SYSTEMS

"(Dictionary definition) Network or assembly of parts that form a whole"
See p23 of Formula Book for notes on laplace transforms.

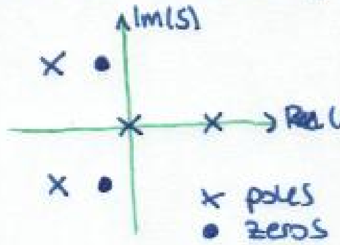
1) **Linear systems** – Dynamics of system described by linear differential equations.
Properties: (i) Solutions to these equations can be superposed to form new solutions.
(ii) Fourier and laplace transforms convert linear differential equations into algebraic ones. Convenient method of analysing system behaviour. (i.e development in time, stability).
Most systems are causal i.e. output response follows input in time. ∴ Need only consider $t>0$ solutions to D.E's describing system. This ⇒ laplace transforms are ideal for system analysis since they nearly always converge as $t \to \infty$. Fourier transforms do not always converge but are more suited to purely periodic systems and sampling. (where time origins are less important).

Gain $= \left|\frac{x(t)}{u(t)}\right|$ though often expressed in $s = i\omega$ domain i.e $H(i\omega)$

* **Analysis of systems** Define input function $u(t)$ and output function $x(t)$. In S domain their respective laplace transforms are $X(s) \left(= \int_0^\infty x(t) e^{-st} dt\right)$ and $U(s)$
Define TRANSFER FUNCTION $H(s) = \frac{X(s)}{U(s)}$ By convolution theorem $x(t) = h(t) * u(t)$ where $h(t) = LT^{-1}[H(s)]$.
Note $h(t)$ is the IMPULSE RESPONSE of the system.

Proof: let $u(t) = \delta(t) \Rightarrow x(t) = \int_0^\infty h(t')u(t-t')dt' = \int_0^\infty h(t')\delta(t-t')dt' = h(t)$ QED.

H(s) yields nearly all the information about the time evolution of the system. Express graphically on a POLE, ZERO diagram

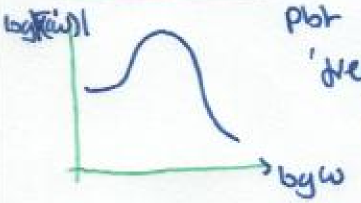


Poles are most useful. Either lie on real axis or appear as conjugate pairs. (Property of roots of algebraic equations).

- Imaginary part of pole ⇒ oscillating solution. ← Simple poles – solution goes $\propto e^{s_{pole}t}$
- real, –ve " " " ⇒ decaying solution.
- real, +ve " " " ⇒ exponentially increasing solution.
- zero pole ⇒ constant solution.

These results stem from inverse transform ⇒ $f(t) = \sum$ residues of $F(s)e^{st}$

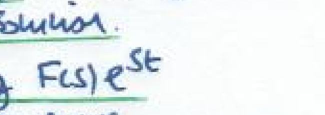
* **BODE plots** are ones of $F[\text{Im}(s)]$. $\text{Im}(s) \equiv i\omega$ c.f. Fourier Transforms.



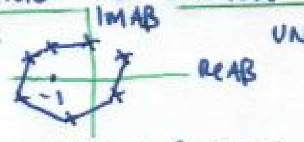
Plot $\log|F(i\omega)|$ and $\arg(F(i\omega))$ vs $\log\omega$ to get a feel for F's 'frequency response'. Preferable to simply Fourier transform though as this will pick out any frequency components.

