2\theta$

* Radiation Resistance and Aerials. Hertzian dipole example of a radiative element – Aerial or Antenna. Device placed in a circuit acts as a resistor with $\langle P\rangle = \langle I^2\rangle R_r$ ← radiation resistance

For Hertzian dipole $\langle P\rangle = \frac{\mu_0\langle[\dot{I}]^2\rangle d^2}{6\pi c} = \frac{\mu_0\omega^2 d^2}{6\pi c}\langle I^2\rangle$ $\therefore R_r = \frac{\mu_0\omega^2 d^2}{6\pi c}$. Now $\omega = 2\pi c/\lambda$

and $Z_0^{-1} = \sqrt{\frac{\epsilon_0}{\mu_0}} = 377\Omega$ $\Rightarrow R_r = \frac{\mu_0 2\pi d^2}{3\sqrt{\epsilon_0\mu_0}\lambda^2}$ $\Rightarrow R_r = \frac{2\pi}{3}Z_0\left(\frac{d}{\lambda}\right)^2 \approx 789\left(\frac{d}{\lambda}\right)^2$ [valid of course if $d \ll \lambda$]

* Hertzian dipole as a receiver and effective area



Now $I = \frac{V}{R+R_r} = \frac{Ed}{R+R_r}$

Absorbed power I^2R: $P_A = \frac{R\langle E^2\rangle d^2}{(R+R_r)^2}$

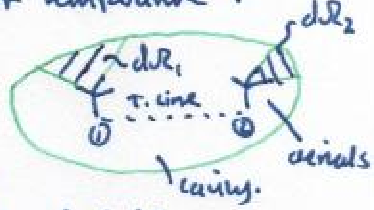
Max power when $\frac{d}{dR}\left(\frac{R}{(R+R_r)^2}\right) = 0$ i.e. $R = R_r$ MATCHED LOAD. GENERAL RESULT.

For Hertzian dipole ($R_r = R$) $\Rightarrow P_A = \frac{\langle E^2\rangle d^2}{4R_r} = \frac{3\lambda^2}{8\pi}\frac{\langle E^2\rangle}{Z_0}$ ← incident power/unit area

$\Rightarrow P_A = A_{eff} \cdot$ incident energy flux. (A_{eff} = "effective area") $\rightarrow$ A_{eff} often $\gg$ geometrical area of aerial seen by incident radiation. Reason? re-radiated power $= R_r \langle I^2 \rangle$ adds to incident field energy and causes the Poynting vector to point towards the aerial. Hence net absorbed power > incident energy flux. aerial actual area. For Hertzian Dipole $A_{eff} = \frac{3\lambda^2}{8\pi}$ // $\uparrow \theta = \pi/2$

* Effective area and power gain $A_{eff} = kG$. $\boxed{k = \frac{\lambda^2}{4\pi}}$ for all antennae.

Proof: Consider two aerials of any type within a black body cavity at temperature T linked by a transmission line.



$\rightarrow \langle I_1^2 \rangle R_1 = \langle I_2^2 \rangle R_2$ (Thermodynamics) and R_r of one aerial acts as the matched load of the other by virtue of the transmission line.

$\Rightarrow$ Power sent into $d\Omega_1$ = Mean power absorbed from $d\Omega_1$ i.e. $\langle I_1^2 \rangle R_1 \frac{d\Omega_1}{4\pi} G_1$

$= {}_1A_{eff} f(T) d\Omega_1$ [$f(T)$ is Black body flux] Similarly $\langle I_2^2 \rangle R_2 \frac{d\Omega_2}{4\pi} G_2 = {}_2A_{eff} f(T) d\Omega_2$

$\therefore {}_1A_{eff}/G_1 = \frac{\langle I_1^2 \rangle R_1}{4\pi f(T)}$ and ${}_2A_{eff}/G_2 = \frac{\langle I_2^2 \rangle R_2}{4\pi f(T)}$ $\Rightarrow {}_1A_{eff}/G_1 = {}_2A_{eff}/G_2$ Now if 1 is Hertzian Dipole with $\theta = \pi/2$

$\Rightarrow {}_1A_{eff}/G_1 = \frac{\lambda^2}{4\pi}$ $\therefore \frac{\lambda^2}{4\pi} = A_{eff}/G$ QED.

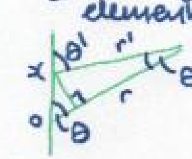
* Half wave dipole

- only use radiation fields
- $I(x,t) = I_0 \cos(kx)\sin(\omega t)$
- $r, r' \gg \lambda \Rightarrow \theta \approx \theta'$

$$B_\phi = \int_{-\lambda/4}^{\lambda/4} \frac{\mu_0 [\dot{I}]}{4\pi c r'} \sin\theta' \, dx$$

[sum of Hertzian dipole elements $p = I dx$]

$r = x\cos\theta + r'\cos(\theta - \theta')$



- only $[\dot{I}]$ x dependence matters to integral.

$$\therefore B_\phi \approx \frac{\mu_0}{4\pi c r} \sin\theta \int_{-\lambda/4}^{\lambda/4} [\dot{I}] \, dx$$

(Excluding $[\dot{I}]$ let $r' = r$ let $\theta \approx \theta'$ hold generally)

$\Rightarrow B_\phi = \frac{\mu_0 I_0}{2\pi r \sin\theta} \cos[\omega(t - r/c)] \cos(\frac{\pi}{2}\cos\theta)$. Now $\underline{E}_{radiation} = E_\theta = c|\underline{B}_\phi| \underline{e}_\theta$ for Hertzian dipole.

