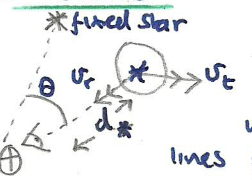
GRAVITATIONAL ASTROPHYSICS AND COSMOLOGY

1) Introduction   This course describes the tools required to model the large scale constituents of the cosmos and indeed the dynamics of the universe as a whole. On such large scales gravitation is the dominant fundamental force. Many objects within the cosmos are described by the extremes of known physics and hence we must use the General Relativistic theory of gravity. (GR)

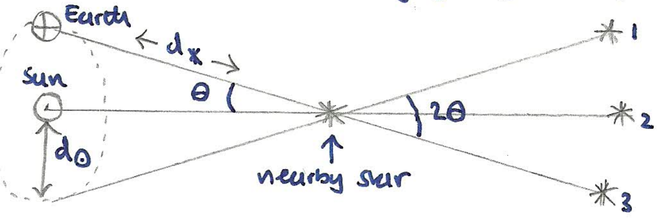
2) Measurement of cosmological parameters

* Velocities – measure radial velocity v_r from Doppler shifts of spectrum of light from star. (compare with characteristic emission lines of H, He on Earth).



– If star is spherically expanding gas cloud $v_r = v_t$. ∴ if $v_t = d_* \dot{\theta}$ can find d_* if measure $\dot{\theta}$.

* Parallax
Measure position of nearby star relative to fixed stars (far away) (1,2,3)



during extremes of Earth's orbit round the sun. (Measure angle θ).

Hence if know $d_\odot$: $d_* = \dfrac{d_\odot}{\sin\theta}$ (GAC1)

* Flux and luminosity.   Energy flux F_* of star radiating with luminosity L_* a distance d_* away is $F_* = \dfrac{L_*}{4\pi d_*^2}$ (GAC2). Use to find distances by comparing to flux from sources of known luminosity. i.e. cepheids, supernovae.

* Redshift   Quantifies Doppler shifts induced by radial motion of cosmological objects by Redshift z.

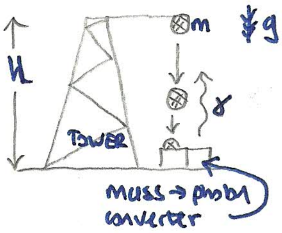
$$z = \frac{\lambda_{obs} - \lambda_{rest}}{\lambda_{rest}} \quad \text{(GAC3)}$$

Now for non relativistic motions $f_{obs} = f_{rest}\left(1 - \frac{v_r}{c}\right) \Rightarrow v_r \approx zc$ (GAC4)

Now ignoring PECULIAR motions of stars etc w.r.t 'cosmic fabric' universe is expanding isotropically with radial velocity $v_r = H_0 d$. (d = distance to point receding from observer). This is the HUBBLE LAW. $H_0 \sim 75\ \mathrm{kms^{-1}Mpc^{-1}}$. ($1\ \mathrm{pc} = 3.086 \times 10^{16}\ \mathrm{m}$).
– can use this as another distance measurer.

3) Need for GR and breakdown of SR in frames where gravitational force exists

– Don't include pressure as a source term for gravity in Newtonian theory. Pressure, energy density have same dimensions. For Earth atmosphere $\frac{p}{\rho c^2} \sim \frac{10^5}{1 \times 9 \times 10^{16}} \sim 10^{-12}$ (Neglect pressure contribution to energy density ⇒ mass via $E = mc^2$)
For radiation $P = \text{energy density}/3$. Cannot neglect!
– Newtonian theory fails to describe light propagation correctly in presence of mass. (light deflection by Sun is twice that predicted by Newtonian theory).
– Gravitational radiation a real possibility. * Einstein Tower 'Gedanken' Experiment – proves that photons moving in a gravitational field must be redshifted. Consider mass m dropped from height H in a uniform gravitational field of strength g. When it reaches the ground it is converted into a photon of energy $E = mc^2 + mgH$ (i.e 100% energy conversion). If it rises, unaffected by g to height h net gain of energy mgh in cycle. (could imagine 100% efficient photon → mass converter at top of tower). ⇒ Perpetual motion, violation of energy conservation.
∴ Photon energies must change by mgh as it rises to height h.



∴ $hf_0 - hf_H = mgH$. Using (GAC3) definition above: ($0 \Leftrightarrow$ rest, $H \Leftrightarrow$ obs) $\dfrac{f_0 - f_H}{f_0} = \dfrac{mgH}{hf_0}$

Now $mc^2 = hf_H \Rightarrow \dfrac{f_0 - f_H}{f_0} = \dfrac{gHhf_H}{c^2 h f_0} = \dfrac{f_H gH}{c^2 f_0}$. Now $c = f\lambda$

$$\therefore \frac{\frac{1}{\lambda_0} - \frac{1}{\lambda_H}}{\frac{1}{\lambda_0}} = \frac{\lambda_0}{\lambda_H}\frac{gH}{c^2} \Rightarrow \frac{\lambda_H - \lambda_0}{\lambda_0} = \frac{gH}{c^2}$$

So $z = \dfrac{gH}{c^2}$ or more generally $z = \dfrac{\Delta\phi}{c^2}$ (GAC5)
where ϕ is gravitational potential. Now if photons are redshifted and photon has period $p = \frac{1}{f}$, $p_0 \neq p_H$. If top and bottom of tower are mutually at rest this is in contradiction with special relativity. (For time dilation frames need to be in relative motion).
→ leads to STRONG and WEAK EQUIVALENCE principle. STRONG: "At any point in a gravitational field, then in a frame moving with the free fall acceleration at that point then all the laws of physics have their usual special relativistic form, except for gravity, which disappears LOCALLY". In this
WEAK: – Same but dynamics of TEST PARTICLES are SR in form in free fall frames. In this case gravitational attraction of particles within the free fall frame can be ignored.

* Free fall frames are found from GEOMETRY of spacetime {itself determined by the distribution of mass within spacetime} in fact free fall frames are GEODESICS of spacetime. (Provable from GR field equations).

4) The GR geometry of spacetime. Metrics, curvature and Geodesics

Note ds has spatial dimensions

The interval between two infinitesimally separated events in spacetime is ds. $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ (GAC 6) (in covariant, contravariant tensor formulation) $g_{\mu\nu}$ is the METRIC. In Euclidean space with cartesian geometry (i.e SR geometry) $ds^2 = c^2t^2 - dx^2 - dy^2 - dz^2$. Metric is derivable from GR field equations and mass distribution. * CURVATURE is an invariant quantity associated with the metric. For any coordinate transforms that change the values of the components $(g_{\mu\nu})$ of the metric, curvature is constant. i.e. for Euclidean space (well Minkowski space-time) transform from cartesian → polar does not change curvature.

For 2D metric can find curvature from $g_{\mu\nu}$ using Gauss' Theorema Egregium

if $g_{\mu\nu} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ curvature

$$K(x_1,x_2) = \frac{1}{2g_{11}g_{22}}\left\{-\frac{\partial^2 g_{11}}{\partial x_2^2} - \frac{\partial^2 g_{22}}{\partial x_1^2} + \frac{1}{2g_{11}}\left[\frac{\partial g_{11}}{\partial x_1}\frac{\partial g_{22}}{\partial x_1} + \left(\frac{\partial g_{11}}{\partial x_2}\right)^2\right] + \frac{1}{2g_{22}}\left[\frac{\partial g_{11}}{\partial x_2}\frac{\partial g_{22}}{\partial x_2} + \left(\frac{\partial g_{22}}{\partial x_1}\right)^2\right]\right\} \quad \text{(GAC 7)}$$

e.g. metric for a 2D manifold with 3D embedding = sphere surface (radius a): $ds^2 = a^2 d\theta^2 + a^2\sin^2\theta\, d\phi^2$

$\theta \equiv x_1 \quad \phi \equiv x_2$

$\Rightarrow g_{\mu\nu} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix} \Rightarrow K(\theta,\phi) = \frac{1}{a^2}$ = constant.

* GEODESICS. As mentioned above, free fall frames are geodesics of spacetime characterised (in differential form) by the metric.

Now, a geodesic between two events A and B is that curve joining A and B for which the interval along the curve is extremal. i.e. the integral $S_{AB} = \int_A^B ds$ is at a stationary value. (i.e. minimum or maximum ....) using (GAC 6) can write $S_{AB} = \int_A^B [g_{\mu\nu}dx^\mu dx^\nu]^{\frac{1}{2}}$

$$= \int_A^B \left[g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\right]^{\frac{1}{2}} ds = \int_A^B G(x^\mu, \dot{x}^\mu)\, ds$$

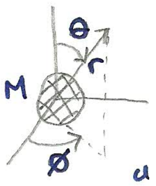
where $G = [g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu]^{\frac{1}{2}}$ $\dot{} \Rightarrow \frac{d}{ds}$.

