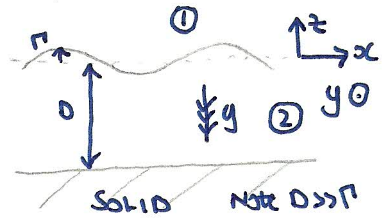
Derivation for dispersion relation for waves on the interface of two fluids



Consider two isotropic, incompressible, Newtonian fluids which are vorticity free, ①, ②

We can $\therefore$ write fluid velocity $\underline{v}$ as $\nabla\phi$ where $\phi = \phi(x,z,t)$. Note incompressibility $\Rightarrow$ $\nabla\cdot\underline{v} = 0$.

Let $\phi_{1,2} = f_{1,2}(z)e^{i(kx-\omega t)}$.

fluids extend infinitely in x and y directions. Fluid ① extends 'infinitely' in z direction (well infinitely relative to D)

Now $\frac{d\Gamma}{dt} = \frac{\partial\Gamma}{\partial t} + (\underline{v}\cdot\nabla)\Gamma \simeq \frac{\partial\Gamma}{\partial t}$ if one ignores terms in wave amplitude^2. Now $\frac{d\Gamma}{dt} = \underline{v}_{1,2}\cdot\hat{\underline{z}}\big|_\Gamma$

$= \frac{\partial f_{1,2}}{\partial z}\Big|_\Gamma e^{i(kx-\omega t)}$ $\therefore$ Integrating w.r.t t and noting initial $t=0$ condition such as $\Gamma = 0$ is not really a valid assertion since we are not concerned with the generation of wave disturbances (simply how they propagate) – let's pretend the (indefinite) integral solution is valid.*

$$\therefore\quad \Gamma = \int \frac{\partial f_{1,2}}{\partial z}\Big|_\Gamma e^{i(kx-\omega t)}\,\partial t = \frac{-1}{i\omega}\frac{\partial f_{1,2}}{\partial z}\Big|_\Gamma e^{i(kx-\omega t)} \quad (*)$$

Better

*[If done definitely i.e from $0 \to t$, integration constant would be $g(x,z)$ which one could set to zero to yield $\Gamma(t=0) = \frac{\partial f_{1,2}}{\partial z}\Big|_\Gamma e^{ikx}$ i.e sinusoidal wave distribution with x].

Now since POTENTIAL FLOW can invoke generalised Bernoulli equation.

$\frac{P}{\rho} + \frac{1}{2}v^2 + gz + \frac{\partial\phi}{\partial t} = \text{constant}$. Neglecting the v^2 term as above and evaluating the LHS at either sides of the interface of 1,2 we find:

$$\rho_1\frac{\partial\phi_1}{\partial t}\Big|_\Gamma + P_1 + \rho_1 g\Gamma = \rho_2\frac{\partial\phi_2}{\partial t}\Big|_\Gamma + P_2 + \rho_2 g\Gamma$$

$$\Rightarrow\quad P_2 - P_1 = (\rho_1-\rho_2)g\Gamma + \rho_1\frac{\partial\phi_1}{\partial t}\Big|_\Gamma - \rho_2\frac{\partial\phi_2}{\partial t}\Big|_\Gamma \quad (+)$$

Now $P_2 - P_1 = -\sigma\frac{\partial^2\Gamma}{\partial x^2}$ where σ is surface tension. [Note $P_2 > P_1 \Rightarrow$ ⌒$_2$, curvature $\therefore \frac{\partial^2\Gamma}{\partial x^2} < 1$ Hence – sign].

So using Γ given in (*), (+) reduces to:

$$\sigma k^2\Gamma = (\rho_1-\rho_2)g\Gamma + \rho_1\frac{\partial\phi_1}{\partial t}\Big|_\Gamma - \rho_2\frac{\partial\phi_2}{\partial t}\Big|_\Gamma \quad \text{since } \frac{\partial^2\Gamma}{\partial x^2} = -k^2\Gamma.$$

Now $\frac{\phi_{1,2}}{f_{1,2}} = \frac{-i\omega\Gamma}{\partial f_{1,2}/\partial z\big|_\Gamma}$, again using (*). $\therefore \frac{\partial\phi_{1,2}}{\partial t} = \frac{-i\omega f_{1,2}}{\partial f_{1,2}/\partial z\big|_\Gamma}\frac{\partial\Gamma}{\partial t}$

and from (*) $\frac{\partial\Gamma}{\partial t} = -i\omega\Gamma \Rightarrow \frac{\partial\phi_{1,2}}{\partial t}\Big|_\Gamma = \frac{-\omega^2 f_{1,2}(\Gamma)}{\partial f_{1,2}/\partial z\big|_\Gamma}\Gamma$

So (+) becomes $\sigma k^2 = (\rho_1-\rho_2)g + \omega^2\left[\rho_2\frac{f_2(\Gamma)}{\partial f_2/\partial z\big|_\Gamma} - \rho_1\frac{\partial f_1(\Gamma)}{\partial f_1/\partial z\big|_\Gamma}\right]$

cancelling Γ.

Now from $\nabla\cdot\underline{v} = 0 \Rightarrow \frac{\partial^2\phi_{1,2}}{\partial x^2} = -\frac{\partial^2\phi_{1,2}}{\partial z^2} = -k^2\phi_{1,2}$.