* **Stability of systems**. Stable system is one where impulse response $\to 0$ as $t \to \infty$. i.e if $H(s) = \frac{A(s)}{B(s)} \Rightarrow B(s)$ must have no poles in the right half plane since there give rise to exponentially increasing solutions.
For 2nd, 3rd order systems (nth order is complicated) use ROUTH-HURWITZ criteria to characterise stability of system. 2nd order system ⇒ $H(s) = \frac{A(s)}{as^2+bs+c}$ if $a,b,c>0$ → STABLE
3rd order system $H(s) = \frac{A(s)}{as^3+bs^2+cs+d}$ if $a,b,c,d>0$ AND $bc>ad$ → STABLE

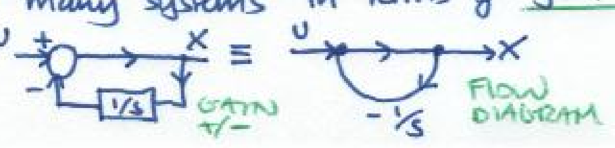
When H(s) is not known in analytic form (i.e only have a data set) use Nyquist criterion. Usually applied to Negative Feedback systems.



If system is stable when $\beta = 0$ i.e $A(s)$ has no right half plane poles: Since $X(s) = (U + \beta X)A \Rightarrow \frac{X}{U} = H(s) = \frac{A}{1+A\beta}$, so feedback system is stable if $1 + A\beta$ has no zeros in the right half plane. Nyquist's criterion is "Data points collected for $A\beta$ must not encircle the point $(-1,0)$ if system is stable".
$(-1,0)$ corresponds to $(\text{Re}\,A\beta, \text{Im}\,A\beta)$.



* **Building blocks**. Can express many systems in terms of gain block, sum/difference diagrams or flow diagrams. i.e

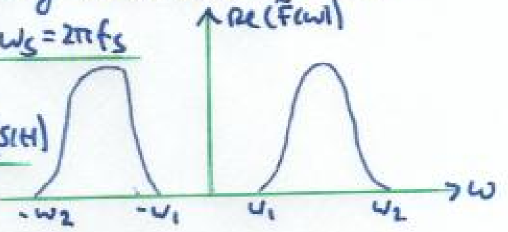


– Add at nodes, multiply along arrows.

* **Sampling.** Consider a real time signal $A(t)$ with frequency spectrum $\tilde{A}(\omega)$ as shown below. $\tilde{A}(\omega)$ is necessarily real and can be computed from the Fourier transform of $A(t)$ $\tilde{F}(\omega)$.

$$\tilde{F}(\omega) = \int_{-\infty}^{\infty} A(t) e^{i\omega t} dt \qquad \tilde{A}(\omega) = \mathrm{Re}(\tilde{F}(\omega)) = \int_{-\infty}^{\infty} A(t)\cos\omega t\, dt \quad 0 < \omega < \infty.$$

Now $\mathrm{Re}(\tilde{F}(\omega))$ itself is even in ω i.e. mirror of $\tilde{A}(\omega)$ about the origin.
Now if $A(t)$ is sampled in time domain at rate $\omega_S = 2\pi f_S$
$\Rightarrow$ resultant sampled signal $A'(t) = A(t)S(t)$



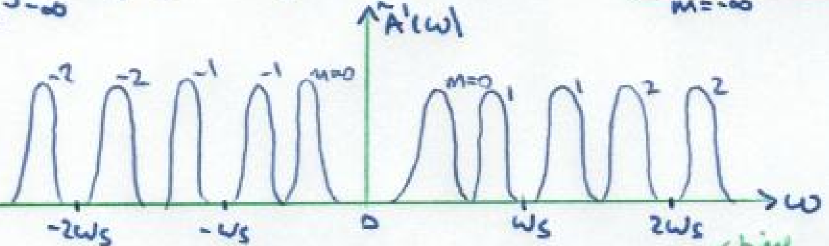
where $S(t) = \sum_{m=-\infty}^{\infty} \delta(t - m/f_S)$. Now $FT(A'(t)) = FT(A(t)) * FT(S(t))$ by convolution theorem.