Note from comparison of $\int_A^B G\, ds = S_{AB}$ and $S_{AB} = \int_A^B ds \Rightarrow G = 1$. (Though $\frac{\partial G}{\partial \xi} \neq 0$ in general where $\xi = x^\mu, \dot{x}^\mu$ ....) If S_{AB} is extremised $\Rightarrow$ G satisfies Euler-Lagrange equation:

$$\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) = \frac{\partial G}{\partial x^\mu} \quad \text{(GAC 8)}$$

[Note 4 equations for spacetime.]

5) The Swarzchild Metric. Geometry: spherical polar coordinates (for space) r, θ, ϕ centered on a spherically symmetric point mass M.



$$ds^2 = c^2\left(1 - \frac{2GM}{c^2r}\right)dt^2 - \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2 \quad \text{(GAC 9)}$$

can use gravitational redshift formula (GAC 5) to ≈ derive temporal part.

In linear g field Einstein Tower $\Rightarrow z = \frac{gH}{c^2} \Rightarrow z = \frac{\Delta\phi}{c^2}$ (GAC 5). Actually a general result. From Newtonian theory $\Delta\phi = \frac{GM}{r}$ in Swarzchild situation

Now $z = \frac{\lambda_{obs} - \lambda_{rest}}{\lambda_{rest}} = \frac{\frac{1}{\nu_{obs}} - \frac{1}{\nu_{rest}}}{1/\nu_{rest}} = \frac{t_{obs} - t_{rest}}{t_{rest}}$ where t = period of photons climbing in gravitational potential well.

ν or f denotes frequency.

Now $t_{rest} = c^{-1}ds$ i.e. proper time in rest frame of photon emitter.

∴ dropping suffix of obs $\Rightarrow z = \frac{c\,dt}{ds} - 1 \Rightarrow \frac{ds}{c\,dt} = \left(1 + \frac{GM}{rc^2}\right)^{-1}$ from (GAC 5).

(and taking limit $t \to dt$

$\Rightarrow ds^2 \approx c^2\left(1 - \frac{2GM}{rc^2}\right)dt^2$ to first order. QED.

Now spatial part of Swarzchild metric can be ≈ derived by assuming spatial interval of $ds^2 = f(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$. (From spherical symmetry of Swarzchild situation).

Now a 2D metric of form $g_{\mu\nu} = \begin{pmatrix} f(r) & 0 \\ 0 & r^2 \end{pmatrix}$ has curvature $K(r) = \frac{f'}{2f^2r}$ $\left(f' \equiv \frac{df}{dr}\right)$

applying (GAC 7). with reference to curvature of sphere surface $= \frac{1}{a^2} \Rightarrow$ expect dimensions of K to be $[L^{-2}]$. Expect in Swarzchild case $K \propto [M]^a[G]^b[c]^c[r]^d$. one combination is $K \propto \frac{GM}{c^2r^3}$. Hence let $K(r) = \frac{\alpha GM}{c^2r^3}$. (GAC 10) Hence $\frac{f'}{2f^2r} = \frac{\alpha GM}{c^2r^3}$

with boundary condition curvature → 1 (flat) as $r \to \infty \Rightarrow \lim_{r\to\infty} f = 1$

$\Rightarrow f(r) = \left(1 + \frac{2\alpha GM}{c^2r}\right)^{-1}$. Guessing $\alpha = -1$ yields spatial part of Swarzchild metric. Q.E.D

* Geodesic equation for Swarzchild metric. General motion of a test particle in Swarzchild spacetime is described by motion resulting from SR compatible forces superimposed upon geodesic trajectories of the Swarzchild spacetime. The latter can be computed from equation (GAC 8). (GAC 11)

Using $G^2 = g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$, in Swarzchild metric $G^2 = \left(1-\frac{2GM}{c^2 r}\right)c^2\dot{t}^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2$

choosing a suitable rotation of coordinates s.t plane of motion corresponds to $\theta = \frac{\pi}{2}$. (Angular momentum conservation $\Rightarrow$ motion planar). Applying (GAC 8) and noting $G = G(\dot{t}, r, \dot{r}, \dot{\phi})$ i.e. independent of t, ϕ $\Rightarrow \frac{\partial G}{\partial t} = \frac{\partial G}{\partial \phi} = 0$. Hence (GAC 8) $\Rightarrow$ $\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{t}}\right) = 0$ and $\frac{d}{ds}\left(\frac{\partial G}{\partial \dot{\phi}}\right) = 0$

$\Rightarrow \frac{\partial G^2}{\partial \dot{t}} = $ constant and $\frac{\partial G^2}{\partial \dot{\phi}} = $ constant. $\left\{ \frac{\partial G^2}{\partial \dot{t}} = 2G\frac{\partial G}{\partial \dot{t}}\right.$ and $G = 1$ $\therefore$ if $\frac{\partial G}{\partial \dot{t}} = $ constant so is $\left.\frac{\partial G^2}{\partial \dot{t}}\right\}$.

using (GAC 11) $\Rightarrow \left(1 - \frac{2GM}{c^2 r}\right)\dot{t} = k$ (GAC 12)

and $r^2\dot{\phi} = h$ (GAC 13) with k, h constants. [RECAST $\dot{}$ as $\frac{d}{d\tau}$, τ = proper time]

Using $G=1$ (GAC 11) becomes (after rearrangement and multiplication by test particle mass m)

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\phi})^2\left(1 - \frac{2GM}{rc^2}\right) - \frac{GMm}{r} = \frac{1}{2}mc^2(k^2-1) \quad \text{(GAC 14)}$$

From comparison to Lagrangian mechanics and assuming conservation laws hold in GR

$\Rightarrow$ h = specific angular momentum ; $\frac{J}{m}$ $\qquad k$ = Total energy/rest mass ; $\frac{E}{mc^2}$

Now $\frac{dr}{ds} = \frac{dr}{d\phi}\frac{d\phi}{ds} = \dot{\phi}\frac{dr}{d\phi} = \frac{h}{r^2}\frac{dr}{d\phi}$ from (GAC 13). Hence (GAC 14) becomes (with $\frac{d}{ds}$ terms substituted)

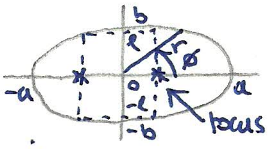
$$\Rightarrow \left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2} = c^2(k^2-1) + \frac{2GM}{r} + \frac{2GMh^2}{c^2r^3} \Rightarrow \left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{c^2}{h^2}(k^2-1) + \frac{2GMu}{h^2} + \frac{2GMu^3}{c^2}$$

using substitution $u \equiv \frac{1}{r}$

Differentiating w.r.t ϕ we arrive at $\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$ (GAC 14†)

Newtonian Gravity produces result $\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2}$ so GR simply adds an extra 'perturbative' term $\frac{3GMu^2}{c^2}$. ↖ unless it is larger than $\frac{GM}{h^2}$.

Newtonian solutions are ELLIPSES $r(\phi) = \frac{\ell}{1+e\cos\phi}$

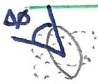


$e^2 = 1 - \frac{b^2}{a^2}$
$\ell = a(1-e^2)$ semi major axis a.

Given a, h, M can find ℓ, e.

$e^2 = -\frac{h^2}{aGM} + 1 \qquad \ell = \frac{h^2}{GM}$.

If GR term is small compared to $\frac{GM}{h^2}$ get slight precession of elliptical orbit. Substitution of Newtonian result $u = \frac{GM}{h^2}(1+e\cos\phi)$ into (GAC 14†) yields another linear ODE which results in modified result $u \simeq \frac{GM}{h^2}(1 + e\cos(\phi[1-\delta]))$ with $\delta = \frac{3(GM)^2}{h^2c^2} \ll 1$. Thus r values repeat on a cycle slightly larger than 2π.



$\Delta\phi$ shown between 'periastra' $= \frac{2\pi}{1-\delta}$

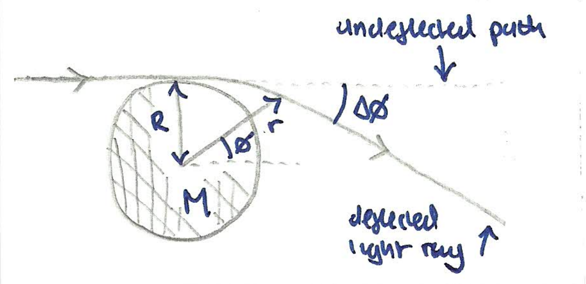
Using e, ℓ definitions above $\Rightarrow \Delta\phi = \frac{6\pi GM}{a(1-e^2)c^2}$ (GAC 15)

For Mercury's orbit around the sun $\Delta\phi = 43''$ / century.

