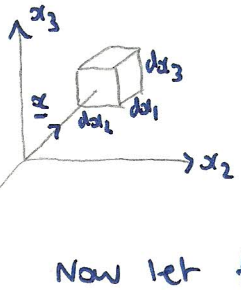
Derivation of Elastic wave equations of isotropic, continuous media

1) Assumptions * Isotropy. (physical properties invariant under rotation of reference frame) * Continuous medium. ($\Rightarrow$ density, force and displacement are continuous and differentiable except at a finite number of interior interfaces). * classical mechanics applies. i.e. speeds $\ll c$.

2) Definitions ① Displacement from some fixed origin: $\underline{x}$
② instantaneous velocity of an element of the medium w.r.t origin: $\underline{u} \equiv \frac{d\underline{x}}{dt}$ (t = time).
③ ρ = local density of the medium.



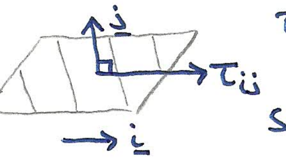
Apply Newton's 2nd law to determine position and velocity of element centre of mass.

$$\underline{f}\, dx_1 dx_2 dx_3 = \frac{d}{dt}(dx_1 dx_2 dx_3 \rho \underline{u})$$ Ignoring derivatives $O(d^2 x_i)$ $i = 1,2,3$

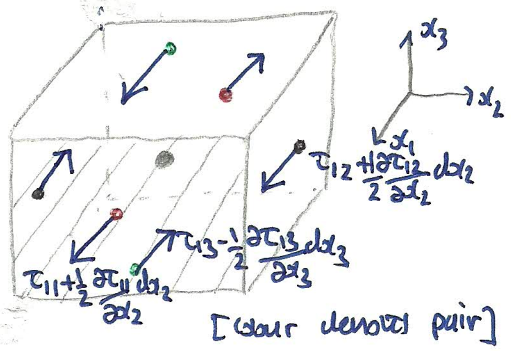
$$\Rightarrow \quad \underline{f} = \rho \frac{d\underline{u}}{dt} + \underline{u}\frac{d\rho}{dt}$$ where $\underline{f}$ is the force / unit volume.

Now let $\underline{f} = \underline{f}_B + \underline{f}_S$. $\underline{f}_B$ are BODY FORCES WHICH ACT ON ALL PARTS of the medium i.e. forces as a result of an external applied field. e.g. gravity. Assume these forces can be suitably defined. $\underline{f}_S$ are SURFACE FORCES. These can be expressed in terms of intrinsic bulk properties of the medium and the current state of motion. (i.e. $\underline{f}_S(K, \sigma, \underline{S})$ where $K, \sigma, \underline{S}$ are BULK MODULUS, POISSON'S RATIO AND strain field $\underline{S} = \underline{x} - \underline{x}_0$ ($\underline{x}_0$ original position at time $t=0$)).

We can decompose $\underline{f}_S$ into * forces through c.o.m and * shearing forces {in general couples - but in solid mechanics one can ignore local rotation}
All forces are evaluated at $\underline{x}$ but the shearing forces and the rest act on the surfaces of the $dx_1 dx_2 dx_3$ cube element defined in the cartesian x_1, x_2, x_3 basis above. Define STRESS TENSOR τ_{ij} to represent all of $\underline{f}_S$.



τ_{ij} is the magnitude of the force / unit area in the ith direction ($\parallel \hat{\underline{x}}_i$) acting on the plane with normal $\hat{\underline{x}}_j$. $\tau_{ij} = \tau_{ij}(\underline{x})$
So actual shearing forces, and c.o.m forces are ($\parallel \hat{\underline{x}}_1$ direction)



$$(\parallel \hat{\underline{x}}_1) \quad \left(\tau_{11} + \frac{1}{2}\frac{\partial \tau_{11}}{\partial x_1} dx_1 - \left[\tau_{11} - \frac{1}{2}\frac{\partial \tau_{11}}{\partial x_1} dx_1\right]\right) dx_2 dx_3$$

$$+ \left(\tau_{12} + \frac{1}{2}\frac{\partial \tau_{12}}{\partial x_2} dx_2 - \left[\tau_{12} - \frac{1}{2}\frac{\partial \tau_{12}}{\partial x_2} dx_2\right]\right) dx_1 dx_3$$

$$+ \left(\tau_{13} + \frac{1}{2}\frac{\partial \tau_{13}}{\partial x_3} dx_3 - \left[\tau_{13} - \frac{1}{2}\frac{\partial \tau_{13}}{\partial x_3} dx_3\right]\right) dx_1 dx_2$$

$$= dx_1 dx_2 dx_3 \left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3}\right) \Rightarrow \underline{f}_S \cdot \hat{\underline{x}}_1 = \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3}$$

