



Consider two isotropic, incompressible, Newtonian fluids which are vorticity free, (1), (2)

We can \therefore write fluid velocity \underline{v} as $\nabla\phi$ where $\phi = \phi(x, z, t)$. Note incompressibility $\Rightarrow \nabla \cdot \underline{v} = 0$.

let $\phi_{1,2} = f_{1,2}(z) e^{i(kx - \omega t)}$

Now $\frac{d\Gamma}{dt} = \frac{\partial \Gamma}{\partial t} + (\underline{v} \cdot \nabla) \Gamma \approx \frac{\partial \Gamma}{\partial t}$ if one ignores terms in wave amplitude². Now $\frac{d\Gamma}{dt} = \underline{v}_{1,2} \cdot \hat{z} / \Gamma$

fluids extend infinitely in x and y directions. Fluid (1) extends 'infinitely' in z direction (well infinitely relative to D)

$= \frac{\partial f_{1,2}}{\partial z} \Big|_{\Gamma} e^{i(kx - \omega t)}$

\therefore Integrating w.r.t t and noting initial $t=0$ condition such as $\Gamma=0$ is not really a valid assertion since we are not concerned with the generation of wave disturbances (simply how they propagate)

let's pretend the (indefinite) integral solution is valid.

$\therefore \Gamma = \int \frac{\partial f_{1,2}}{\partial z} \Big|_{\Gamma} e^{i(kx - \omega t)} dt = \frac{-1}{i\omega} \frac{\partial f_{1,2}}{\partial z} \Big|_{\Gamma} e^{i(kx - \omega t)} \quad (*)$

Better

* [if done definitely we from $0 \rightarrow t$, integration constant would be $g(x, z)$ which one could set to zero to yield $\Gamma(t=0) = \frac{\partial f_{1,2}}{\partial z} \Big|_{\Gamma} e^{i k x}$ i.e sinusoidal wave distribution with x].

Now since POTENTIAL FLOW can involve generalized Bernoulli equation.

$\frac{P}{\rho} + \frac{1}{2} v^2 + gz + \frac{\partial \phi}{\partial t} = \text{constant}$

Neglecting the v^2 term as above and evaluating the LHS at either sides of the interface of 1, 2 we find:

$P_1 \frac{\partial \phi_1}{\partial t} \Big|_{\Gamma} + P_1 + \rho_1 g \Gamma = P_2 \frac{\partial \phi_2}{\partial t} \Big|_{\Gamma} + P_2 + \rho_2 g \Gamma$

$\Rightarrow P_2 - P_1 = (\rho_1 - \rho_2) g \Gamma + \rho_1 \frac{\partial \phi_1}{\partial t} \Big|_{\Gamma} - \rho_2 \frac{\partial \phi_2}{\partial t} \Big|_{\Gamma}$ (†) Now $P_2 - P_1 = -\sigma \frac{\partial^2 \Gamma}{\partial x^2}$

where σ is surface tension. [Note $\rho_2 > \rho_1 \Rightarrow$ hence - sign].

(2) concave $\therefore \frac{\partial^2 \Gamma}{\partial x^2} < 1$

so using Γ given in (*), (†) reduces to:

$\sigma k^2 \Gamma = (\rho_1 - \rho_2) g \Gamma + \rho_1 \frac{\partial \phi_1}{\partial t} \Big|_{\Gamma} - \rho_2 \frac{\partial \phi_2}{\partial t} \Big|_{\Gamma}$ since $\frac{\partial^2 \Gamma}{\partial x^2} = -k^2 \Gamma$

Now $\frac{\partial \phi_{1,2}}{\partial t} = \frac{-i\omega \Gamma}{\frac{\partial f_{1,2}}{\partial z} \Big|_{\Gamma}}$, again using (*). $\therefore \frac{\partial \phi_{1,2}}{\partial t} = \frac{-i\omega f_{1,2}}{\frac{\partial f_{1,2}}{\partial z} \Big|_{\Gamma}} \frac{\partial \Gamma}{\partial t}$

and from (†) $\frac{\partial \Gamma}{\partial t} = \frac{-i\omega \Gamma}{\frac{\partial f_{1,2}}{\partial z} \Big|_{\Gamma}} \Rightarrow \frac{\partial \phi_{1,2}}{\partial t} \Big|_{\Gamma} = \frac{-\omega^2 f_{1,2}(\Gamma)}{\frac{\partial f_{1,2}}{\partial z} \Big|_{\Gamma}} \Gamma$

so (†) becomes $\sigma k^2 = (\rho_1 - \rho_2) g + \omega^2 \left[\rho_2 \frac{f_2(\Gamma)}{\frac{\partial f_2}{\partial z} \Big|_{\Gamma}} - \rho_1 \frac{\frac{\partial f_1(\Gamma)}{\partial t} \Big|_{\Gamma}}{\frac{\partial f_1}{\partial z} \Big|_{\Gamma}} \right]$ cancelling Γ .

Now from $\nabla \cdot \underline{v} = 0 \Rightarrow \frac{\partial^2 \phi_{1,2}}{\partial x^2} = -\frac{\partial^2 \phi_{1,2}}{\partial z^2} = -k^2 \phi_{1,2}$.

Now $\frac{\partial^2 \phi_{1,2}}{\partial z^2} = e^{i(kx - \omega t)} \frac{d^2 f_{1,2}}{dz^2}$ and $\phi_{1,2} = f_{1,2} e^{i(kx - \omega t)} \Rightarrow \frac{d^2 f_{1,2}}{dz^2} - k^2 f_{1,2} = 0$

Solutions of the form $f_{1,2}(z) = A e^{\frac{kz}{2}} + B e^{-\frac{kz}{2}}$ (A, B constants).