* Bending of light

For light $c\,ds = 0$ $\therefore \dot{\phi} = \infty$ $\therefore$ (GAC 13) $\Rightarrow h = \infty$. Hence (GAC 14†) becomes

$\frac{d^2u}{d\phi^2} + u = \frac{3GMu^2}{c^2}$. Now solution to $\frac{d^2u}{d\phi^2} + u = 0$ is $u = \frac{\sin\phi}{R}$ (w.r.t diagram on the right – i.e. follows undeflected path).

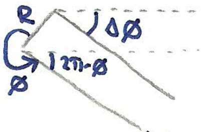


Substitution of this into above yields $\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2R^2}\sin^2\phi$

'Particular integral' for this is

$u_{PI} = \frac{3GM}{2c^2R^2}\left(1 + \frac{1}{3}\cos 2\phi\right)$ Hence 'guess' full solution to be

$u(\phi) = \frac{\sin\phi}{R} + \frac{3GM}{2c^2R^2}\left(1 + \frac{1}{3}\cos 2\phi\right)$ (GAC 16) Now in limit $r \to \infty$ ($\Rightarrow u \to 0$)



Σ, and ray path are $\parallel$. Clearly $\Delta\phi = 2\pi - \phi$ $\therefore \sin\phi = \sin(2\pi - \Delta\phi) \simeq -\Delta\phi$ if $\Delta\phi \ll 2\pi$. Similarly $\cos 2\phi = \cos(4\pi - 2\Delta\phi) \simeq 1$. $\therefore$ Rearranging above $\Rightarrow \Delta\phi = \frac{2GM}{Rc^2}$ (GAC 17)

(Note diagram different from notes)

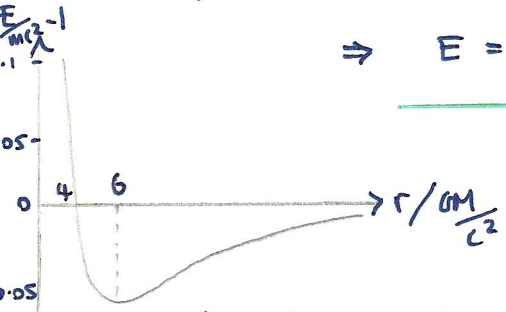
* Circular orbits and accretion discs in Swarzchild spacetime. Regarding (GAC 7) Swarzchild metric blows up when $\frac{2GM}{c^2 r}=1$ i.e. r = SWARZCHILD RADIUS, $R_S = \frac{2GM}{c^2}$. Incidently this is the radius of a spherical mass M which results in (Newtonian) escape velocity to equal the speed of light. Compact objects with $R < R_S$ are called BLACK HOLES. Matter accreting onto Black holes will usually do so in circular orbits since the infalling material nearly always has angular momentum and direct radial infall is less likely.

Referring to the orbit equation (GAC 14*) and noting for circular orbits $r =$ constant $\Rightarrow u =$ constant

$\Rightarrow u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2 \Rightarrow h^2 = \frac{GMr^2}{r - 3GM/c^2}$ (GAC 18). Now with $\dot r = 0$ $\left\{\dot r = \frac{h}{r^2}\frac{dr}{d\phi},\ \frac{dr}{d\phi}=0\right\}$

Energy equation, (GAC 14) becomes $\frac12 m\left(\frac{h}{r}\right)^2\left(1-\frac{2GM}{rc^2}\right) - \frac{GMm}{r} = \frac12 mc^2(k^2-1)$ $\left\{\text{using } r^2\dot\phi = h\right\}$

Substitution for h above (GAC 18) and recognising $k = \frac{E}{mc^2}$



$$\Rightarrow E = \frac{1-2GM/rc^2}{\sqrt{1-3GM/rc^2}}\,mc^2 \quad \text{(GAC 19)}$$

{In large r limit binomial expansion $\Rightarrow E \sim mc^2 - \frac{GMm}{2r}$

$E_{Newtonian} = \frac12 mv^2 - \frac{GMm}{r} + mc^2$

$\frac{mv^2}{r} = \frac{GMm}{r^2} \Rightarrow \frac12 mv^2 = \frac12\frac{GMm}{r}$

$\therefore E_{Newtonian} = mc^2 - \frac{GMm}{2r}$ ✓}

BOUND orbits for $E < mc^2$ (less energy than stationary particle at infinity)

(GAC 19) $\Rightarrow$ bound orbits for $\frac{4GM}{c^2} < r < \infty$.

Now suppose we have a small departure from a circular orbit i.e. need to use full form of (GAC 14) but assume $E \sim E_{circ}$ and h is the same as for a circular orbit.

$$\therefore \frac12 m(r\dot\phi)^2\left(1-\frac{2GM}{rc^2}\right) - \frac{GMm}{r} = \frac12 mc^2\left[\frac{E_{circ}^2}{m^2c^4}-1\right] \xrightarrow{(GAC\ 14)} \frac12 m\dot r^2 + \frac12 mc^2\left[\frac{E_{circ}^2}{m^2c^4}-1\right] = \text{const.}$$

$\left\{h = E_{circ}/mc^2\right\}$ Differentiating w.r.t s $\Rightarrow m\dot r\ddot r + \frac{E_{circ}}{mc^2}\frac{dE_{circ}}{dr}\dot r = 0$

Now $ds = c\,d\tau$ (τ = proper time) BUT in derivation of (GAC 14) we re cast ' as $\frac{d}{d\tau}$ to make physical correspondence of k, h with energy, angular momentum. {Algebraically we get rid of the c^2 since $\frac{d}{ds}\left(\frac{\partial G}{\partial \dot t}\right)=0 \Rightarrow \frac{d}{d\tau}\left(\frac{\partial G}{\partial(\frac{dt}{d\tau})}\right)=0$, similarly with $\frac{d}{ds}\left(\frac{\partial G}{\partial\dot\phi}\right)=0$. It probably would have made sense to define ds to have units of proper time.....}

So $m\ddot r + \frac{E_{circ}}{mc^2}\frac{dE_{circ}}{dr} = 0$. Get restoring force ($m\ddot r < 0$) if $\frac{dE_{circ}}{dr} > 0$ since $E_{circ}/mc^2 > 0$. i.e. STABLE orbits if $\frac{dE_{circ}}{dr} > 0$.

Using (GAC 19) this occurs for $r > \frac{6GM}{c^2}$. Innermost stable orbit (i.e. inner edge of accretion disc) most likely to occur at $r = 3R_S$.

Now for photons $\frac{d^2u}{d\phi^2} + u = \frac{3GMu^2}{c^2}$

circular orbits $\Rightarrow u = \frac{c^2}{3GM} \Rightarrow r = \frac{3GM}{c^2}$. i.e. can find accreted photons within inner edge of matter accretion disc. NOTE - accretion disc are very energy efficient. Removing particle from inner edge of disc liberates ~5% of rest mass energy! (>> nuclear processes). For KERR co-rotating Black Holes (GAC 19 + metric slightly modified) this can be as much as 40%. NOTE(2) SWARZCHILD should be spelt SCHWARZCHILD!!

6) The Friedmann-Robertson-Walker (FRW) metric - applying Extended Copernican principle that all positions are, on the largest scales in the universe, equivalent from the point of view of cosmology; we deduce a metric for the universe at large must embrace fundamental assumptions of ISOTROPY, HOMOGENEITY, EXPANSION. ISOTROPY assumes angular symmetry so in spherical polars express proper time interval $ds^2 = \frac{f(r)}{c^2}dr^2 + \frac{r^2}{c^2}(d\theta^2 + \sin^2\theta\, d\phi^2)$ {spatial part}

HOMOGENEITY $\Rightarrow$ k (curvature) should be constant spatially. Curvature of metric $g_{\mu\nu} = \begin{pmatrix} f(r) & 0 \\ 0 & r^2 \end{pmatrix}$ is $k(r) = \frac{f'}{2f^2 r}$ $\Rightarrow$ if k constant $f(r) = \frac{1}{1-kr^2}$. Now Expansion $\Rightarrow$ k varies with time i.e. let $k(t) = \frac{k}{R^2(t)}$ with $k = (-1, 0, 1)$ for +ve, flat, -ve curvature. R(t) is scale factor.