Why are the forces in pairs? - well in the limit $dx_1 dx_2 dx_3 \to 0$ the net force on the element must $\to 0$ since if ρ is finite, Newton's 2nd law $\Rightarrow$ infinite acceleration. One can $\therefore$ imagine equal and opposite forces acting at the c.o.m being 'Taylor-projected' to the surfaces. The limit volume $\to 0$ effectively squeezes these projected forces back to the c.o.m where, because they are equal and opposite, result in no net force.

Now in the limit $dx_1 dx_2 dx_3 \to 0$ there must be no net torque also. Need and extra condition in addition to above. Consider torque ($\underline{r} \times \underline{F}$) in $\hat{\underline{x}}_2$ direction

Torque $\parallel \hat{\underline{x}}_2$: $\left(\tau_{13} + \frac{1}{2}\frac{\partial \tau_{13}}{\partial x_3} dx_3\right) dx_2 dx_1 \frac{1}{2} dx_3 + \left(\tau_{13} - \frac{1}{2}\frac{\partial \tau_{13}}{\partial x_3} dx_3\right) dx_2 dx_1 \frac{1}{2} dx_3$
$- \left(\tau_{31} + \frac{1}{2}\frac{\partial \tau_{31}}{\partial x_1} dx_1\right) dx_2 dx_3 \frac{1}{2} dx_1 - \left(\tau_{31} - \frac{1}{2}\frac{\partial \tau_{31}}{\partial x_1} dx_1\right) dx_2 dx_3 \frac{1}{2} dx_1 = (\tau_{13} - \tau_{31}) dx_1 dx_2 dx_3$

So if in limit $dx_1 dx_2 dx_3 \to 0$ Torque $\to$ $\Rightarrow \tau_{13} = \tau_{31}$ More general result: τ_{ij} symmetric

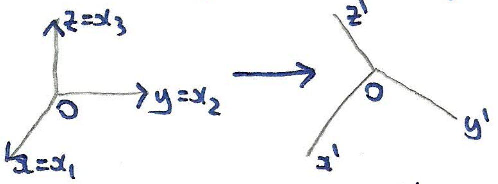
[More exact analysis considers length a for each side of cube to prove torque, force must $\to 0$ as volume (well a) shrinks to zero]

So: $\underline{f}_S = \left(\frac{\partial\tau_{11}}{\partial x_1} + \frac{\partial\tau_{12}}{\partial x_2} + \frac{\partial\tau_{13}}{\partial x_3},\ \frac{\partial\tau_{21}}{\partial x_1} + \frac{\partial\tau_{22}}{\partial x_2} + \frac{\partial\tau_{23}}{\partial x_3},\ \frac{\partial\tau_{31}}{\partial x_1} + \frac{\partial\tau_{32}}{\partial x_2} + \frac{\partial\tau_{33}}{\partial x_3}\right)$

or in tensor formulation (noting summation convention) $f_i = \frac{\partial\tau_{ij}}{\partial x_j}$ (dropping S suffix)

Now further progress can be made invoking a constitutive equation. In SOLID MECHANICS THIS IS HOOKES LAW $\tau_{ij} = C_{ijkl}e_{kl}$ where e_{kl} is the strain tensor (defined below) and in FLUID MECHANICS this is $\tau_{ij} = 2\eta_{ijkl}e_{kl}$. (2 there for elimination of $\frac{1}{2}$'s later on). We will consider solid mechanics here.

Strain is 'extension/original length' and strain tensor e_{kl} can be defined thus: consider basis x,y,z // to our basis x_1,x_2,x_3. Let deformation of medium be monitored by modification of basis x,y,z to x',y',z' i.e. they are set in the material.



