STEP II. 2000

1/ let N be an integer greater than 1. Define unit fraction $\frac{1}{N}$

we want to prove that $\frac{1}{N} = \frac{1}{a} + \frac{1}{b}$ where $a \neq b$ and

a, b are both integers > 1, ie $\frac{1}{a}$ and $\frac{1}{b}$ are both distinct

unit fractions.

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

Guess: $\boxed{\frac{1}{N} = \frac{1}{N+1} + \frac{1}{N(N+1)}}$ (1.1)

$$= \frac{N}{N}\frac{1}{N+1} + \frac{1}{N(N+1)}$$

$$= \frac{N+1}{N(N+1)}$$

$$= \frac{1}{N} \checkmark$$

Write $\frac{1}{N} = \frac{1}{a} + \frac{1}{b} = \frac{b}{ab} + \frac{a}{ab}$

$$\frac{1}{N} = \frac{b+a}{ab}$$

$\therefore$ $\boxed{ab = N(a+b)}$ (1.2)

Now consider $(a-N)(b-N) = ab - Nb - Na + N^2$

$$= ab - N(a+b) + N^2$$

using (*) $ab = N(a+b)$ $\therefore$ $\boxed{(a-N)(b-N) = N^2}$ (1.3)

(so $ab - N(a+b) = 0$)

Let N be a prime number, ie only factors are N and 1

$\therefore$ factors of $(a-N)(b-N)$ are $\boxed{1, N \text{ or } N^2}$

options are

(i) $\left.\begin{array}{l} a-N = 1 \\ b-N = N^2 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} a = N+1 \\ b = N^2+N = N(N+1) \end{array}}$

(ii) $\left.\begin{array}{l} a-N = N \\ b-N = N \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} a = 2N \\ b = 2N \end{array}}$ {Not allowed since $a \neq b$}

(iii) $\left.\begin{array}{l} a-N = N^2 \\ b-N = 1 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} a = N^2+N = N(N+1) \\ b = N+1 \end{array}}$

So (i) & (iii) are essentially the same and (ii) is not allowed since $a \neq b$

Hence $\boxed{\frac{1}{N} = \frac{1}{N+1} + \frac{1}{N(N+1)}}$ is the only way of expressing $\frac{1}{N}$ as the sum of two distinct unit fractions, if N is prime

(1.4)

Now consider a fraction of the form $2/N$ where N is prime and > 2. This means N must be odd, so can be written as $\boxed{N = 2n-1}$

Since $N > 2$ (i.e. $N \geq 3$) $\boxed{n \geq 2}$

using (1.4) $\frac{2}{N} = \frac{2}{N+1} + \frac{2}{N(N+1)}$

$\therefore \frac{2}{2n-1} = \frac{2}{2n} + \frac{2}{(2n-1)(2n)}$

$$\boxed{\frac{2}{2n-1} = \frac{1}{n} + \frac{1}{n(2n-1)}} \quad (1.5)$$

Now is there, as before only one way of expressing $\frac{2}{N}$ as $\frac{2}{N} = \frac{1}{a} + \frac{1}{b}$ where N is prime, ≥ 3 and a, b are integers > 1 and $a \neq b$?

Consider $(2a-N)(2b-N) = 4ab - 2bN - 2aN + N^2$
$= 4ab - 2N(a+b) + N^2$ (1.6)

Now $\frac{2}{N} = \frac{1}{a} + \frac{1}{b} \Rightarrow \frac{b+a}{ab} = \frac{2}{N} \Rightarrow N(b+a) = 2ab$
$2N(b+a) = 4ab$

$\therefore$ In (1.6) $\boxed{(2a-N)(2b-N) = N^2}$ (1.7)

So since N is prime, $N \geq 3$ and $a \neq b$

W.L.O.G $2a - N = N^2$, $2b - N = 1$ $\Rightarrow$ $\boxed{a = \frac{1}{2}N(N+1)}$

$$\boxed{b = \tfrac{1}{2}(N+1)}$$

So $\dfrac{2}{N} = \dfrac{2}{N(N+1)} + \dfrac{2}{N+1}$

and hence since we can write $N = 2n-1$ [n integer ≥ 2]

$$\boxed{\frac{2}{N} = \frac{1}{n} + \frac{1}{n(2n-1)}}$$

which is definitely the form of the sum of two distinct unit fractions, since $n \neq n(2n-1)$ if $n \geq 2$

of the expression in (1.5)

2/ Consider polynomial $p(x)$. Let $(x-a)^2$ be a factor

$\therefore$ $\boxed{p(x) = (x-a)^2 q(x)}$ where $q(x)$ is a polynomial of two orders lower than $p(x)$

So $\boxed{p(a) = 0}$ (factor theorem)