Note $1/r^2$ dimensions of k

spatial part of FRW metric becomes $c^2 ds^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) = R^2(t)\left\{\frac{d\sigma^2}{1-k\sigma^2} + \sigma^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right\}$
with $\sigma = r/R(t)$. Now objects with fixed σ have no "peculiar" motions of their own – strictly obey uniform expansion. These are called "fundamental observers". Fundamental observers are those with zero velocity relative to the frame defined by the Cosmic Microwave Background Radiation (CMBR) – see later. Now fundamental observers measure proper/"cosmic" time since if $d\sigma = d\theta = d\phi = 0$ spatial part of FRW metric $= 0$. Hence if t label corresponds to cosmic time full FRW metric is

$$ds^2 = dt^2 - \frac{R^2(t)}{c^2}\left\{\frac{d\sigma^2}{1-k\sigma^2} + \sigma^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right\} \quad \text{(GAC 20)}$$

Fundamental (or "comoving") observers follow geodesics of the FRW metric i.e. are freely falling frames and $\therefore$ all SR laws hold. (In their local frame). Can re write GAC 20 by redefining the radial coordinate. Let $d\chi = \frac{d\sigma}{\sqrt{1-k\sigma^2}} \Rightarrow \chi = \begin{cases} \arcsin\sigma & k=+1 \\ \sigma & k=0 \\ \operatorname{arcsinh}\sigma & k=-1 \end{cases}$

$\Rightarrow$ FRW metric is:

$$ds^2 = dt^2 - \frac{R^2(t)}{c^2}\left\{d\chi^2 + [S(\chi)]^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right\} \quad \text{(GAC 20)} \qquad S(\chi) = \begin{cases} \sin\chi & k=+1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$$

* Redshift of photons in FRW metric (compared to Schwarzschild metric).
In Schwarzschild metric we can compare the difference in time measured at spatial coordinate r,θ,ϕ between a clock sitting in a gravitational potential of a spherically symmetric mass and one sitting in empty space. i.e. $dr = d\theta = d\phi = 0 \Rightarrow d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right)dt^2$ $\quad d\tau =$ proper time interval.
Hence $\frac{t}{\tau} = \left(1-\frac{2GM}{rc^2}\right)^{-\frac{1}{2}}$. Now if t, τ are periods of photon waves $\Rightarrow$ redshift $z = \frac{-\lambda_{obs} + \lambda_{em}}{\lambda_{obs}}$
$= \frac{t-\tau}{\tau} = \frac{t}{\tau} - 1$ Hence $z + 1 = \left(1 - \frac{2GM}{rc^2}\right)^{-\frac{1}{2}}$ (GAC 21)

Now in FRW metric would like to calculate the redshift of a photon emitted by a fundamental observer (1) at cosmic time t_1 and recieved by another fundamental observer (0) (some distance away) at cosmic time t_0. Assuming radial motion ($d\theta = d\phi = 0$) from (GAC 20)
$ds^2 = dt^2 - \frac{R}{c^2}d\chi^2$. Now for photons $ds = 0$, so for propagating photons $\frac{d\chi}{dt} = \pm\frac{c}{R(t)}$ (GAC 21)
Hence w.r.t fundamental observer (0), (1) has χ coordinate $\chi_1 = \int_{t_1}^{t_0}\frac{c\,dt}{R(t)}$ (taking +ve root).
NOTE χ is fixed. (FUNDAMENTAL OBSERVER)
Now if interval between successive wavecrests at reception is δt_0 and at emmission is δt_1 can write χ_1 as

$$\chi_1 = \int_{t_1}^{t_0}\frac{c\,dt}{R(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0}\frac{c\,dt}{R(t)} \text{ since } \chi_1 \text{ is fixed.} \quad \int_{t_1+\delta t_1}^{t_0+\delta t_0}\frac{c\,dt}{R(t)} = \left(\int_{t_1+\delta t_1}^{t_1} + \int_{t_1}^{t_0} + \int_{t_0}^{t_0+\delta t_0}\right)\frac{c\,dt}{R(t)}$$

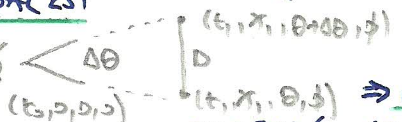
Hence $\left(\int_{t_1+\delta t_1}^{t_1} + \int_{t_0}^{t_0+\delta t_0}\right)\frac{c\,dt}{R(t)} = 0 \Rightarrow \int_{t_1}^{t_1+\delta t_1}\frac{c\,dt}{R(t)} = \int_{t_0}^{t_0+\delta t_0}\frac{c\,dt}{R(t)}$ Now if $R(t)$ changes little over the period of an EM wave (Good approx!!)

$\Rightarrow \frac{\delta t_0}{R(t_0)} = \frac{\delta t_1}{R(t_1)}$ $\quad\therefore$ Redshift $z = \frac{\delta t_0 - \delta t_1}{\delta t}$ $\quad\left(\nu = \frac{1}{\delta t},\ \lambda = \frac{c}{\nu} = c\delta t\right)$.

$\Rightarrow 1 + z = \frac{R(t_0)}{R(t_1)}$ (GAC 22)

* Distances in FRW spacetime – Proper distance (measured at particular cosmic time t_0) $d_{prop}(t_0,\chi) = \int_0^{\chi} R(t_0)\,d\chi'$
$\Rightarrow d_{prop}(t_0,\chi) = R_0\chi$. (GAC 23)

– Radar distance : reflection of light off object. $d_{radar} = \frac{1}{2}c(t_{recieve} - t_{send})$ (GAC 24) cosmic times.
(Above hopelessly useless) – luminosity distance. With analogy to Euclidean space, define distance measure $d_L = \left(\frac{L}{4\pi F}\right)^{\frac{1}{2}}$ relating d_L to luminosity and recieved flux from source d_L away. Now expression $F(t_0) = \frac{L(t_0)}{4\pi r^2(t_0)}$ is valid in FRW spacetime BUT observe sources emitted at cosmic time t_1. Now $r^2 = \sigma R(t_0) = [R_0 S(\chi_1)]^2$ by symmetry of observer, emitter coordinate systems.
Also photon frequencies are redshifted – reducing L by $\frac{1}{1+z}$ + discrete pulse intervals are reduced in rate by $\frac{1}{1+z}$. Hence $L(t_0) = \frac{L(t_1)}{(1+z)^2} \Rightarrow F = \frac{L(t_1)}{4\pi[R_0 S(\chi_1)]^2}\frac{1}{(1+z)^2}$
Hence $d_L(t_0,\chi_1) = \left(\frac{L(t_1)}{4\pi F(t_0)}\right)^{\frac{1}{2}} = R_0 S(\chi_1)(1+z)$ (GAC 25)



– Angular diameter distance $d_A = \frac{D}{\Delta\theta} = \frac{\text{assumed proper size}}{\text{measured angular diameter}}$ $\quad D = R(t_1)S(\chi_1)\Delta\theta$
$\Rightarrow d_A(t_0,\chi_1) = R(t_1)S(\chi_1)$
$= R(t_0)R(t_1)/R(t_0)\,S(\chi_1) = R(t_0)S(\chi_1)$ (GAC 26)

7) The cosmological Field Equations – further use of the FRW metric – need to determine k, $R(t)$ for use in FRW metric derived expressions above. From GR field equations can show:

$$\text{(GAC 26)} \quad \frac{\ddot{R}}{R} + \frac{4\pi G\rho}{3}(1+\varepsilon) - \frac{\Lambda}{3} = 0 \qquad \left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} - \frac{\Lambda}{3} = -\frac{kc^2}{R^2} \quad \text{(GAC 27)}$$

(Dynamical) (Energy)

Approximate derivations are given below: (GAC 26): (i) For photons $ds^2 = dt^2 - \frac{R^2(t)}{c^2}d\chi^2$ curvature of metric $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -R^2/c^2 \end{pmatrix}$ is $k(t) = -\frac{\ddot{R}}{R}$ (Note $\frac{1}{\text{time}^2}$ by new definition of interval being proper time).
(ii) Assume gravitating matter results in curvature $\Rightarrow k(t) \propto \rho(t)$ $\therefore$ let $k(t) = C\rho G^x c^y \Rightarrow x = 1, y = 0$ $\therefore -\frac{\ddot{R}}{R} = C\rho G$. By analogy to Newtonian gravity consider expanding sphere of radius R_χ and mass M. Acceleration of small mass at the boundary is $\ddot{R}_\chi = -\frac{GM}{R_\chi^2}$ (χ fixed – f.o.). Now $M = \frac{4}{3}\pi(R_\chi)^3\rho \Rightarrow \frac{\ddot{R}_\chi}{R_\chi} = -\frac{4}{3}\pi G\rho$ same term. This $\Rightarrow C = -\frac{4}{3}\pi$. (iii) Include pressure of radiation as gravitational radiation pressure $= \frac{1}{3}$ energy density. $\therefore$ if matter energy density is ρc^2 (at rest) Define ratio $\varepsilon = \frac{3p}{\rho c^2}$. $\varepsilon = 1 \Rightarrow$ radiation dominated, $\varepsilon \simeq 0 \Rightarrow$ matter dominated.
$\therefore$ let $\rho \to \rho(1+\varepsilon)$ to take into account matter and radiation. $\to$ yields full (GAC 26) if COSMOLOGICAL CONSTANT is also included. (Possibility of free spacetime curvature).