(unit vectors $\hat{\underline{x}},\hat{\underline{y}},\hat{\underline{z}}$)

let
$$\underline{x}' = (1+e_{xx})\hat{\underline{x}} + e_{xy}\hat{\underline{y}} + e_{xz}\hat{\underline{z}}$$
$$\underline{y}' = e_{yx}\hat{\underline{x}} + (1+e_{yy})\hat{\underline{y}} + e_{yz}\hat{\underline{z}}$$
$$\underline{z}' = e_{zx}\hat{\underline{x}} + e_{zy}\hat{\underline{y}} + (1+e_{zz})\hat{\underline{z}}$$

Note ' basis only unit length to 1st order in e_{kl}.

Define e_{kl}.)

∴ we can write this

Now centering the $x,y,z/x_1,x_2,x_3$ basis on the c.o.m of our medium element (so deformation is local and a function of $\underline{x}$ in general) initial position vector of a point in the local vicinity is $\underline{x}_0$, after deformation this becomes $\underline{x}$ (w.r.t to original element c.o.m) ↑ we assume this (O) to be unchanged. i.e NO FLOW OF MEDIUM *

so
$$\underline{x}_0 = x\hat{\underline{x}} + y\hat{\underline{y}} + z\hat{\underline{z}}$$
$$\underline{x} = x\underline{x}' + y\underline{y}' + z\underline{z}'$$

* otherwise we would have to include a translation vector $\underline{R}$ here

∴ 'strain field' $\underline{S} = \underline{x} - \underline{x}_0 = x(\underline{x}'-\hat{\underline{x}}) + y(\underline{y}'-\hat{\underline{y}}) + z(\underline{z}'-\hat{\underline{z}})$

$= (xe_{xx} + ye_{yx} + ze_{zx})\hat{\underline{x}} + (xe_{xy} + ye_{yy} + ze_{zy})\hat{\underline{y}} + (xe_{xz} + ye_{yz} + ze_{zz})\hat{\underline{z}}$

$\equiv S_x\hat{\underline{x}} + S_y\hat{\underline{y}} + S_z\hat{\underline{z}}$. Hence we can compute e_{kl} in terms of elements of $\underline{S}$ as:

$$\underline{\underline{e}} = \begin{pmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{pmatrix} = \begin{pmatrix} \partial S_x/\partial x & \partial S_y/\partial x & \partial S_z/\partial x \\ \partial S_x/\partial y & \partial S_y/\partial y & \partial S_z/\partial y \\ \partial S_x/\partial z & \partial S_y/\partial z & \partial S_z/\partial z \end{pmatrix}$$

{Note matrix notation $e_{ij} \Rightarrow e_{\text{rows, colums}}$)

Now we can write $\underline{\underline{e}}$ in terms of symmetric and antisymmetric forms since $e_{ij} = \partial S_j/\partial x_i = \frac{1}{2}\left(\frac{\partial S_j}{\partial x_i} + \frac{\partial S_i}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial S_j}{\partial x_i} - \frac{\partial S_i}{\partial x_j}\right)$

{ $x,y,z = x_1,x_2,x_3$ in terms of suffixes so $i,j = 1,2,3$ }

Now the antisymmetric part corresponds to rotation which we will ignore in solid mechanics of a continuum element.

Hence define $e_{ij} = \frac{1}{2}\left(\frac{\partial S_j}{\partial x_i} + \frac{\partial S_i}{\partial x_j}\right)$.

Now in our Hookes law expression C_{ijkl} must be an isotropic 4th rank tensor since the medium which it applies is isotropic. ∴ C_{ijkl} will take general form $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \eta\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}$ [Mathematical Methods.... pp 701]

∴ $\tau_{ij} = \lambda\delta_{ij}\delta_{kl}e_{kl} + \eta\delta_{ik}\delta_{jl}e_{kl} + \nu\delta_{il}\delta_{jk}e_{kl}$
$= \lambda\delta_{ij}e_{kk} + \eta e_{ij} + \nu e_{ji}$. Now since $e_{ij} = e_{ji}$

'lamé constants'

$\Rightarrow \tau_{ij} = e_{ij}(\eta+\nu) + \lambda e_{kk}\delta_{ij} \equiv 2\mu e_{ij} + \lambda e_{kk}\delta_{ij}$ (re casting coeffecients)

can use this equation to determine relations between μ,λ and σ, K.

$$\mu(K,\sigma) = \frac{3K(1-2\sigma)}{2(1+\sigma)} \qquad \lambda(K,\sigma) = \frac{3K\sigma}{1+\sigma}$$

For most seismic problems $\sigma = 0.25$, $\mu \approx 3\times10^{10}$ (Earth's crust) comparable to steel.