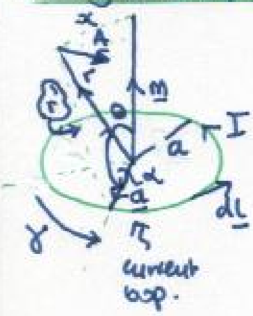
$\Rightarrow B_\phi c = E_\theta$ for half wave dipole (since it is a superposition of Hertzian dipoles). $\therefore |\underline{N}| = E_\theta B_\phi / \mu_0$

$= \frac{c}{\mu_0} B_\phi^2$. $G = \frac{r^2 N}{\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 N \, d\Omega} = \frac{2}{1.219} \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$ using $\int_0^\pi \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta} = 1.219$

Mean power $\langle P \rangle = \int_r \underline{N} \cdot d\underline{S} = \int_0^\pi \frac{c \langle B_\phi \rangle^2}{\mu_0} r^2 \sin\theta \, d\theta \cdot 2\pi = \frac{1.219 \mu_0 I_0^2 c}{4\pi}$ $\therefore R_r = \frac{\langle P \rangle}{\langle I^2 \rangle} = \frac{1.219 \mu_0 c}{2\pi}$

$\approx 73.1\Omega$ ($\langle I^2 \rangle = \frac{1}{2} I_0$).

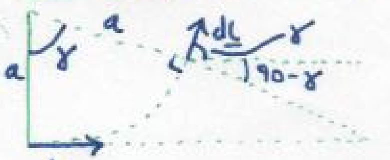
* Magnetic Dipoles are generated from current loops. $|\underline{m}| = I\pi a^2$ in this case. Vector potential must be azimuthal (angle γ) and azimuthally symmetric to follow 'constraints' and symmetry of I. (well $\underline{j}$) So in spherical polars (r, θ, γ)



$\underline{A} = (0, 0, A_\gamma)$. $\uparrow$ with ϕ not time.

$$A_\gamma = \frac{\mu_0}{4\pi} \int_{loop} \frac{[I] \, dl \cos\gamma}{r'} = \frac{\mu_0}{4\pi} \int_0^{2\pi} I_0 e^{i\omega(t - r'/c)} \frac{a\cos\gamma \, d\gamma}{r'}$$

Need to sum over contributions of $\underline{j} \parallel A_\gamma$



Now $r'^2 = r^2 + a^2 - 2ar\cos\alpha$ (cosine rule)

$\Rightarrow r' \approx r(1 - \frac{a}{r}\cos\alpha)$ $\leftarrow r \gg a$

and $\frac{1}{r'} \approx \frac{1}{r}(1 + \frac{a}{r}\cos\alpha)$ $\uparrow$ Substitute for α.

Also $\underline{r} \cdot \underline{a} = ra\cos\alpha = (x, 0, z) \cdot (a\cos\gamma, a\sin\gamma, 0) = xa\cos\gamma$.
$\uparrow x = r\sin\theta$

$\Rightarrow A_\gamma = \frac{\mu_0 \sin\theta}{4\pi} \left(\frac{[m]}{r^2} + \frac{[\dot{m}]}{rc} \right)$ Now since $\nabla \cdot \underline{A} = -\dot{\phi}/c^2$ and $\frac{\partial}{\partial\gamma} A_\gamma = 0 \Rightarrow \dot{\phi} = 0$

$\Rightarrow \phi$ = constant. $\therefore \nabla\phi = 0$. Hence $\underline{E} = -\dot{\underline{A}}$

$= \left(0, 0, -\frac{\mu_0 \sin\theta}{4\pi}\left(\frac{[\dot{m}]}{r^2} + \frac{[\ddot{m}]}{rc}\right)\right)$. $\underline{B} = \nabla \times \underline{A} = \left(\frac{2\mu_0\cos\theta}{4\pi}\left(\frac{[m]}{r^3} + \frac{[\dot{m}]}{r^2 c}\right), \frac{\mu_0 \sin\theta}{4\pi}\left(\frac{[m]}{r^3} + \frac{[\dot{m}]}{r^2 c} + \frac{[\ddot{m}]}{rc^2}\right), 0 \right)$

As for Hertzian Dipole only $\frac{1}{r}$ terms contribute to radiative field.

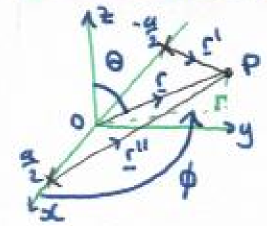
Radiation field: $\underline{E} = (0, 0, -\frac{\mu_0 \sin\theta}{4\pi r c}[\ddot{m}])$ $\underline{B} = (0, \frac{\mu_0 \sin\theta}{4\pi r c^2}[\ddot{m}], 0)$ Similar to H.D radiation fields!

H.D	M.D.
E	$\longleftrightarrow$ B
B	$\longleftrightarrow$ $-E$
$[p]$	$\longleftrightarrow$ $[m]$
$1/\varepsilon_0$	$\longleftrightarrow$ μ_0

Now $\frac{P_{HD}}{P_{MD}} = \left|\frac{B_\gamma E_\theta}{B_\theta E_\gamma}\right| = \frac{c^2[\ddot{p}]^2}{[\ddot{m}]^2} = \frac{c^2(\omega I \cdot \pi a)^2}{(\omega^2 I \cdot \pi a^2)^2} \approx \frac{c^2}{\omega^2 a^2} = \left(\frac{\lambda}{2\pi a}\right)^2 \gg 1$

(Dipole $\frac{1}{2}$ length of current loop) $\Rightarrow$ rates of electric dipole transitions $\gg$ magnetic dipole " in atoms.

