

STEP II. 2000

1/ let N be an integer greater than 1. Define unit fraction $\frac{1}{N}$
 we want to prove that $\frac{1}{N} = \frac{1}{a} + \frac{1}{b}$ where $a \neq b$ and
 a, b are both integers > 1 , i.e. $\frac{1}{a}$ and $\frac{1}{b}$ are both distinct
 unit fractions.

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

Guess:

$$\boxed{\frac{1}{N} = \frac{1}{N+1} + \frac{1}{N(N+1)}} \quad (1.1)$$

$$= \frac{N}{N} \frac{1}{N+1} + \frac{1}{N(N+1)}$$

$$= \frac{N+1}{N(N+1)}$$

$$= \frac{1}{N} \checkmark$$

write $\frac{1}{N} = \frac{1}{a} + \frac{1}{b} = \frac{b}{ab} + \frac{a}{ab}$

$$\frac{1}{N} = \frac{b+a}{ab}$$

$$\boxed{ab = N(a+b)} \quad (1.2)$$

Now consider $(a-N)(b-N) = ab - Nb - Na + N^2$
 $= ab - N(a+b) + N^2$

using (*) $ab = N(a+b)$ $\therefore \boxed{(a-N)(b-N) = N^2}$ (1.3)
 (So $ab - N(a+b) = 0$)

let N be a prime number, i.e. only factors are N and 1
 \therefore factors of $(a-N)(b-N)$ are $\boxed{1, N \text{ or } N^2}$

options are (i) $\left. \begin{matrix} a-N = 1 \\ b-N = N^2 \end{matrix} \right\} \Rightarrow \boxed{\begin{matrix} a = N+1 \\ b = N^2 + N = N(N+1) \end{matrix}}$

(ii) $\left. \begin{matrix} a-N = N \\ b-N = N \end{matrix} \right\} \Rightarrow \boxed{\begin{matrix} a = 2N \\ b = 2N \end{matrix}}$ } Not allowed
since $a \neq b$

(iii) $\left. \begin{matrix} a-N = N^2 \\ b-N = 1 \end{matrix} \right\} \Rightarrow \boxed{\begin{matrix} a = N^2 + N = N(N+1) \\ b = N+1 \end{matrix}}$

So (i) & (iii) are essentially the same and (ii) is not allowed since $a \neq b$

Hence
$$\boxed{\frac{1}{N} = \frac{1}{N+1} + \frac{1}{N(N+1)}} \quad (1.4)$$

is the only way of expressing $\frac{1}{N}$ as the sum of two distinct unit fractions, if N is prime

Now consider a fraction of the form $\frac{2}{N}$ where N is prime and $N > 2$. This means N must be odd, so can be written as

$$\boxed{N = 2n-1}$$

using (1.4)
$$\frac{2}{N} = \frac{2}{N+1} + \frac{2}{N(N+1)}$$

Since $N > 2$
(i.e. $N \geq 3$)

$$\boxed{n \geq 2}$$

$$\frac{2}{2n-1} = \frac{2}{2n} + \frac{2}{(2n-1)(2n)}$$

$$\boxed{\frac{2}{2n-1} = \frac{1}{n} + \frac{1}{n(2n-1)}} \quad (1.5)$$

Now is there, as before only one way of expressing $\frac{2}{N}$ as $\frac{2}{N} = \frac{1}{a} + \frac{1}{b}$ where N is prime, $N \geq 3$ and a, b are integers and $a \neq b$?

Consider
$$\begin{aligned} (2a-N)(2b-N) &= 4ab - 2bN - 2aN + N^2 \\ &= 4ab - 2N(a+b) + N^2 \end{aligned} \quad (1.6)$$

Now
$$\frac{2}{N} = \frac{1}{a} + \frac{1}{b} \Rightarrow \frac{b+a}{ab} = \frac{2}{N} \Rightarrow \begin{aligned} N(b+a) &= 2ab \\ 2N(b+a) &= 4ab \end{aligned}$$

\therefore In (1.6)
$$\boxed{(2a-N)(2b-N) = N^2} \quad (1.7)$$

So since N is prime, $N \geq 3$ and $a \neq b$

W.L.O.G
$$2a-N = N^2, \quad 2b-N = 1 \Rightarrow \boxed{a = \frac{1}{2}N(N+1)}$$

$$b = \frac{1}{2}(N+1)$$

So $\frac{2}{N} = \frac{2}{N(N+1)} + \frac{2}{N+1}$

and hence since we can write

$$N = 2n - 1 \quad [n \text{ integer } \geq 2]$$

$$\frac{2}{N} = \frac{1}{n} + \frac{1}{n(2n-1)}$$

← which is definitely the sum of the sum of two distinct unit fractions, since $n \neq n(2n-1)$ if $n \geq 2$

is the expression in (1.5)