Best way is to firstly express τ_{ij} in terms of σ, Young's modulus E. Using $K = \frac{E}{3(1-2\sigma)}$ (Derived in PIB mechanics) one can then express in terms of σ, K.

Consider simple tension in the x_1 direction i.e. $\tau_{11} = T$, all other $\tau_{ij} = 0$.
Let e_{kk} (sum over k) $= \theta$ $\therefore$ (in addition to $e_{ij} = 0$ for $i \neq j$)

(1) $T = 2\mu e_{11} + \lambda\theta$ (2) $0 = 2\mu e_{22} + \lambda\theta$ (3) $0 = 2\mu e_{33} + \lambda\theta$.

(1)+(2)+(3) $\Rightarrow$ $T = \theta(3\lambda + 2\mu)$. Now $E = \tau_{11}/e_{11}$ $\Rightarrow$ $e_{11} = \frac{T}{E}$.

$\therefore$ in (1): $T = 2\mu\frac{T}{E} + \frac{\lambda T}{3\lambda+2\mu}$ $\Rightarrow$ $1 - \frac{\lambda}{3\lambda+2\mu} = \frac{2\mu}{E}$ $\Rightarrow$ $E = \frac{\mu(3\lambda+2\mu)}{\mu+\lambda}$

Now $\sigma = -e_{22}/e_{11}$ (or $-e_{33}/e_{11}$) $= \frac{1}{e_{11}}\frac{\lambda\theta}{2\mu} = \frac{1}{e_{11}}\frac{\lambda}{2\mu}\frac{Ee_{11}}{3\lambda+2\mu}$ $\left(\theta = \frac{T}{3\lambda+2\mu}\right.$ and $\left.T = Ee_{11}\right)$

$=$ (using E above) $\frac{\lambda}{2\mu}\frac{\mu(3\lambda+2\mu)}{\mu+\lambda}\frac{1}{3\lambda+2\mu} = \frac{\lambda}{2(\mu+\lambda)}$

So if $E = \frac{\mu(3\lambda+2\mu)}{\mu+\lambda}$ and $\sigma = \frac{\lambda}{2(\mu+\lambda)}$ and $k = \frac{E}{3(1-2\sigma)}$

$\Rightarrow$ $k = \frac{\mu(3\lambda+2\mu)}{3(\mu+\lambda)}\left(1 - \frac{\lambda}{\mu+\lambda}\right)^{-1} = \frac{\mu(3\lambda+2\mu)}{3\cancel{(\mu+\lambda)}}\frac{\cancel{\mu+\lambda}}{\mu} = \lambda + \frac{2}{3}\mu$

$\therefore$ $k(\lambda,\mu) = \lambda + \frac{2}{3}\mu$ and $\sigma(\lambda,\mu) = \frac{\lambda}{2(\mu+\lambda)}$: $\mu = \frac{3}{2}(k-\lambda)$

$\therefore$ $\sigma = \frac{\lambda}{2(\mu+\lambda)} = \frac{\lambda}{2\lambda+3k-3\lambda} = \frac{\lambda}{3k-\lambda}$ $\therefore$ $\lambda = \sigma(3k-\lambda)$ $\Rightarrow$ $\lambda(1+\sigma) = 3k\sigma$

$\Rightarrow$ $\lambda = \frac{3k\sigma}{1+\sigma}$ as required. $\therefore$ $\mu = \frac{3}{2}\left(k - \frac{3k\sigma}{1+\sigma}\right) = \frac{3}{2}k\left(\frac{1+\sigma-3\sigma}{1+\sigma}\right) = \frac{3k(1-2\sigma)}{2(1+\sigma)}$
as required.

So in terms of k, σ can write constitutive equation for solid material as

$$\tau_{ij} = \frac{3k(1-2\sigma)}{1+\sigma}e_{ij} + \frac{3k\sigma}{1+\sigma}\delta_{ij}e_{kk}$$

Now, in terms of $\underline{s}$ (and μ,λ since this forms the simpler constitutive relation)

$$\tau_{ij} = 2\mu\frac{1}{2}\left(\frac{\partial s_j}{\partial x_i} + \frac{\partial s_i}{\partial x_j}\right) + \lambda\delta_{ij}\frac{1}{2}\left(\frac{\partial s_k}{\partial x_k} + \frac{\partial s_k}{\partial x_k}\right)$$

$\Rightarrow$ $\tau_{ij} = \mu\left(\frac{\partial s_j}{\partial x_i} + \frac{\partial s_i}{\partial x_j}\right) + \lambda\frac{\partial s_k}{\partial x_k}\delta_{ij}$ Now $f_i = \frac{\partial \tau_{ij}}{\partial x_j}$.