Electric Quadrupoles consider two dipoles, separation a, out of phase. Phase difference of radiation from each dipole as viewed looking down radial vector $\underline{r}$ is $\frac{\pi}{2}+k|(\underline{r}''-\underline{r}')\cdot\hat{\underline{r}}|(=\delta) = k|(\underline{r}-\frac{a}{2}-(\frac{a}{2}+\underline{r}))\cdot\hat{\underline{r}}|+\frac{\pi}{2}$ (wave vector k)



(defining $\underline{a}=(a,0,0)$ and $\underline{r}=(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\phi)$).

$= k|-\underline{a}\cdot\hat{\underline{r}}| = \underline{ka\sin\theta\cos\phi+\frac{\pi}{2}}$. $\therefore$ Since $\underline{B}_{rad} = \frac{\mu_0}{4\pi}\sin\theta\frac{[\ddot{p}]}{rc}\hat{\underline{\phi}}$ for dipole, $\underline{B}_{rad}$ for quadrupole will be $2\frac{\mu_0}{4\pi}\sin\theta\frac{[\ddot{p}]}{rc}\mathrm{Re}[e^{i\delta}]$ (i.e. two dipoles with phase δ at origin) one dipole of double strength with extra phase.

Now $\mathrm{Re}[e^{i\delta}] = -\sin(ka\sin\theta\cos\phi)$ and since $\lambda \gg a$ (≈ for dipole formula) $\Rightarrow 1 \gg \frac{a}{\lambda} \Rightarrow 1 \gg \frac{ka}{2\pi} \Rightarrow ka\sin\theta\cos\phi$ is SMALL.

$\Rightarrow \mathrm{Re}[e^{i\delta}] \simeq -ka\sin\theta\cos\phi$. $\therefore \underline{B}_{quad} = \frac{-2\mu_0}{4\pi}\sin^2\theta\cos\phi\frac{[\ddot{p}]}{rc}ka\hat{\underline{\phi}}$

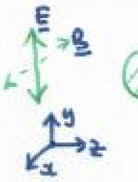
$\therefore$ maximum $|B|$ compares $B_{quad} = 2\pi a B_{dipole}$. $\Rightarrow \langle P_{quad}\rangle \simeq \left(\frac{4\pi a}{\lambda}\right)^2\langle P_{dipole}\rangle$

$G(\theta,\phi)$ has different θ dependence $(\sin^4\theta)$ and no longer cylindrically symmetric.

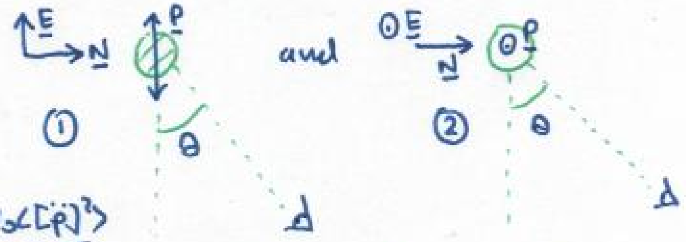
Scattering by particles. Energy incident upon an aerial or dipole etc is absorbed then re-radiated. Angular dependence of re-radiated energy flux is not in general same as incident flux. $\Rightarrow$ Scattering.

- If EM radiation is incident upon a small particle (length dimension $a \ll \lambda$) $\Rightarrow$ dipole induced.
- Particle radiates energy taken from the field with power $\langle P\rangle = \frac{\mu_0\langle[\ddot{p}]^2\rangle}{6\pi c}$
- Define CROSS SECTION $\frac{\langle P\rangle}{\text{Incident EM Flux}} = \sigma$. $\sigma = \frac{\langle P\rangle}{E_0^2/2\mu_0 c}$ $\Rightarrow \sigma = \frac{\mu_0^2}{3\pi E_0^2}\langle[\ddot{p}]^2\rangle$

$\underline{E} = |E_0|e^{i(kz-\omega t)}\hat{y}$
$\underline{B} = \frac{|E_0|}{c}e^{i(kz-\omega t)}\hat{x}$



Polarisation. Consider $\underline{E}$ incident on a small particle to be resolved into $\perp$ components.



For ① θ is the polar angle so w.r.t dipole

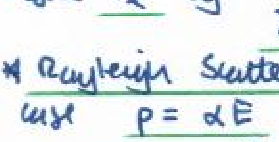
$$|\underline{N}|_{scattered} = \frac{\mu_0\langle[\ddot{p}]^2\rangle\sin^2\theta}{16\pi^2r^2c}$$

For ② polar angle is $\frac{\pi}{2}$ $\Rightarrow |\underline{N}|_{scattered} = \frac{\mu_0\langle[\ddot{p}]^2\rangle}{16\pi^2r^2c}$ $(\sin(\frac{\pi}{2})=1)$.