Now all FT's will be real if done by mechanical / electronic means $\Rightarrow$ $\tilde{A}'(\omega) = \mathrm{Re}(\tilde{F}(\omega)) * \tilde{S}(\omega)$. Now $\tilde{S}(\omega) = \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m f_S)$

$$\Rightarrow \tilde{A}'(\omega) = \int_{-\infty}^{\infty} \mathrm{Re}(\tilde{F}(\omega'))\tilde{S}(\omega - \omega')d\omega' \equiv \int_{-\infty}^{\infty} \tilde{S}(\omega')\mathrm{Re}(\tilde{F}(\omega - \omega'))d\omega'$$

$$= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{Re}(\tilde{F}(\omega - \omega'))\delta(\omega' - 2\pi m f_S)d\omega' = \sum_{m=-\infty}^{\infty} \mathrm{Re}(\tilde{F}(\omega - 2\pi m f_S))$$

i.e. $\tilde{A}(\omega)$ will have ALIASES of $\mathrm{Re}(\tilde{F}(\omega))$!



(in this case $\omega_S = 2\omega_2$)
$\omega_S = 2\pi f_S$.

If $\omega_S < 2(\omega_2 - \omega_1)$ then overlap will occur and errors will result when trying to synthesise a time signal from $\tilde{A}'(\omega)$. $\omega_S > 2(\omega_2 - \omega_1)$ ← Bandwidth
for no overlap is called the SHANNON SAMPLING CRITERION. Now if this criterion is met we can recover the original signal by multiplying $\tilde{A}'(\omega)$ by a top hat function of width $\omega_S/2$ with one edge at the origin. $T(\omega) = \begin{cases} 1 & 0 < \omega < \omega_S/2 \\ 0 & \text{otherwise} \end{cases}$

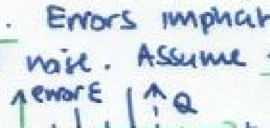
Now if $A'(t) = \sum_{m=-\infty}^{\infty} a_m \delta(t - m/f_S) \Rightarrow A'(\omega) = \sum_{m=-\infty}^{\infty} a_m e^{-im\omega/f_S}$

and $\mathrm{Re}(\tilde{F}(\omega)) = \sum_{m=-\infty}^{\infty} a_m e^{-im\omega/f_S} T(\omega) = \tilde{A}(\omega)$. $\therefore A(t) = \mathrm{Re}\left\{\sum_{m=-\infty}^{\infty} a_m \int_{-\infty}^{\infty} T(\omega) e^{-im\omega/f_S} e^{i\omega t} d\omega\right\}$
↑ real $\omega > 0$

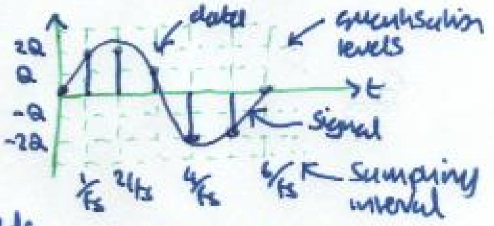
$$\Rightarrow A(t) = \sum_{m=-\infty}^{\infty} \frac{\omega_S a_m}{2} \cos\left(\frac{\omega_S}{4}\left(t - \frac{m}{f_S}\right)\right) \mathrm{sinc}\left(\frac{\omega_S}{4}\left(t - \frac{m}{f_S}\right)\right)$$

So if restrict m can ≈ reconstruct $A(t)$ from $\{a_m\}$.

* **Quantised sampling.** Errors implicit in discrete sampling. - give rise to signal noise. Assume errors are distributed uniformly.



$|\text{Max error}| = Q/2$.



Fraction of samples with error in $d\varepsilon = d\varepsilon/Q$
contribution to mean^2 error $= \varepsilon^2 d\varepsilon/Q$ $\therefore$ mean^2 error $= \int_{-Q/2}^{Q/2} \frac{\varepsilon^2 d\varepsilon}{Q} = Q^2/12$
$\therefore$ RMS error $= Q/\sqrt{12}$

Systematic errors arise from other effects such as unequal quantisation levels.