$\Rightarrow$ $f_i = \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial s_j}{\partial x_i} + \frac{\partial s_i}{\partial x_j}\right) + \lambda\frac{\partial s_k}{\partial x_k}\delta_{ij}\right] = \mu\frac{\partial^2 s_i}{\partial x_j^2} + \mu\frac{\partial^2 s_j}{\partial x_j \partial x_i} + \lambda\frac{\partial^2 s_k}{\partial x_j \partial x_k}\delta_{ij}$

$= \mu\frac{\partial^2 s_i}{\partial x_j^2} + \mu\frac{\partial^2 s_j}{\partial x_j \partial x_i} + \lambda\frac{\partial^2 s_k}{\partial x_i \partial x_k} = \mu\frac{\partial^2 s_i}{\partial x_j^2} + (\mu+\lambda)\frac{\partial^2 s_j}{\partial x_i \partial x_j}$

Same $k \leftrightarrow j$ (summed over). Since these are cartesian tensors we can immediately represent $\underline{f}_s$ in vector form.

$$\underline{f}_s = \mu\nabla^2\underline{s} + (\lambda+\mu)\nabla(\nabla\cdot\underline{s})$$

So, since $\frac{d\underline{s}}{dt} = \frac{d\underline{x}}{dt} = \underline{u}$, complete equation becomes: (if $\frac{d\rho}{dt} = 0$)

$$\rho\frac{d^2\underline{s}}{dt^2} = \underline{f}_B + \mu\nabla^2\underline{s} + (\lambda+\mu)\nabla(\nabla\cdot\underline{s})$$ (Navier equation).

Now $\underline{u} = \underline{u}(x_1, x_2, x_3, t)$ in general. Since scalar $A(x_1,x_2,x_3,t)$ results in
$dA = \frac{\partial A}{\partial x_1}dx_1 + \frac{\partial A}{\partial x_2}dx_2 + \frac{\partial A}{\partial x_3}dx_3 + \frac{\partial A}{\partial t}dt$ by chain rule $\Rightarrow$ $\frac{dA}{dt} = (\underline{u}\cdot\nabla)A + \frac{\partial A}{\partial t}$

$\Rightarrow \frac{d\underline{u}}{dt} = (\underline{u}.\nabla)\underline{u} + \frac{\partial \underline{u}}{\partial t} \Rightarrow \left(\frac{d\underline{s}}{dt}.\nabla\right)\frac{d\underline{s}}{dt} + \frac{\partial^2 \underline{s}}{\partial t^2} = \frac{d^2\underline{s}}{dt^2}$. Now can throw away first (non-linear) term if $\left(\frac{d\underline{s}}{dt}.\nabla\right)\frac{d\underline{s}}{dt} \simeq 0$ i.e. variations of velocity field spatially are $\perp$ to the velocity field itself. (i.e. Poiseulle or pipe flow). .... not clear in notes why this is so but ignoring the non-linear terms we arrive at the Navier equation given in the notes.

$$\rho \frac{\partial^2 \underline{s}}{\partial t^2} = (\lambda+\mu)\nabla(\nabla.\underline{s}) + \mu\nabla^2\underline{s} + \underline{f}_B$$

($\underline{f}_B$ is $-\rho g\hat{\underline{z}}$ where $\hat{\underline{z}}$ is vertical unit vector (in Notes I)).

Now using identity $\nabla^2\underline{A} = \nabla(\nabla.\underline{A}) - \nabla\times(\nabla\times\underline{A}) \ \exists \underline{A}$

$$\Rightarrow \rho\frac{\partial^2 \underline{s}}{\partial t^2} = (\lambda+2\mu)\nabla(\nabla.\underline{s}) - \mu\nabla\times(\nabla\times\underline{s}) + \underline{f}_B$$

(More useful form since $\nabla^2\underline{s}$ is difficult to evaluate in non cartesian coordinates).