$\therefore$ Total $|\underline{N}|_{scattered}$ in direction θ is average of ①, ② $= \frac{\mu_0\langle[\ddot{p}]^2\rangle}{16\pi^2r^2c}\cdot\frac{1+\sin^2\theta}{2}$

Define POLARISATION $= \frac{1}{2}\frac{|\underline{N}|^{scattered}_{②} - |\underline{N}|^{scattered}_{①}}{\text{Total }|\underline{N}|_{scattered}} = \frac{1-\sin^2\theta}{1+\sin^2\theta}$

Thomson Scattering from Free Electrons



$m_e\ddot{z} = -eE_0e^{i\omega t}$ - electron like dipole, with $p=-ez$

$\therefore \ddot{p} = -e\ddot{z} = \frac{e^2}{m}E_0e^{i\omega t}$ $\Rightarrow \sigma_T = \frac{\mu_0^2}{3\pi E_0^2}\frac{e^4}{2m^2}E_0^2$

$\therefore \langle[\ddot{p}]^2\rangle = \frac{e^4}{2m^2}E_0^2$ $\Rightarrow \sigma_T = \frac{\mu_0^2e^4}{6\pi m^2} = 6.65\times10^{-29}\,\text{m}^2$

$\underline{E} = E_0e^{i\omega t}\hat{\underline{z}}$

THOMSON CROSS SECTION IS CONSTANT - DOES NOT DEPEND ON λ. Breaks down at high frequencies $\Rightarrow$ need to consider individual photons. In that case use Compton scattering. Can write σ_T in terms of classical electron radius. Define r_e by $\frac{e^2}{4\pi\varepsilon_0 r_e} = m_ec^2$ (rest mass energy = electrostatic) $\Rightarrow \sigma_T = \frac{8\pi}{3}r_e^2$

Rayleigh Scattering. Consider EM scattering from small neutral particles. In this case $p=\alpha E$ (α = polarisability) $\therefore \ddot{p} = -\omega^2E_0e^{i\omega t}\alpha$ $\therefore \langle[\ddot{p}]^2\rangle = \omega^4\frac{E_0^2}{2}\alpha^2$

$\therefore \sigma_R = \frac{\mu_0^2}{3\pi E_0^2}\omega^4\frac{E_0^2\alpha^2}{2} = \frac{\mu_0^2\omega^4\alpha^2}{6\pi} = \frac{8\pi^3}{3}\mu_0^2c^4\alpha^2/\lambda^4$

Note $\alpha_{dielectric} = \varepsilon_0(\varepsilon-1)a^3$, $\alpha_{conductor} = 4\pi\varepsilon_0a^3$ (spheres radius a)

- Explains why sky is blue and sunsets are red. Sunset look towards (near) sun. (Away from Sun Blue light is most strongly scattered in visible spectrum)

How does scattering depend on particle separation? – Two limits. ① $L \gg d \gg \lambda$ ② $L \gg \lambda \gg d$

① Expect incoherent scattering from each particle. ∴ Total power $\propto$ # particles, n.

② Actually the same result. Proof: (i) Divide L^3 into equal volumes ΔV where $\Delta V \ll \lambda^3$ (ii) All particles in ΔV scatter with same phase since $\lambda \gg d$ (iii) Random (Poisson) particle number distribution. Mean # in each ΔV is $\bar{n}$. Variance $\overline{\delta n_i^2} = \bar{n}$ and $\overline{\delta n_i} = 0$

(iv) write $n_i = \bar{n} + \delta n_i$ (explains origin of δn_i term used in variance above).

(v) $\underline{E}_i = \Delta \underline{E}_i + \delta \underline{E}_i$ $\quad \Delta\underline{E}_i$ is mean $\underline{E}$ field in each ΔV. $\delta\underline{E}_i$ is deviation from $\Delta\underline{E}_i$.

(vi) Total scattered power $\propto \left(\sum_i \underline{E}_i\right)^2 = \left(\sum_i \Delta\underline{E}_i + \delta\underline{E}_i\right)^2$. Now $\sum_i \Delta\underline{E}_i = 0$ since amplitude $\propto \bar{n}$ and same for each ΔV_i but phase is random.

$$\therefore \left(\sum_i \underline{E}_i\right)^2 = \left(\sum_i \delta\underline{E}_i\right)^2 = \sum_i \delta\underline{E}_i^2 + \sum_{i \neq j} \delta\underline{E}_i \cdot \delta\underline{E}_j$$

Now $\sum_{i\neq j} \delta\underline{E}_i \cdot \delta\underline{E}_j = 0$ since amplitude and phase of $\delta\underline{E}_i$ is random.

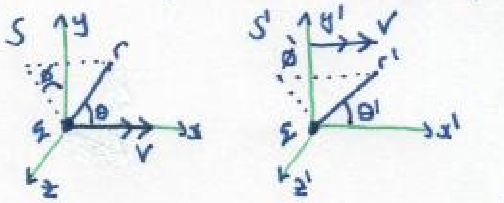
$$\Rightarrow \left(\sum_i \underline{E}_i\right)^2 = \sum_i \delta\underline{E}_i^2.$$

Now since all in ΔV_i scatter with same phase $\Rightarrow \delta\underline{E}_i^2 \propto \delta n_i^2$. $\therefore \left(\sum_i \underline{E}_i\right)^2 \propto \sum_i \delta n_i^2 \propto \overline{\delta n_i^2} = \bar{n}$

Now $\bar{n} = \dfrac{n\Delta V}{L^3} \Rightarrow \left(\sum_i \underline{E}_i\right)^2 \propto n$. i.e. Total scattered power $\propto$ # particles. QED.