* **Noise in oversampled systems.** Sampling M times over Shannon rate reduces RMS error by $1/\sqrt{M}$ // **Proof:** compare system sampling at ω_{Sh} ↑ ω_{Sh} and one at $M\omega_{Sh}$. Both cases mean^2 noise error is $Q^2/12$ but oversampled system occupies $\frac{1}{M}$ × bandwidth $\Rightarrow$ RMS error reduced by $1/\sqrt{M}$ // Note reconstruction of signals from samples is less prone to PHASE ERROR and other problems since $T(\omega)$ need not be as sharp as a top hat for an oversampled signal.

* **Sampling in frequency domain** leads to CIRCULAR CONVOLUTION (periodic time functions convolved with signal) - reduce by padding out original signal with zeros.

* Can 'window' in the time domain to make finite length sample. - leads to extra frequency LEAKAGE due to the window. Minimise by using special window functions ⊓⊓ not ⊓⊓

* **Phase space.** A general n^{th} order linear system will be described by an equation of the form

$$(+)\quad \frac{d^n}{dt^n}x(t) - \sum_{i=n-1}^{0} a_i \frac{d^i}{dt^i}x(t) = u(t).$$

To solve this equation we take the L.T and find $H(s) = \left(U(s)/X(s)\right)^{-1}$. From poles of $H(s)$ we can describe the time evolution and stability of the system. For explicit solution we invert $H(s)$ and convolve with input $u(t)$ to find $x(t)$. For 1st, 2nd and 3rd order systems we have an added graphical tool for system evolution – phase portraits. (Higher order systems require more than 3 dimensions – difficult to draw!) Construct an orthonormal phase space from dynamic variables $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ (for a 3rd order system). i.e n^{th} order system comprises of n dynamic variables. The equation governing the system then becomes n coupled D.E's in the dynamic variables. i.e, for a 2nd order system $\ddot{x}_1 - a\dot{x}_1 - bx_1 = u \Rightarrow \begin{cases} x_2 = \dot{x}_1 \\ \dot{x}_2 - ax_2 - bx_1 = u \end{cases}$

Note these D.E's are 1st order. Now can write the set of coupled equations in MATRIX form $\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x}$ where $\underline{\dot{x}} = (\dot{x}_1, \dot{x}_2 \ldots\ldots) \equiv (\dot{x}_1, \ddot{x}_1, \dddot{x}_1 \ldots\ldots)$ and $\underline{x} = (x_1, x_2 \ldots\ldots)$ in the 2nd order case $\underline{\dot{x}} = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}\underline{x}$. This of course assumes $u = 0$. Useful feature of this formalism is solutions to the equation $\underline{\dot{x}} = \lambda\underline{x}$ i.e $\lambda\underline{x} = \underline{\underline{A}}\,\underline{x}$ ⇒ i.e, eigenvector equation of $\underline{\underline{A}}$. The eigenvectors and eigenvalues show lines in phase space where time evolution follows. – Aid to drawing phase portraits. If the poles of the system are also known, this gives information to whether the system decays to the origin or diverges or oscillates (and the relative rates).

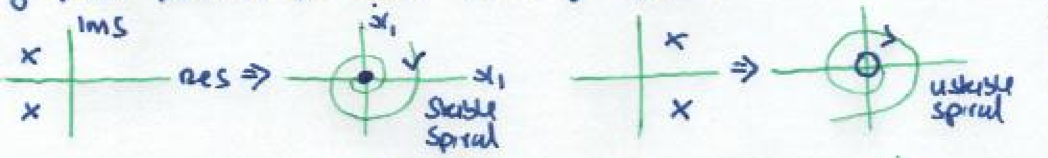
↳ Finite type of phase portraits for each order of system.