(GAC 27): (i) consider Newtonian energy of small mass on surface of expanding sphere of mass M (as used above). $\frac{1}{2}m\dot{R}^2 - \frac{GMm}{R} = mc^2 \Rightarrow \left(\frac{\dot{R}}{R}\right)^2 - \frac{2GM}{R^3} = \frac{c^2}{R^2} \Rightarrow \left(\frac{\dot{R}}{R}\right)^2 - \frac{8\pi G\rho}{3} = \frac{c^2}{R^2}$
Adding in $\frac{\Lambda}{3}$ and $-k$ from GR get complete expression.

* A note about ρ and radiation temperature. For matter $\rho \propto 1/R^3$.
For radiation energy density ρc^2 × volume V = constant by 1st law of thermo. Hence during an expansion $\rho \to \rho + d\rho$, $V \to V + dV \Rightarrow \rho c^2 V = \underbrace{(\rho+d\rho)c^2(V+dV)}_{\text{heat}} + \underbrace{p\,dV}_{\text{work}}$
$V = \frac{4}{3}\pi R^3$, $p = \frac{\rho c^2}{3} \Rightarrow V d\rho + \rho dV + \frac{1}{3}\rho dV = 0 \Rightarrow R^3 d\rho + \frac{4}{3}\rho 3R^2 dR = 0 \Rightarrow \rho \propto R^{-4}$
[Note since Black body energy density $\propto T^4$ and energy density $\propto \rho \Rightarrow T_{rad} \propto 1/R$]
Hence using ε above $\rho \propto R^{-(3+\varepsilon)}$ (GAC 28)

8) Cosmological Field Equation solutions, Static and Friedmann models. Using (GAC 26, 27):

STATIC solution

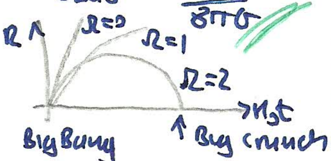
$\varepsilon = 0$ "Dust filled" $\Rightarrow \Lambda = \frac{kc^2}{R^2} = 4\pi G\rho$ Hence $k = +1$ for $R \in \mathbb{R}, \neq 0 \Rightarrow R = \frac{c}{\sqrt{4\pi G\rho}}$
$\dot{R} = \ddot{R} = 0$
[Note $\rho \sim 3\times10^{-28}\,\text{kgm}^{-3}$, average universe density $\Rightarrow R \sim 6\times10^{26}\text{m} \mid 20{,}000\,\text{Mpc}$]
↑ size of known universe.

FRIEDMANN models
$\Lambda = 0$

(GAC 26) $\Rightarrow \ddot{R} < 0 \;\forall t \Rightarrow R(t)$ convex towards t axis. Let $t = 0 \Rightarrow R = 0$ i.e. Big Bang. (GAC 27) + $\rho \propto R^{-(3+\varepsilon)}$ $\Rightarrow \dot{R}^2 + kc^2 \propto R^{-(1+\varepsilon)}$

when $t \to 0$, $R \to 0 \Rightarrow \dot{R} \to \infty$ $\therefore$ at early times $\dot{R}^2 \gg kc^2$ (or if $k = 0$)
$\Rightarrow \dot{R} \propto R^{-(1+\varepsilon)/2} \Rightarrow R \propto t^{2/(3+\varepsilon)}$. $\Lambda = \varepsilon = k = 0$ is EINSTEIN DE SITTER universe (EdS) and has $\frac{R(t)}{R(t_0)} = \left(\frac{t}{t_0}\right)^{2/3}$

* Hubble, deceleration and density parameters are often used to simplify (GAC 26, 27). (i) Hubble: $H(t) = \dot{R}/R$ (ii) deceleration: $q(t) = -\frac{\ddot{R}R}{\dot{R}^2}$ (iii) density $\Omega(t) = \frac{8\pi G}{3H^2(t)}\rho(t)$. For EdS $H^2(t) = 8\pi G\rho/3$
$\Rightarrow \rho_{0\,EdS} = \frac{3H_0^2}{8\pi G}$ 'critical density'.
Also get $q = \frac{\Omega}{2}(1+\varepsilon)$, $\Omega - 1 = \frac{kc^2}{H^2R^2}$ so since $(c/HR)^2 > 0$
$\Omega > 1 \Rightarrow k = +1$, $\Omega = 1 \Rightarrow k = 0$ and $\Omega < 1 \Rightarrow k = -1$.
closed, finite flat, Euclidean open, infinite



Ω controls geometry and dynamics.

For $\Omega > 1$ get "Big Crunch"!
$\Omega <$ get continuous expansion for all cosmic time.

{ Note also from (GAC 27), $\Lambda = 0$
{ $\Rightarrow H(z) = H_0(1+z)(1+\Omega_0 z)^{\frac{1}{2}}$

[use $\rho = \rho_0(1+z)^3$; $\Omega_0 = \rho_0 . 8\pi G/3H_0^2$ and evaluating (GAC 27) at present time $H_0^2(1-\Omega_0) = -kc^2/R_0^2$]

9) The Cosmic Background Radiation - Introduction. The CMBR is a surprisingly uniform sea of radiation $\sim 2.7K$ with anisotropy of order $\frac{\Delta T}{T} \sim 10^{-5}$ on scales of $10°$. COBE results indicate a near perfect Black body spectrum. In the early universe (when the CMBR was emitted) matter and radiation were in equilibrium so this would be expected. Not so in todays matter dominated universe...... In fact one can show that the Black Body spectrum is preserved using results derived above for FRW metric.

# of photons in comoving (constituents are f.o's) volumes are conserved, Hence:

$n_{\nu_1}(t_1) d\nu_1(t_1) V(t_1) = n_{\nu_0}(t_0) d\nu_0(t_0) V(t_0)$ where n_ν = # density of photons. t_1 is emission cosmic time.

Hence $n_{\nu_0}(t_0) = n_{\nu_1}(t_1) \frac{d\nu_1(t_1)}{d\nu_0(t_0)} \frac{V(t_1)}{V(t_0)}$ Now comoving volumes $\Rightarrow$ $V \propto R^3(t)$

and $1+z = R(t_0)/R(t_1)$ $\Rightarrow$ $d\nu \propto \frac{1}{R}$ $\left\{ \frac{\delta t_0}{R(t_0)} = \frac{\delta t_1}{R(t_1)} \text{ more useful} \right\}$

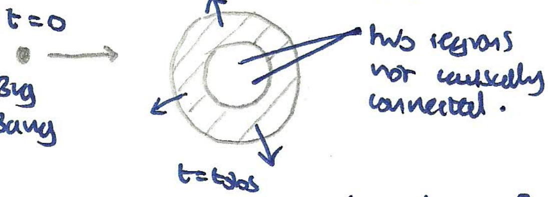
$\therefore$ $n_{\nu_0}(t_0) = n_{\nu_1}(t_1) \frac{R_0}{R_1} \left(\frac{R_1}{R_0}\right)^3 = \left(\frac{R_1}{R_0}\right)^2 n_{\nu_1}(t_1)$ Now $n_{\nu_1}(t_1)$ is black body

$\Rightarrow n_{\nu_1}(t_1) = \frac{8\pi \nu_1(t_1)^2}{c^3 (e^{h\nu_1(t_1)/kT_1} - 1)}$ so using $\frac{\nu_1}{\nu_0} = \frac{R_0}{R_1} \Rightarrow n_{\nu_0}(t_0) = \frac{8\pi \nu_0^2}{c^3} \left[\exp\left[\frac{h\nu_0}{kT_1 R_1/R_0}\right] - 1 \right]^{-1}$

i.e. $n_{\nu_0}(t_0)$ is also Black body BUT with $T_0 = T_1 R_1/R_0 = \frac{T_1}{1+z}$. So note $T \propto \frac{1}{R}$.

10) Particle Horizons and ages * A particle Horizon is the χ coordinate of the most distant particle/object that can be seen by a given observer at a given time. Let this $= \chi_p$. Using (GAC 21) $\chi_p = \int_0^{t_{obs}} \frac{c\,dt}{R(t)}$ for observer at cosmic time t_{obs} bearing for photons emitted at the Big Bang ($t=0$). For Friedmann models $R(t) \propto t^{\frac{2}{3+\epsilon}}$

$\Rightarrow \chi_p \propto \lim_{\delta \to 0} \left[\frac{-1}{1-\frac{2}{3+\epsilon}} t^{-\frac{2}{3+\epsilon}+1} \right]_\delta^{t_{obs}}$ so if $\frac{2}{3+\epsilon} < 1 \Rightarrow \chi_p$ is FINITE. i.e. points not in causal contact with observer.



* $\chi(z)$ we can compute $\chi(z)$ using a handy trick. $dz = d(1+z) = d\left(\frac{R_0}{R}\right) = -\frac{R_0}{R^2}\dot{R}\,dt = -(1+z)H(z)dt$

Now $\chi(t) = \int_0^t \frac{c\,dt'}{R(t')}$ and from above $\frac{dt}{R} = \frac{-dz}{(1+z)H(z)R}$

Now $1+z = \frac{R_0}{R}$ $\Rightarrow$ $\chi(t) = \int_0^z \frac{c\,dz'}{R_0 H(z')}$ (Note $z=0 \Rightarrow t$ so reversal of integration range cancels -ve sign).