*** Field of a uniformly moving charge**

Consider instant $t = t' = 0$ when charge is at the origin of both frames.



S' is rest frame of charge. Charge moves at velocity $V\hat{x}$ in S.

only since $\underline{B}' = 0$

STATIC fields in S' $\quad \underline{E}' = \dfrac{q}{4\pi\epsilon_0 r'^3}(x', y', z')$ $\quad \underline{B}' = (0,0,0)$ $\quad$ Applying LT:

$$x' = \gamma_v x \qquad E_x = E_x'$$
$$y' = y \qquad E_y = \gamma_v E_y'$$
$$z' = z \qquad E_z = \gamma_v E_z'$$

AND: $B_x = B_x'$ $\quad B_y = -\dfrac{V}{c^2}\gamma_v E_z'$ $\quad B_z = \dfrac{V}{c^2}\gamma_v E_y'$ (Easier to calculate $\underline{E}$ then find $\underline{B}$)

Now $r'^2 = x'^2 + y'^2 + z'^2 = \gamma_v^2 x^2 + y^2 + z^2$. Now $x = r\cos\theta$ $\quad y = r\sin\theta\cos\phi$ $\quad z = r\sin\theta\sin\phi$

$$\Rightarrow r'^2 = \gamma_v^2 r^2\cos^2\theta + r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi = \gamma_v^2 r^2(1-\sin^2\theta) + r^2\sin^2\theta$$
$$= \gamma_v^2 r^2 + \sin^2\theta(-\gamma_v^2 r^2 + r^2) = \gamma_v^2 r^2\left(1 - \left(1 - \tfrac{1}{\gamma^2}\right)\sin^2\theta\right) = \gamma_v^2 r^2\left(1 - \tfrac{v^2}{c^2}\sin^2\theta\right)$$

$$\therefore \underline{E} = \frac{q}{4\pi\epsilon_0 \gamma_v^3 r^3}\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{-\frac{3}{2}}(\gamma_v x, \gamma_v y, \gamma_v z) = \frac{q}{4\pi\epsilon_0\gamma_v^3 r^2}\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{-\frac{3}{2}}(\gamma_v\cos\theta, \sin\theta\cos\phi\,\gamma_v, \gamma_v\sin\theta\sin\phi)$$

$$\Rightarrow \underline{B} = \left(0, -\frac{V}{c^2}E_z, \frac{V}{c^2}E_y\right) = \frac{q}{4\pi\epsilon_0\gamma_v^2 r^2}\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{-\frac{3}{2}}\left(0, -\frac{V}{c^2}\sin\theta\sin\phi, \frac{V}{c^2}\sin\theta\cos\phi\right)$$

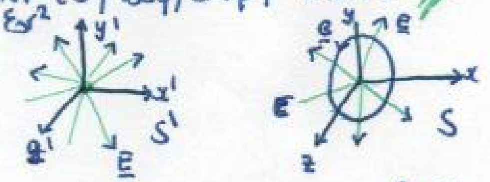
So $\underline{B}$ is always confined to the y,z plane – forms circular field lines.

Now if $\gamma \sim 1$ i.e. $V \sim 0 \Rightarrow \underline{E} = \dfrac{q}{4\pi\epsilon_0 r^3}(x,y,z)$ $\quad \underline{B} = \dfrac{q}{4\pi\epsilon_0 r^2}\dfrac{V}{c^2}(0, -\sin\theta\sin\phi, \sin\theta\cos\phi)$

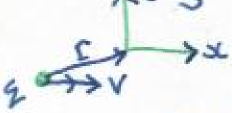
so $\underline{E}$ isotropic, $\underline{B}$ small. If $\gamma \gg 1$, near $\theta = 0$ $\quad \underline{E} \simeq \dfrac{q}{4\pi\epsilon_0\gamma_v^2 r^2}(1,0,0)$ SMALL

near $\theta = \frac{\pi}{2}$ $\quad \underline{E} \simeq \dfrac{q}{4\pi\epsilon_0 r^2}\gamma_v(0, \sin\theta\cos\phi, \sin\theta\sin\phi) = \dfrac{q\gamma_v}{4\pi\epsilon_0 r^2}(0, \cos\phi, \sin\phi)$ LARGE.

so in limit $\gamma \gg 1$ $\underline{E}$ gets flattened into y,z plane.



*** Potentials due to a moving charge** – More general result.



Let S' be IRF of charge, positioned at (x', y'). At $t' = t = 0$ 4-potential at origin in IRF is $\underline{A}' = \left(\frac{\phi'}{c}, \underline{A}'\right) = \left(\frac{q}{4\pi\epsilon_0 r' c}, 0, 0, 0\right) \Rightarrow$ in S: $\underline{A} = \left(\dfrac{\gamma_v q}{4\pi\epsilon_0 r' c}, \dfrac{\beta\gamma_v q}{4\pi\epsilon_0 r' c}, 0, 0\right)$