Now let $\underline{s} = \nabla\phi + \nabla\times\underline{A}$ AND IGNORE f_B (with letter reference to electromagnetism for $\underline{E}$ and $\underline{B}$ fields) ϕ - scalar potential. $\underline{A}$ - vector potential.

Substitution into above $\Rightarrow (\lambda+2\mu)\nabla[\nabla.\{\nabla\phi+\nabla\times\underline{A}\}] - \mu\nabla\times\nabla\times\{\nabla\phi+\nabla\times\underline{A}\} - \rho\frac{\partial^2 \underline{s}}{\partial t^2} = 0$

Now
$$\left.\begin{aligned}\nabla.\nabla\times\underline{A} &= 0\\ \nabla\times\nabla\phi &= 0\\ \nabla.\nabla\phi &= \nabla^2\phi\end{aligned}\right\} (\lambda+2\mu)\nabla(\nabla^2\phi) - \mu\nabla\times\nabla\times\nabla\times\underline{A} - \rho\frac{\partial^2}{\partial t^2}[\nabla\phi+\nabla\times\underline{A}]$$

Now $\nabla\times\nabla\times\nabla\times\underline{A} = \nabla(\nabla.(\nabla\times\underline{A})) - \nabla.\nabla(\nabla\times\underline{A}) = -\nabla^2(\nabla\times\underline{A})$

$$\Rightarrow (\lambda+2\mu)\nabla(\nabla^2\phi) + \mu\nabla\times\nabla^2\underline{A} - \rho\frac{\partial^2}{\partial t^2}\{\nabla\phi+\nabla\times\underline{A}\} = 0$$

$$\Rightarrow \nabla\left[(\lambda+2\mu)\nabla^2\phi - \rho\frac{\partial^2\phi}{\partial t^2}\right] + \nabla\times\left[\mu\nabla^2\underline{A} - \rho\frac{\partial^2\underline{A}}{\partial t^2}\right] = 0$$

both terms in [ ] must be zero

$$\therefore \quad (1)\ \nabla^2\phi - \frac{\rho}{\lambda+2\mu}\frac{\partial^2\phi}{\partial t^2} = 0 \qquad (2)\ \nabla^2\underline{A} - \frac{\rho}{\mu}\frac{\partial^2\underline{A}}{\partial t^2} = 0$$

WAVE EQUATIONS

wave equation (either vector or scalar) of form $\nabla^2\underline{\varepsilon} - \frac{1}{c^2}\frac{\partial^2\underline{\varepsilon}}{\partial t^2} = 0$ has solutions $\underline{\varepsilon} = \underline{E}e^{\pm i(\underline{k}.\underline{r}\pm\omega t)}$ where $\underline{k}$ is wavevector $k = |\underline{k}| = \frac{2\pi}{\lambda}$ and $\omega = ck$. c is the wave speed. [well this is certainly a valid solution in cartesian geometry].

c is a general property (i.e coordinate independent) and hence we can identify P WAVES (in seismic applications) with

$$c_p = \left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}}$$ (longitudinal, pressure waves hence scalar potential)

and S WAVES with $c_s = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}$ (shear, transverse waves hence with vector potential)

using $\lambda = \frac{3K\sigma}{1+\sigma}$ and $\mu = \frac{3K(1-2\sigma)}{2(1+\sigma)}$

Note absence of $\underline{f}_B$!!!

$$\Rightarrow c_p = \left(\frac{3K\sigma + 3K(1-2\sigma)}{\rho(1+\sigma)}\right)^{\frac{1}{2}} = \left(\frac{3K(1-\sigma)}{\rho(1+\sigma)}\right)^{\frac{1}{2}}$$

$\rightarrow$ In a particular geometry solve (1), (2) to yield ϕ, $\underline{A}$

$$c_s = \left(\frac{3K(1-2\sigma)}{2\rho(1+\sigma)}\right)^{\frac{1}{2}}$$

then $\underline{s}_p = \nabla\phi$, $\underline{s}_s = \nabla\times\underline{A}$.

For non cartesian geometries application of $\nabla^2\underline{A} = \nabla(\nabla.\underline{A}) - \nabla\times(\nabla\times\underline{A}) \Rightarrow (2)\ \nabla(\nabla.\underline{A}) - \nabla\times(\nabla\times\underline{A}) - \frac{\rho}{\mu}\frac{\partial^2\underline{A}}{\partial t^2} = 0$ is a better wave equation to attempt to solve.