2nd order

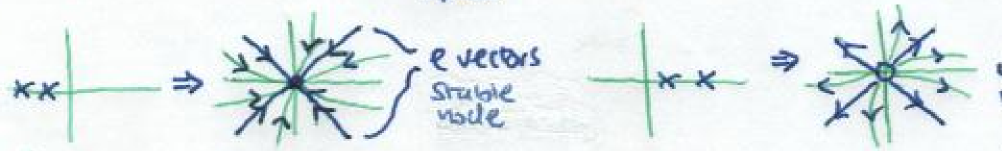
● ⇒ stable FIXED POINT

O ⇒ unstable " "

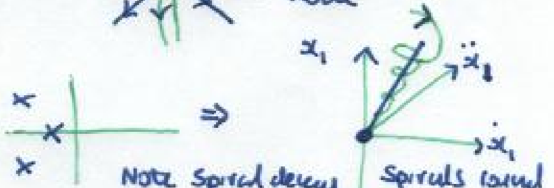
otherwise known as "attractor" and "repeller".



Note λ is imaginary here – no real e. vectors.



And as an example of a stable 3rd order system



Note spiral decay rate > overall decay rate. ↑ well linear // e. vector

Spirals round e vector.

Note eigenvalue = pole so spiral solutions all have $\lambda \in \mathbb{C}$.

2) **Non linear systems** – General case of equation (+) above

$$(*)\quad \frac{d^n}{dt^n}x(t) - \sum_{i=0}^{n-1} f_i\left(\frac{d^i}{dt^i}x(t)\right) = u(t) \qquad \text{for any \{function\} } f_i.$$

NOT usually solvable (i.e by transform methods etc).

→ Exhibit new features – limit cycles → isolated closed phase space trajectory
– chaos
– strange attractors .....

Most useful to write (*) in $\underline{\dot{x}} = \underline{f}(\underline{x})$ i.e $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2 \ldots) \\ f_2(x_1, x_2 \ldots) \\ \vdots \end{pmatrix}$

→ FIXED POINTS are the solutions to $\underline{\dot{x}} = 0$

A useful mathematical device is to find $V(\underline{x})$ s.t $\underline{\dot{x}} = -\nabla V$ ⇒ system is stable i.e has a non zero # of fixed points. If V exists (called the Lyapunov function – SCALAR). For 1D systems (1st order) $\dot{x} = -\frac{dV}{dx}$. So $\frac{dV}{dx} = 0$ yields set of fixed points x^*. $\left.\frac{d^2V}{dx^2}\right|_{x=x^*}$ classifies fixed points as stable or not. If $\left.\frac{d^2V}{dx^2}\right|_{x=x^*} > 0$ ⇒ stable (converse for unstable). Particularly for non-linear systems – i.e the constants in (*). Can use useful plot is x^* vs a control parameter knowledge of V and it's derivative to test which regions are stable or not.

Consider $V(x) = \alpha x^2$ (α ↑ control parameter) $\frac{dV}{dx} = 2\alpha x$ ⇒ fixed point at $x^* = 0$. Now $\frac{d^2V}{dx^2} = 2\alpha$ so if $\alpha > 0$ ⇒ stable, $\alpha < 0$ ⇒ unstable.

* Control parameter, x^* plots for different Lyapunov functions. — can always construct $V(x)$ using a Taylor series. By shifting of the coordinate system and forming the series about a fixed point, $V(x) = \sum_{i=2}^{\infty} a_i x^i$. We have seen the case for harmonic potential $V = \alpha x^2$

→ others:

(i) limit point instability / fold / saddle node bifurcation

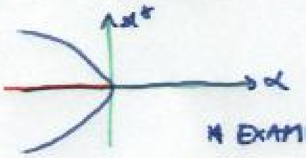
$V(x) = \alpha x + x^3$ $\quad \frac{dV}{dx} = 0 \Rightarrow x^* = \pm\sqrt{\frac{-\alpha}{3}}$

$\left.\frac{d^2V}{dx^2}\right|_{x=x^*} = \pm 6\sqrt{\frac{-\alpha}{3}}$

EXAMPLES: * Flexible steel rule
* weighted pulley....

(ii) Stable symmetric transition / Pitchfork bifurcation.