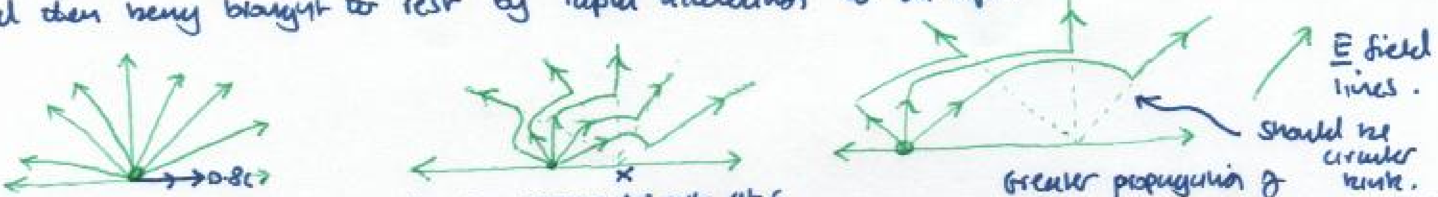
Now potentials at $t' = t = 0$ are attributable to the charge at times $t' = -r'/c$ and $t = -r/c$ respectively. Events which give rise to potential are $(-r'/c, x', y', z')$ $(-r, x, y, z)$ in both frames. $\xrightarrow{LT}$ $-r' = -\gamma r - \gamma\beta x = \gamma(-r - \beta(-\underline{r}\cdot\hat{\underline{V}})) = -\gamma(r - \underline{\beta}\cdot\underline{r})$

where $\underline{\beta} = \underline{V}/c$. Now this β, r' correspond to those in $\underline{A}$ at times $t = -r/c$, $t' = -r'/c$ not $t = t' = 0$ $\therefore$ Need to evaluate $\underline{\beta}$ and $r - \underline{\beta}\cdot\underline{r}$ at RETARDED TIMES.

$$\Rightarrow \phi = \frac{q}{4\pi\epsilon_0[r - \underline{\beta}\cdot\underline{r}]} \qquad \underline{A} = \frac{\mu_0 q[\underline{V}]}{4\pi[r - \underline{\beta}\cdot\underline{r}]}$$

Liénard–Wiechert Potentials.

* Radiation by an accelerated charge. Diagram below shows how the field lines develop for a charge initially in steady motion (with a speed $v = 0.8c$ to the right) and then being brought to rest by rapid acceleration to the left.



uniformly moving charge. Field lines stay with charge.

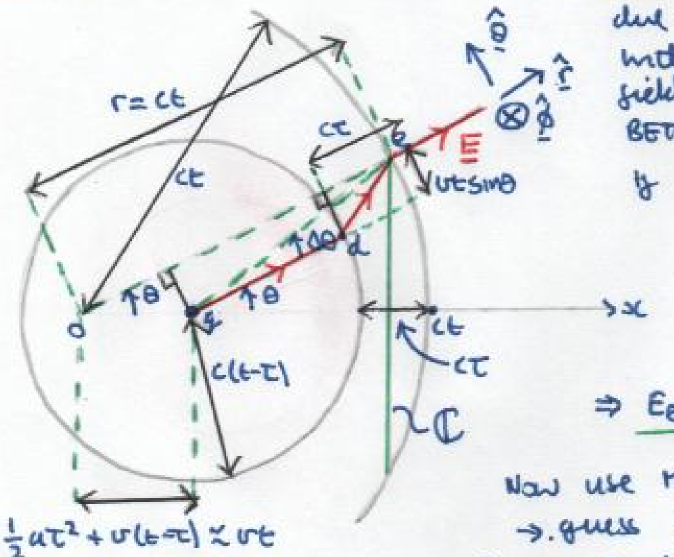
kink propagates out at c as charge accelerates. To distant observer field emanates from location the charge would have moved to

Greater propagation of kink

Radius of kink $= c\tau$ where $\tau = 0$ corresponds to start of acceleration. Beyond ct fields have not had time for info about new information to reach them.

* x calculation of $\underline{E}, \underline{B}$ in kink.

Consider charge q accelerating at rate a for a time τ before travelling at constant velocity. Beyond circle of radius ct centered on O - field due to a stationary charge at O. Within circle radius $c(t-\tau)$ centered on charge q field due to a charge in motion. FIELD KINKED BETWEEN THE TWO.



If $\underline{E} = (E_r, E_\theta)$ then in kink expect (if kink straight)

$$\frac{E_r}{E_\theta} = \frac{c\tau}{vt\sin\theta} = \frac{c\tau}{a\tau t\sin\theta} = \frac{c}{at\sin\theta} = \frac{c^2}{ar\sin\theta}$$

Now assuming $v \ll c$, estimate E_r from Gauss' theorem $E_r \sim q/4\pi\varepsilon_0 r^2$

$$\Rightarrow E_\theta \sim \frac{ar\sin\theta}{c^2}\frac{q}{4\pi\varepsilon_0 r^2} = \frac{\mu_0 q a\sin\theta}{4\pi r}$$

Now use Maxwell 4 to estimate $\underline{B}$ in the kink $\Rightarrow$ guess $\underline{B} = B_\phi\hat{\underline{\phi}}$. $\oint_{\mathcal{C}} \underline{B}\cdot d\underline{l} = \varepsilon_0\mu_0\frac{\partial}{\partial t}\int_S \underline{E}\cdot d\underline{S}$ (No $\underline{J}$ → moving charge constitutes v. small current). Take $\mathcal{C}$ as circular loop with projection shown on diagram above. S is bulge of ct sphere poking out of $\mathcal{C}$ ~S

Area $S = \int_{\phi=0}^{2\pi}\int_{\theta'=0}^{\theta} r^2\sin\theta'\,d\phi\,d\theta = 2\pi r^2(1-\cos\theta)$