Now for EdS universe: $R(t) = R_0\left(\frac{t}{t_0}\right)^{2/3} = (1+z)^{-1}R_0$

$\Rightarrow H(t) = \frac{\dot{R}}{R} = \frac{2}{3t}$ Hence $(1+z)^{-1} = \left(\frac{2}{3Ht_0}\right)^{2/3} \Rightarrow 1+z = \left(\frac{H}{H_0}\right)^{2/3} \Rightarrow H(z) = H_0(1+z)^{3/2}$

$\Rightarrow t = \frac{2}{3H}$, $t_0 = \frac{2}{3H_0}$ Hence for EdS $\chi(z) = \frac{2c}{R_0 H_0}\left\{1 - \frac{1}{\sqrt{1+z}}\right\}$ (GAC 28)

* Universe age estimates can be found from
- Globular cluster position in H.R. diagram
- Radioactive dating of uranium produced in supernovae
- Estimates of the Hubble constant, H_0. Note for EdS $R(t) = R_0\left(\frac{t}{t_0}\right)^{\frac{2}{3}}$ $\therefore$ $H = \frac{\dot{R}}{R} = \frac{\frac{2}{3}t^{-\frac{1}{3}}R_0 t_0^{-\frac{2}{3}}}{R_0 t_0^{-\frac{2}{3}} t^{\frac{2}{3}}}$
$= \frac{2}{3}t^{-1}$ $\therefore$ $t_0 = \left(\frac{3}{2}H_0\right)^{-1}$ $\Rightarrow$ Age of present epoch $= \frac{2}{3H_0}$. Find H_0 from - 'distance ladder' using cepheid variables - Type Ia Supernovae "Standard candles" - help measure q_0 Hubble Space Telescope - Sunyaev-Zel'dovich effect {Slight diminution of CMB temp in Rayleigh-Jeans region of spectrum due to inverse compton scattering off hot galactic clusters - comparison with X ray spectrum can yield H_0!} - Gravitational lensing {image of quasar flickers - time delay between flickers is $\propto \frac{1}{H_0}$ (!) If mass distribution which gives rise to lensing arrangement is known can find H_0}.

$\Rightarrow H_0 \sim 75\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ ($\pm \sim 20\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$).

11) Cosmological constant not zero! cosmological field equations (GAC 26, 27) with $\epsilon = 0$ (i.e. matter dominated) Define $\Omega_m = \frac{8\pi G\rho}{3H^2}$. $\therefore$ (GAC 27) $\Rightarrow H^2 - \Omega_m H^2 - \frac{\Lambda}{3} = -\frac{kc^2}{R^2}$

Write $\frac{\Lambda}{3H^2} = \Omega_\Lambda$ $\Rightarrow$ $1 - \Omega_m - \Omega_\Lambda = -\frac{kc^2}{R^2H^2}$ $\Rightarrow$ $1 + \frac{kc^2}{R^2H^2} = \Omega_m + \Omega_\Lambda$

Hence $\Omega_m + \Omega_\Lambda > 1 \Rightarrow$ closed $k=+1$
$\Omega_m + \Omega_\Lambda = 0 \Rightarrow$ flat $k=0$
$\Omega_m + \Omega_\Lambda < 1 \Rightarrow$ open $k=-1$

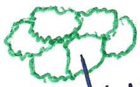
Now $\left(\frac{R}{\dot{R}}\right)^2 \cdot$ (GAC 26) $\Rightarrow \frac{\ddot{R}R}{\dot{R}^2} + \frac{4\pi G\rho}{3H^2} - \frac{\Lambda}{3H^2} = 0$

$\Rightarrow -q + \frac{\Omega_m}{2} - \Omega_\Lambda = 0 \Rightarrow q = \frac{\Omega_m}{2} - \Omega_\Lambda$

Ω_Λ could reverse acceleration if large enough

2) Mass distribution in the universe: clusters, voids and Dark matter

Heirarchy: Clusters | Voids $\rightarrow$ Galaxies $\rightarrow$ Star systems $\rightarrow$ planets | moons



Temperatures of cluster gas - Virial theorem result. $k_B T \propto GM_{cluster} m_p / R_{cluster}$

voids "soap bubbles?" elliptical spiral irregular

See "Structure and Evolution of Stars" course.

For galaxies $10^8 M_\odot < M < 10^{12} M_\odot$

spiral L $10^8 L_\odot < L < 10^{12} L_\odot$

$\frac{M}{L}\Big|_{\text{typical galaxy}} \sim 1-10 \ \frac{M_\odot}{L_\odot}$

* Dark Matter

- From above typical galaxy has mass $M_* \sim 10^{10} M_\odot$. Mass estimate from virial theorem $[\frac{1}{2}v^2 \sim GM_{galaxy}/R$ where v is radial galaxy velocity (or stars within it)$]$ >> this. $\Rightarrow$ MISSING MASS. Possible explanation is significant thermal energy released during gravitational collapse resulting in cluster formation. However does not account for all the missing mass.
- Measurements of star rotational velocity about galactic centre do not agree with Keplers law if most of galactic mass is distributed in proportion to the visible light emitted. $\Rightarrow$ missing mass
- large elliptical galaxies: mass at large radii inferred from temperature of hot interstellar medium seen in X ray spectrum. $\Rightarrow$ much higher than can be accounted for by detectable stars alone.

$\Rightarrow$ Assume galaxies are surrounded by massive halos of dark matter.

[Most convincing argument is if universe is spatially flat $\Rightarrow \Omega = 1 \Rightarrow \rho \sim \rho_{0 EdS} \sim \frac{3H_0}{8\pi G}$

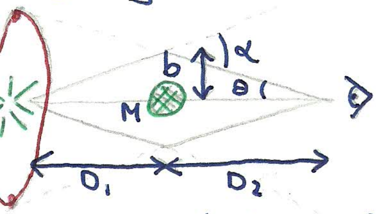
$\Rightarrow \frac{M}{L} \sim 1500 \, M_\odot / L_\odot$ $\Rightarrow$ most of universe is dark matter].

* What is dark matter? Possible candidates are: (1) Very low mass stars $< 0.08 M_\odot$ (2) Black holes (3) Neutrinos (only few eV/c^2 is needed since neutrino density $> 10^9$ Baryon density) (4) Weakly interacting massive particles (WIMPS) $> 10 \, eV/c^2$

3) Gravitational lensing. As shown at the foot of GAC(3) light is 'bent' by gravitational fields. In Swarzchild metric (GAC 17) $\Rightarrow$ deflection angle $\Delta\phi = 2GM/Rc^2$. This in fact wrong by a factor of 2. (?) Let R become 'collision parameter' b and $\Delta\phi \rightarrow \alpha$.

Hence $\alpha = \frac{4GM}{bc^2}$ (GAC 30) In terms of Swarzchild radius $\alpha = 2R_S/b$

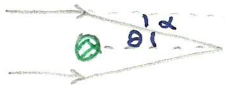
A galaxy behind a large mass is $\therefore$ observed as a ring. "Einstein ring".



Angular radius of ring $\theta = \arctan\left(\frac{b}{D_2}\right) \approx \frac{b}{D_2}$

If $D_1 \gg D_2$ then light arrives at lens almost //.

$\therefore \alpha \sim \theta \Rightarrow b \cong \frac{4GM}{\theta c^2}$



$\therefore \theta \approx \sqrt{\frac{4GM}{D_2 c^2}}$ (GAC 31) Note $\theta \propto \sqrt{M}$.

(Note typical resolution θ of a ground based telescope is $\gtrsim 1$ arcsec)

* microlensing - individual stars in the lensing galaxy can perturb light rays by a few microarcsec. Only detectable if star has transverse velocity (v) on a timescale $\sim \frac{b}{v}$.

* Surface density lensing criterion

- For a mass M of radius R to lens $R < b$. If light rays are // at lens $\theta \sim \frac{b}{D_2}$ and $\theta = \alpha \Rightarrow \frac{b}{D_2} = \frac{4GM}{bc^2} \Rightarrow b^2 = \frac{4GM}{c^2}D_2$. So if $b^2 > R^2 \Rightarrow R^2 < \frac{4GM}{c^2}D_2$

Now define surface mass density $\Sigma = \frac{M}{\pi R^2}$. From above $\frac{M}{\pi R^2} > \frac{c^2}{4\pi G D_2} \Rightarrow \Sigma > \frac{c^2}{4\pi G D_2}$ (GAC 32)

For cosmic distances $D_2 \sim \frac{c}{H_0} \Rightarrow \Sigma \sim 1 \, kg m^{-2}$ i.e. typical hand density.