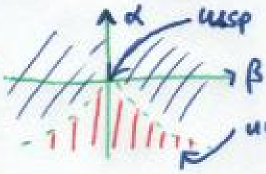
$V(x) = \alpha x^2 \pm x^4$



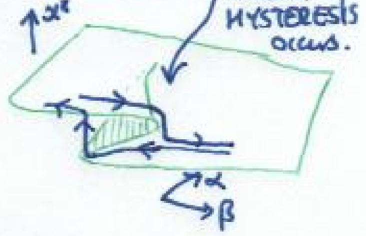
* EXAMPLE: Euler strut

(iii) Cusp catastrophe / imperfect bifurcation

$V(x) = \alpha x^2 + x^4 + \beta x$



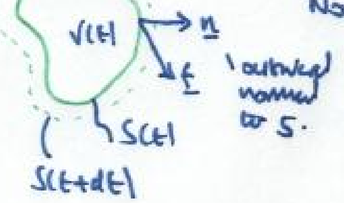
⊙ x^*



* EXAMPLES * Zeeman's catastrophe machine
* Van-der-Waals equation

-ve compressibility can't happen! ⇒ Two state region.

* Dissipation in phase space. Consider system described by set of equations $\dot{\underline{x}} = \underline{f}(\underline{x})$. Now $\underline{f}$ is the instantaneous velocity of points in phase space. Assume at time t volume $V(t)$ contains all solutions to $\dot{\underline{x}} = \underline{f}(\underline{x})$ (i.e. all trajectories). Let this volume be bounded by surface $S(t)$.



Now $dV = \int_S \underline{f}.\underline{n}\, dt\, dS \Rightarrow \dot{V} = \int_S \underline{f}.\underline{n}\, dS = \int_S \underline{f}.d\underline{S} = \int_V \nabla.\underline{f}\, dV$ using Divergence theorem.

⇒ System is DISSIPATIVE if $\dot{V} < 0$ i.e. $\int_V \nabla.\underline{f}\, dV < 0$

⇒ "Volume of phase space occupied by any set of trajectories decreases with time".

* Attractors Definition: "A set to which all neighbouring trajectories converge" i.e. fixed points, limit cycles. Properties: * INVARIANCE — trajectory $\underline{x}(t)$ that starts in the set stays in the set for all time * ATTRACTION — Attracts an open set of initial conditions. i.e. if $\underline{x}(0) \in$ open set, distance between $\underline{x}(t)$ and attractor $\to 0$ as $t \to \infty$. Largest open set which is drawn into the attractor is called the BASIN of attraction. * MINIMAL — There is no proper subset of the attractor set which satisfies axioms above.

* Strange attractors and chaos. A strange attractor is often a property of chaotic systems. (i.e. ones which result in highly aperiodic behaviour and are v. sensitive to the initial conditions of the system.) Strange attractor has following properties beyond that of an attractor. * Zero volume in phase space (⇒ dissipative equations) * Infinite length line (aperiodic trajectories)

Often takes the "Butterfly wing" form — i.e. from the 3rd order Lorenz system of equations.

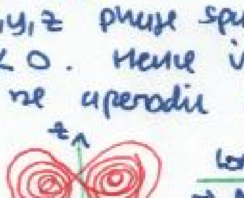
The Lorenz equations

$\dot{x} = \sigma(y - x)$
$\dot{y} = rx - y - xz$
$\dot{z} = xy - bz$

with $\sigma, r, b > 0$

↑ Simple model of atmospheric convection.

→ in $\dot{\underline{x}} = \underline{f}(\underline{x})$ formalism $= \underline{f} = (\sigma(y-x), rx - y - xz, xy - bz)$ in x, y, z phase space. $\nabla.\underline{f} = -\sigma - 1 - b = -(\sigma + 1 + b)$ which < 0. Hence $\dot{V} < 0 \Rightarrow$ Dissipative. Turns out to be aperiodic too ⇒ Chaotic with strange attractor.



Lorenz attractor
→ Note projection onto z, x plane. No trajectories ever cross.

Other chaos related topics are * Period doubling — Feigenbaum #'s * Logistic maps, etc....