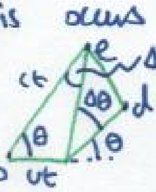
$$\therefore \int_S \underline{E}\cdot d\underline{S} \approx E_r\cdot 2\pi r^2(1-\cos\theta) = \frac{q}{4\pi\varepsilon_0 r^2}\cdot 2\pi r^2(1-\cos\theta) = \frac{q}{2\varepsilon_0}(1-\cos\theta) \equiv \Phi_E \text{ (Electric flux).}$$

This is the flux just before arrival of kink. When kink arrives flux increases by $\Delta\Phi_E$ st. $\Phi_E + \Delta\Phi_E = \frac{q}{2\varepsilon_0}(1-\cos(\theta+\Delta\theta))$. Now $\cos(\theta+\Delta\theta) = \cos\theta\cos\Delta\theta - \sin\theta\sin\Delta\theta \approx \cos\theta - \sin\theta\Delta\theta$. This occurs in time τ so $\frac{\partial}{\partial t}\int_S \underline{E}\cdot d\underline{S}$

$$\therefore \Delta\Phi_E = \frac{q}{2\varepsilon_0}(1-\cos\theta+\sin\theta\Delta\theta-1+\cos\theta) = \frac{q}{2\varepsilon_0}\sin\theta\Delta\theta.$$

$$\approx \Delta\Phi_E/\tau = \frac{q}{2\varepsilon_0}\sin\theta\frac{\Delta\theta}{\tau}.$$

Now from geometry: (and sine rule)



$$\frac{ct}{\sin(\pi-\theta-\Delta\theta)} = \frac{vt}{\sin\Delta\theta} \approx \frac{vt}{\Delta\theta}$$

$$\Rightarrow \Delta\theta \approx \frac{v}{c}\sin\theta = \frac{a\tau}{c}\sin\theta$$

$$\Rightarrow \Delta\Phi_E/\tau = \frac{q}{2\varepsilon_0}\sin^2\theta\frac{a}{c}.$$ Now $\oint_{\mathcal{C}} \underline{B}\cdot d\underline{l} = 2\pi r\sin\theta B_\phi$

$$\Rightarrow B_\phi = \frac{\mu_0\varepsilon_0}{2\pi r\sin\theta}\cdot\frac{q}{2\varepsilon_0}\cdot\frac{a}{c}\sin^2\theta = \frac{\mu_0 q a\sin\theta}{4\pi r c} = \frac{E_\theta}{c}$$

$\Rightarrow$ E_θ, B_ϕ vary like $\frac{1}{r}$ radiation fields, are $\perp$ to one another and have correct E_θ, B_ϕ relationship.

$\therefore$ Accelerated charge radiates with $\underline{N} = \frac{B_\phi E_\theta}{\mu_0}\hat{\underline{r}} = \frac{\mu_0 q^2 a^2\sin^2\theta}{16\pi^2 r^2 c}$

Total power radiated is $2\pi r^2\int_{\theta=0}^{\pi}|\underline{N}|\sin\theta\,d\theta$ $= \frac{\mu_0 q^2 a^2}{6\pi c}$. Exact result in IRF of charge. Now using $a^2 \equiv \alpha^2 = \gamma_v^4 a^2 + (\underline{a}\cdot\underline{v})^2\gamma_v^6/c^2$ (a^2 refers to lab a.) i.e. q moves at velocity $\underline{v}$

$\Rightarrow$ LARMOR'S FORMULA $P = \frac{\mu_0 q^2}{6\pi c}\left(\gamma_v^4 a^2 + (\underline{a}\cdot\underline{v})\gamma_v^6/c^2\right)$

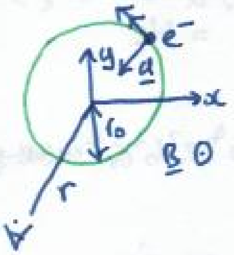
consider charge describing circular motion about a uniform $\underline{B}$ field. → let $q = -e$. ELECTRON.

v, a $\underline{B}\odot$ $-e\underline{v}\times\underline{B} = \gamma m\underline{a}$ $(\underline{f}\cdot\underline{u} = 0)$ $\Rightarrow \gamma m\frac{v^2}{r} = evB \Rightarrow v = \frac{reB}{\gamma m_e} \Rightarrow \omega = \frac{eB}{\gamma m_e}$. $a = \frac{evB}{\gamma m}$

$\Rightarrow P = \frac{\mu_0 e^2\gamma^2 B^2 v^2}{6\pi m^2 c}$. using $U_B = \frac{B^2}{2\mu_0}$, $\sigma_T = \frac{\mu_0^2 e^4}{6\pi m^2}$ $\Rightarrow P = 2c\sigma_T\beta^2\gamma^2 U_B$

SRand EM.

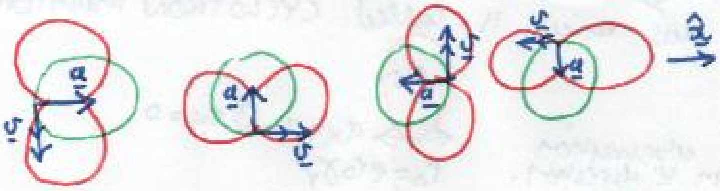
* cyclotron and synchrotron radiation. Uniformly accelerating charges radiate like dipoles. i.e. $N_{e^-} = \frac{\mu_0 e^2 a^2 \sin^2\theta}{16\pi^2 r^2 c}\gamma_v^4$; $N_{dipole} = \frac{\mu_0 \omega^4 p_0^2 \cos^2\omega(t-\frac{r}{c})}{16\pi^2 r^2 c}\sin^2\theta$



For our rotating charge around $\underline{B}$ lines
$a^2 = r_0^2\omega^4\cos^2\omega t + r_0^2\omega^4\sin^2\omega t \rightarrow$ evaluate at retarded time for observer at r away $\rightarrow$ two dipoles radiating in quadrature. In this case $p_0 = e r_0 \gamma_r^2$.