4) Summary of universe Evolution

* Transition from radiation dominated to matter dominated Friedmann, flat (or early) universe.

As shown above $\rho_{matter} \propto \frac{1}{R^3}$, $\rho_{rad} \propto \frac{1}{R^4}$ $\Rightarrow \rho(t) = \rho_{0,m}\left(\frac{R_0}{R}\right)^3 + \rho_{0,rad}\left(\frac{R_0}{R}\right)^4$

EdS today i.e. $\rho_{0,m}/\rho_{0,rad} \gg 1$. Get transition from matter - radiation when

$\frac{R_0}{R} > \frac{\rho_{0,m}}{\rho_{0,rad}} \Rightarrow z > \frac{\rho_{0,m}}{\rho_{0,rad}} - 1$ (GAC 33) $\rho_{0,m} \sim \frac{3H_0^2}{8\pi G}$, $\rho_{0,rad} \sim 4.5 \times 10^{-31}\left(\frac{T}{2.7K}\right)^4 \, kg m^{-3}$

$H_0 \sim 75 \, km s^{-1} Mpc^{-1} \sim 2.43 \times 10^{-18} s^{-1}$. $\Rightarrow z \sim 10^5$.

* Cosmic time, temperature relation for radiation dominated universe

If $k=0$ (or term ignorable), $\Lambda=0$: (GAC 27) for $\varepsilon=1 \Rightarrow \left(\frac{\dot R}{R}\right)^2 = \frac{8\pi G\rho}{3}$. Now if radiation dominated $\Rightarrow T \propto \frac{1}{R} \Rightarrow \frac{\dot R}{R} = -\frac{\dot T}{T}$. Hence re-write above as $\left(\frac{\dot T}{T}\right)^2 = \frac{8\pi G a T^4}{3c^2}$

using $\rho c^2 = aT^4$ (Black body energy density) $\Rightarrow T(t) = \left(\frac{3c^2}{32\pi G a}\right)^{\frac14} t^{-\frac12}$ (GAC 34)

$\Rightarrow t = 2.3\left(\frac{10^{10}K}{T}\right)^2 s$. (and if $E = k_BT$)

$\Rightarrow t = \left(\frac{1.3\,MeV}{E}\right)^2 s$. So 'recombination' occurs at $\sim 4000K \Rightarrow t_{recom} \sim 500{,}000$ years

[Not strictly valid since matter dominated....] radiation $\Rightarrow$ matter crossover at $z \sim 10^5$.

Since $T \propto \frac{1}{R} \Rightarrow \frac{T_{crossover}}{T_0} = \frac{R_0}{R_{crossover}} \Rightarrow T_{crossover} = T_0(1+z_{crossover}) \sim 2.7 \times 10^5 K$

$\Rightarrow t_{crossover} \sim 100$ years. GUT unification at $\sim 10^{14}$ GeV $\Rightarrow t_{GUT} \sim 2 \times 10^{-34}$ seconds.

* Correction to above result to take into account lepton, antilepton pairs.

Lepton-antilepton pairs produced in the early universe annihilate to add to overall radiation density.

Now a massless particle with g spin states has energy density $\rho c^2 = \frac{g}{(2\pi\hbar)^3}\frac{4\pi}{c^3}\int_0^\infty \frac{E^3\,dE}{e^{E/kT} \pm 1}$

+1 $\Rightarrow$ Fermions. In units of $\frac{\rho c^2}{aT^4}$:
−1 $\Rightarrow$ Bosons.

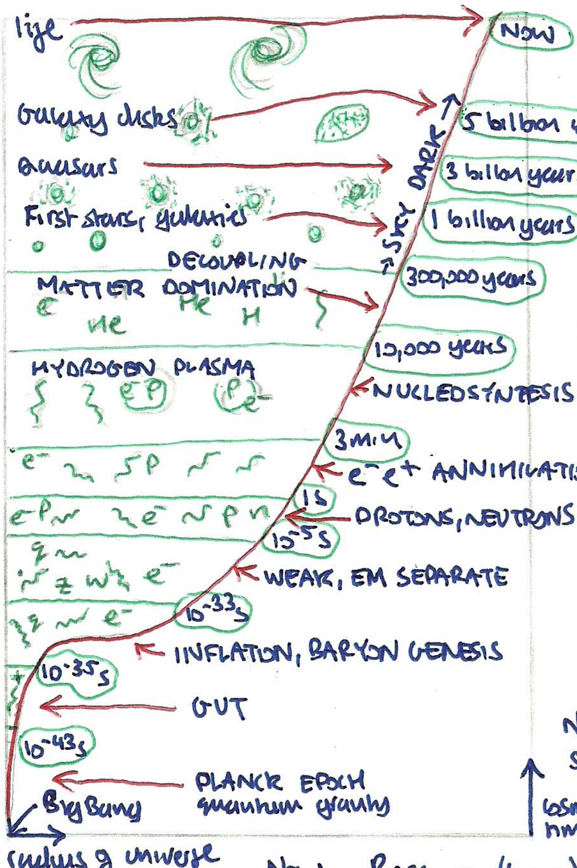
photons	1
electrons	7/8
positrons	7/8
e neutrinos	7/16 ($\uparrow, -1$)
μ "	7/16 ($\uparrow, -1$)
τ "	7/16 .....

$\therefore$ a in GAC 34 should be

$$fa = \left(1 + (2 + \#\,\text{neutrinos species})\frac{7}{8}\right)a$$

i.e. $t = 2.3 f^{-\frac12}\left(\frac{10^{10}K}{T}\right)^2 s$. (GAC 35).

* Universe chronology



* Horizon problem - need for inflation

Angular separation of objects in causal contact since Big Bang $\sim 1°$. Observe uniform CMBR (to 1 in 10^5) on much larger scales.

[Proof of angular separation $\sim 1°$: Particle horizon at recombination

$\chi_{rec} = \int_0^{t_{rec}} \frac{c\,dt}{R(t)}$. $\Lambda = k = 0$, $\varepsilon = 1$ $\Rightarrow R = R_{rec}\left(\frac{t}{t_{rec}}\right)^{\frac12} \Rightarrow \chi_{rec} = \frac{2ct_{rec}}{R_{rec}}$

Proper distance separating points on horizon $= R_{rec}\chi_{rec}$

$\Rightarrow d_{rec} = 2ct_{rec}$

Now $d_{rec} = R_{rec}\,S(\chi_h)\Delta\theta$ (FRW)

$\Rightarrow \Delta\theta = \frac{d_{rec}}{S(\chi_h)}\frac{R_0}{R_{rec}R_0} = \frac{d_{rec}(1+z_{rec})}{R_0 S(\chi_h)}$ $\quad \int_{t_{rec}}^{t_0}\frac{c\,dt}{R(t)}$ for $\varepsilon = 0$

Now $k=0 \Rightarrow S(\chi_h) = \chi_h$. $R_0\chi_h = 2ct_{rec} + 3c(t_0 - t_{rec}) \approx 3ct_0$ since $t_0 \gg t_{rec}$. Now $H_0 = \frac{2}{3t_0} \Rightarrow 3t_0 = \frac{2}{H_0} \Rightarrow \chi_h = \frac{2c}{H_0R_0}$

$\therefore \Delta\theta = d_{rec}(1+z_{rec})\frac{H_0}{2c} = t_{rec}(1+z_{rec})H_0$

Now $\frac{R_{rec}}{R_0} = \left(\frac{t_{rec}}{t_0}\right)^{2/3}$ and $t_0 = \frac{2}{3H_0} \Rightarrow$ (using $1+z_{rec} = \frac{R_0}{R_{rec}}$) $(1+z)^{-1} = \left(\frac{3H_0t_{rec}}{2}\right)^{\frac23}$

$\Rightarrow t_{rec}H_0 = \frac23(1+z_{rec})^{-\frac32}$. Hence $\Delta\theta = \frac23(1+z_{rec})^{-\frac12}$. $z_{rec} \sim \frac{4000K}{2.7K} \sim 1400$

$\Rightarrow \Delta\theta \sim 1°$ QED]

Inflation solution? Consider vacuum expansion: if $\rho = 0$, $\Lambda \neq 0$ (GAC 26) $\Rightarrow \frac{3\ddot R}{R} = \Lambda$

$\Rightarrow R(t) = A\exp\left[t\sqrt{\Lambda/3}\right] + B$ (A, B fit R at beginning and end points of expansion)

* Fluctuations

– structure seen today is thought to grow from density fluctuations. Consider sphere of radius r containing mass M. If EdS at critical density $\frac{GM}{r} - \frac{v^2}{2} = 0$ (I) ← v is surface velocity

$\Rightarrow$ Perturbation: $\frac{GM}{r} - \frac{(v-\Delta v)^2}{2} = \Delta E$. Ignoring $\Delta v^2 \Rightarrow v\Delta v = \Delta E$ using (I). If M fixed (I) $\Rightarrow v \propto r^{-\frac12}$