Now $\frac{\partial^2\phi_{1,2}}{\partial z^2} = e^{i(kx-\omega t)}\frac{d^2 f_{1,2}}{dz^2}$ and $\phi_{1,2} = f_{1,2}e^{i(kx-\omega t)} \Rightarrow \frac{d^2 f_{1,2}(z)}{dz^2} - k^2 f_{1,2}(z) = 0$

Solutions of the form $f_{1,2}(z) = A_{1,2}e^{kz} + B_{1,2}e^{-kz}$ (A, B constants).

For our scenario: $\underline{v}\cdot\hat{\underline{z}}|_{z=\infty}=0$ (1) $\quad \underline{v}\cdot\hat{\underline{z}}|_{z=-D}=0$ (2)

So (1) applies to region ① (2) " " " ②.

For (1) to be satisfied $A_1 = 0$ and for (2) " " " :

Since $\underline{v}_2\cdot\hat{\underline{z}} = \frac{\partial\phi_2}{\partial z} = \frac{\partial f_2}{\partial z} e^{i(kx-\omega t)} \Rightarrow$ if $\underline{v}\cdot\hat{\underline{z}}|_{z=-D}=0$

$\therefore\ kA_2 e^{-kD} - kB_2 e^{+kD} = 0 . \quad \therefore A_2 = B_2 e^{2kD}$.

So $f_2(z) = B_2\left(e^{2kD}e^{kz} + e^{-kz}\right) = 2B_2 e^{kD}\cdot\frac{1}{2}\left(e^{(kz+kD)} + e^{-(kz+kD)}\right)$

$\Rightarrow \underline{f_2(z) = 2B_2 e^{kD}\cosh(kz+kD)}$.

From above, $f_1(z)$ is clearly $B_1 e^{-kz}$.

$\therefore \underline{\frac{\partial f_1}{\partial z}\Big|_\Gamma} = \underline{-kf_1(\Gamma)}$ and $\frac{\partial f_2}{\partial z}\Big|_\Gamma = 2kB_2 e^{kD}\sinh(kz+kD)$
$= k\tanh(k\Gamma+kD) f_2(\Gamma)$

Now since $D \gg \Gamma \Rightarrow \underline{\frac{\partial f_2}{\partial z}\Big|_\Gamma \simeq k\tanh(kD) f_2(\Gamma)}$ $\therefore$ Putting these into (+) we arrive at an equation for $\omega(k)$. The Dispersion relation.

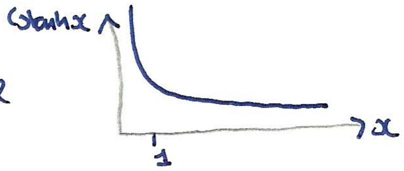
$\sigma k^2 = (\rho_1-\rho_2)g + \omega^2\left[\rho_2/k\tanh(kD) + \rho_1/k\right] \Rightarrow \omega^2 = \dfrac{\sigma k^3 + \left(\frac{\rho_2}{\rho_1}-1\right)gk}{1+\frac{\rho_2}{\rho_1}\coth(kD)}$

Special cases:

① Deep water waves on a water: air interface. i.e. $2 \Rightarrow$ water, $1 \Rightarrow$ air.

clearly $\frac{\rho_2}{\rho_1} \gg 1$ and we can ignore surface tension term if $\left(\frac{\rho_2}{\rho_1}-1\right)gk \gg \sigma k^3 \Rightarrow$ (in this case) $\frac{\rho_{water}}{\rho_{air}}\frac{g}{\sigma} \gg k^2 \Rightarrow 2\pi\sqrt{\frac{\sigma\rho_{air}}{g\rho_{water}}} \ll \lambda$

$2\pi\sqrt{\frac{\sigma\rho_{air}}{g\sigma_{water}}} \sim 1.7$cm for water so for deep ocean waves we can ignore σk^3 term.



Since $\Rightarrow \lim_{D\to\infty}\coth(kD) = 0$

$\therefore\ \underline{\omega^2 = gk}$

(Note kD must be $\gg 1$ for this to occur i.e. $\frac{D}{\lambda} \gg \frac{1}{2\pi}$).

② Shallow water waves on water: air interface where $\frac{D}{\lambda} \ll \frac{1}{2\pi}$ but still $\lambda \gg 2\pi\sqrt{\frac{\sigma\rho_{air}}{g\rho_{water}}}$

i.e. $\lambda \gg \frac{D}{(2\pi)^{-1}}$ and $2\pi\sqrt{\frac{\sigma\rho_{air}}{g\rho_{water}}}$. In this case $kD \ll 1$ so $\coth(kD) \gg 1$.

$\therefore\ \omega^2 = gk\tanh(kD)$. Now $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ so $\lim_{x\to 0}\tanh x$

$\simeq \lim_{x\to 0}\left\{\frac{1+x+\frac{x^2}{2} - 1 + x - \frac{x^2}{2}}{1+x+\frac{x^2}{2}+1-x+\frac{x^2}{2}}\right\} \simeq \frac{2x}{2} + O(x^2) \simeq x . \quad \therefore \underline{\omega^2 = gk^2D}$

③ Ripples on water: air interface where $\lambda \ll \frac{D}{(2\pi)^{-1}}$ but $\lambda \ll 2\pi\sqrt{\frac{\sigma\rho_{air}}{g\rho_{water}}}$

$\Rightarrow$ Can't ignore σk^3 term but can $\coth(kD) \Rightarrow \underline{\omega^2 = \frac{\sigma k^3}{\rho_2} + gk}$ $\quad \left(\frac{4g\sigma}{\rho_{water}}\right)^{\frac{1}{4}}$

(multiply by ρ_2 [illegible]) $\Rightarrow$ minimum phase velocity $c^* =$