In this case get cyclotron radiation.

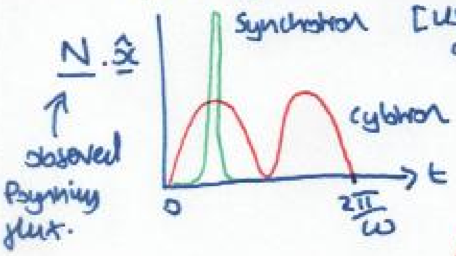
Now one if $\gamma_v \sim 1$.



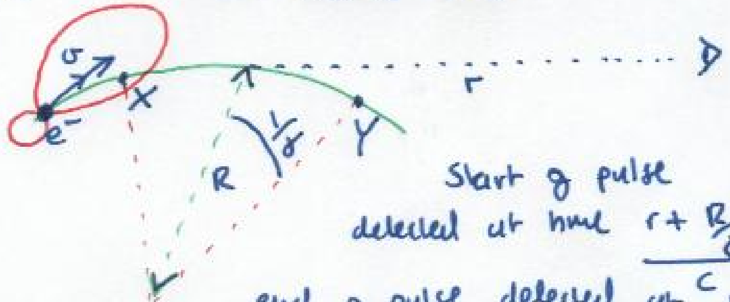
i.e. get a maximum twice / cycle. (Equivalent to dipoles in quadrature).

If $\gamma_v \gg 1$, power distribution severely aberrated in $\underline{v}$ direction.
$\rightarrow$ SYNCHROTRON radiation.

In this case get a definite PULSE once per cycle.



How long does a synchrotron pulse last for an observer $\parallel \hat{x}$?



For heavily aberrated pulse $\rightarrow$ width $\sim \frac{2}{\gamma}$
$\therefore$ observer sees pulse when electron orbits through angle $2/\gamma$

Start of pulse detected at time $\frac{r+\frac{R}{\gamma}}{c}$ from e^- reaching X.

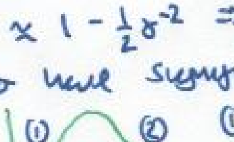
end of pulse detected at time $\frac{2R}{\gamma v} + \frac{r-R/\gamma}{c}$

$\Rightarrow$ pulse duration $\delta t = \frac{1}{c}\left(r-\frac{R}{\gamma}-r-\frac{R}{\gamma}\right)+\frac{2R}{\gamma v} = \frac{2R}{\gamma}\left(\frac{1}{v}-\frac{1}{c}\right) = \frac{2}{\gamma\omega}\left(1-\frac{v}{c}\right) \simeq \frac{1}{\gamma^3\omega}$
($v = R\omega$ and $1-\frac{v}{c} \sim \frac{1}{2\gamma^2}$)

Proof: $\gamma^{-2} = 1-\frac{v^2}{c^2} \Rightarrow v = c(1-\gamma^{-2})^{\frac{1}{2}}$ Highly relativistic $\Rightarrow \gamma \gg 1$ $\therefore$ can binomially expand since $\gamma^{-2} \ll 1$. $\Rightarrow \frac{v}{c} \simeq 1-\frac{1}{2}\gamma^{-2} \Rightarrow 1-\frac{v}{c} \simeq \frac{1}{2\gamma^2}$ QED. $\therefore$ expect spectrum of synchrotron radiation to have significant contributions at $\frac{1}{\delta t} = \gamma^3\omega$.

{ Find synchrotron in SUPERNOVAE RADIO GALAXIES etc.

Actual spectrum like:



① $\sim (\nu/\nu_s)^{1/3}$ ② $\exp(-2\nu/3\nu_s)$
$2\pi\nu_s = \gamma^3\omega$.

* Inverse Compton scattering - back scattering of photons from moving electrons

LAB e^- ⟶ ⟵ frequency ν, arrival rate / unit area N

Now since ν', N are both rates $\nu' = \gamma(1+\beta)\nu \simeq 2\gamma\nu$ $N' = \gamma(1+\beta)N \simeq 2\gamma N$

e^- rest frame e^- ⟵ ν', N'

- Assume elastic collision in rest frame of e^-.
$\therefore$ Scattered power $= \sigma_T N' h\nu' \equiv P$ (Scattered flux $= \sigma_T N'$, photon energy $= h\nu'$)

Power scattered must be Lorentz invariant $\therefore P = P' = 4\gamma^2\sigma_T N h\nu = 4\gamma^2\sigma_T c n h\nu$
(n = number density) $= 4\gamma^2 c\sigma_T U_{photon}$. Compare with synchrotron radiating power $= 2c\sigma_T\beta^2\gamma^2 U_B \simeq 2c\sigma_T\gamma^2 U_B$. Synchrotron radiation thought to be inverse compton scattering off virtual photons instead of static $\underline{B}$ field.