$\therefore \frac{\Delta v}{v} = \frac{\Delta E}{v^2} \propto r$ since ΔE is fixed. $\therefore \frac{\Delta v}{v} \propto R(t)$. Now (I) $\Rightarrow v^2 \propto \frac{M}{r}$, $M \propto \rho r^3 \Rightarrow v^2 \propto \rho r^2$

r fixed so $2v\Delta v \propto r^2\Delta\rho$ $\therefore \frac{\Delta v}{v} \propto \frac{\Delta\rho}{v^2}r^2 \Rightarrow \frac{\Delta v}{v} \propto \frac{\Delta\rho}{\rho}$ since $v^2 \propto \rho r^2$. $\therefore \frac{\Delta\rho}{\rho} \propto R(t)$ (GAC 36)

Now for matter dominated universe radius of particle horizon at cosmic time t is $r = R(t)\chi \propto Rt^{1/3}$. For radiation dominated $r \propto Rt^{\frac12}$

Now mass $M \propto \rho r^3$. (contained within particle horizon) For matter $\rho \propto \frac{1}{R^3}$ and radiation $\rho \propto \frac{1}{R^4}$ ∴

matter : $M \propto t$ $\quad (R^{-3}(Rt^{1/3})^3 = t)$

radiation : $M \propto \frac{t^{3/2}}{R}$ $\quad (R^{-4}(Rt^{1/2})^3 = t^{3/2}/R)$

Now for matter $R \propto t^{2/3}$, radiation $R \propto t^{1/2}$. ∴

matter : $M \propto R^{3/2} \Rightarrow R \propto M^{2/3}$

radiation : $M \propto R^2 \Rightarrow R \propto M^{1/2}$

Hence fluctuations are: radiation : $\frac{\delta\rho}{\rho} \propto M^{\frac{1}{2}}$

matter : $\frac{\delta\rho}{\rho} \propto M^{\frac{2}{3}}$

* Observations that help validate Big Bang model

- Helium abundance not less than 23%
- Black body CMBR
- Stable neutrinos are very low mass $< 10\,eV/c^2$

* Big Questions

- why matter, antimatter asymmetry in favour of matter?
- what resulted in n_{baryon}/n_{γ} rather of $\sim 10^9$
- why is $\Omega \sim 1, \Rightarrow k = 0$ (i.e. a flat universe)
- why $\alpha_G = \frac{GM_p^2}{\hbar c} \ll$ than for other fundamental forces i.e. EM $\frac{1}{137}$.

* Planck quantities – describe instant after big bang?

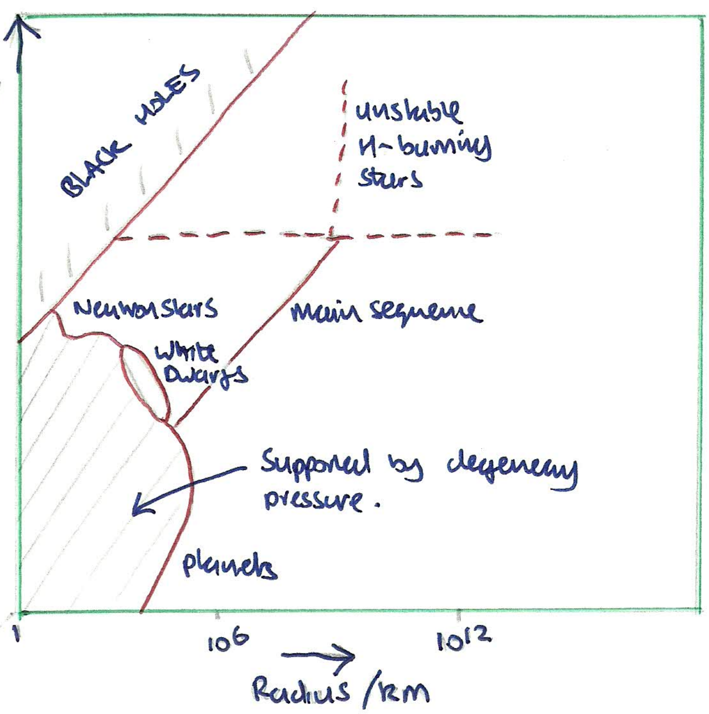
$$t_{planck} = \left(\frac{G\hbar}{c^5}\right)^{\frac{1}{2}} \sim 5\times10^{-44}\,s$$

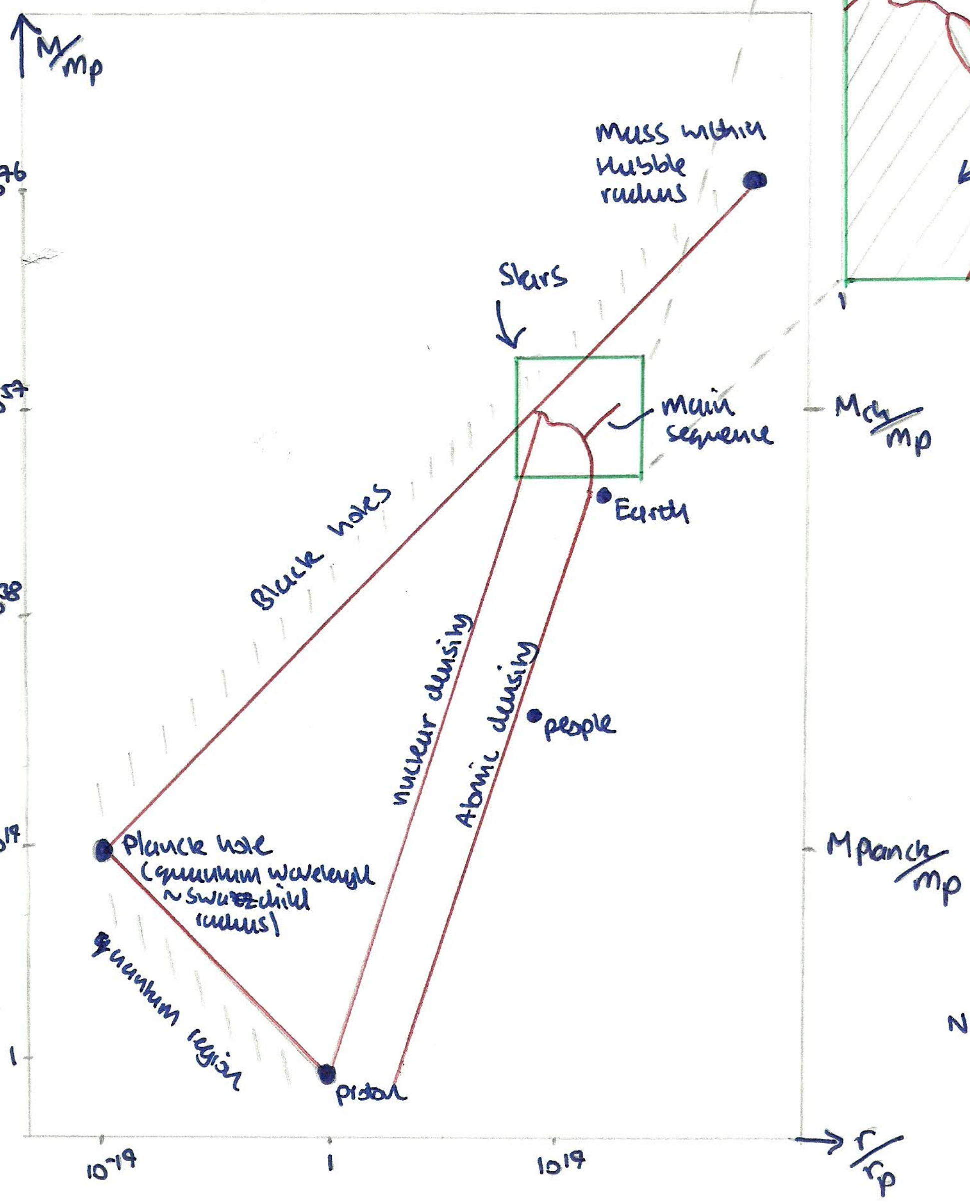
$$kT_{planck} = \left(\frac{c^5\hbar}{G}\right)^{\frac{1}{2}} \sim 1.2\times10^{19}\,GeV$$

$$M_{planck} = \left(\frac{\hbar c}{G}\right)^{\frac{1}{2}} \sim 10^{-8}\,kg$$

[Planck length $\ell_{planck} = ct_{planck}$]

15) Overall mass, radius diagram for structures within the universe





[The Stars box is described in the "Structure and Evolution of Stars" course].

M_p = proton mass

M_{ch} = Chandrasekhar mass $\sim 1.4\,M_{\odot}$

Note $10^{19} = \alpha_G^{-\frac{1}{2}}$