Now $p'(x) = 2(x-a)q(x) + (x-a)^2 q'(x)$

$\therefore$ $\boxed{p'(a) = 0}$

Now let $(x-a)^4$ be a factor of $p(x)$

$\therefore$ $p(x) = (x-a)^4 r(x)$ where $r(x)$ is a polynomial

$p'(x) = 4(x-a)^3 r(x) + (x-a)^4 r'(x)$

$\therefore$ $\boxed{p'(a) = 0}$ $\boxed{\text{Clearly } p''(a) = p'''(a) \text{ must also be } 0}$

Now $\boxed{p(x) = x^6 + 4x^5 - 5x^4 - 40x^3 - 40x^2 + 32x + k}$

has a factor $\boxed{(x-a)^4}$. Motivated by idea above (namely $p'''(a) = 0$)

$p'(x) = 6x^5 + 20x^4 - 20x^3 - 120x^2 - 80x + 32$

$p''(x) = 30x^4 + 80x^3 - 60x^2 - 240x - 80$

$\boxed{p'''(x) = 120x^3 + 240x^2 - 120x - 240}$

So if $p'''(a) = 0$ $\Rightarrow$ $\boxed{a^3 + 2a^2 - a - 2 = 0}$

$\Rightarrow$ $a^2(a+2) - (a+2) = 0$

$(a+2)(a^2-1) = 0$

$(a+2)(a+1)(a-1) = 0$

So $\boxed{a = -2, -1, 1}$

Now if $(x-a)^4$ is a factor of $p(x)$

$p(a)=0$
$p'(a)=0$
$p''(a)=0$
$p'''(a)=0$

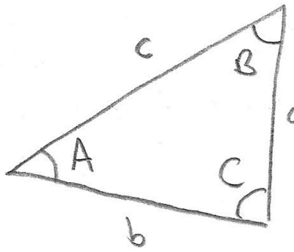
	-2	-1	1
$p(a)$	$-48+k$	$-40+k$	$-48+k$
$p'(a)$	0	26	-162
$p''(a)$	0	50	-270
$p'''(a)$	0	0	0
	✓	X	X

← possible values of a from $p'''(a)=0$

So the only value of a which satisfies $p'''(a)=p''(a)=p'(a)=0$ is $\boxed{a=-2}$

$\therefore$ Since $p(a)=0 \Rightarrow \boxed{k=48}$

3/



$A = \pi/3 + \varepsilon_3$ (radians obviously!)

$b = 8 + \varepsilon_1$

$c = 3 + \varepsilon_2$

$\varepsilon_1, \varepsilon_2, \varepsilon_3 \ll 1$

Cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\therefore \boxed{a = \left[(8+\varepsilon_1)^2 + (3+\varepsilon_2)^2 - 2(8+\varepsilon_1)(3+\varepsilon_2)\cos\left(\tfrac{\pi}{3}+\varepsilon_3\right)\right]^{\frac{1}{2}}} \quad (3.1)$$

$$\cos\left(\tfrac{\pi}{3}+\varepsilon_3\right) = \cos\tfrac{\pi}{3}\cos\varepsilon_3 - \sin\tfrac{\pi}{3}\sin\varepsilon_3$$

Expansions for $\sin x$ and $\cos x$ are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

So since $\varepsilon_3 \ll 1$ $\quad \boxed{\sin\varepsilon_3 \simeq \varepsilon_3} \quad \boxed{\cos\varepsilon_3 \approx 1} \quad (3,2)$

we will ignore terms in $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of higher powers than 1 from now on! (And also $\varepsilon_1\varepsilon_2$ etc)

$$\therefore (8+\varepsilon_1)^2 \simeq 64 + 16\varepsilon_1 \quad (3,3)$$

$$(3+\varepsilon_2)^2 \simeq 9 + 6\varepsilon_2 \quad (3.4)$$

$$(8+\varepsilon_1)(3+\varepsilon_2) \simeq 24 + 3\varepsilon_1 + 8\varepsilon_2 \quad (3.5)$$

$$\therefore \text{In } (3.1) \quad a \simeq \left[64 + 16\varepsilon_1 + 9 + 6\varepsilon_2 - (48 + 6\varepsilon_1 + 16\varepsilon_2)\cos\left(\tfrac{\pi}{3}+\varepsilon_3\right)\right]^{\frac{1}{2}}$$

$$\cos\left(\tfrac{\pi}{3}+\varepsilon_3\right) \cong \frac{1}{2} - \frac{\sqrt{3}}{2}\varepsilon_3$$

$$\left[\cos\tfrac{\pi}{3} = \tfrac{1}{2}, \quad \sin\tfrac{\pi}{3} = \sqrt{3}/2\right]$$

$$\therefore a \simeq \left[73 + 16\varepsilon_1 + 6\varepsilon_2 - (24 + 3\varepsilon_1 + 8\varepsilon_2)(1 - \sqrt{3}\varepsilon_3)\right]^{\frac{1}{2}}$$

$$\simeq \left[49 + 13\varepsilon_1 - 2\varepsilon_2 + 24\sqrt{3}\varepsilon_3\right]^{\frac{1}{2}}$$

Define $\boxed{\eta = \dfrac{13\varepsilon_1 - 2\varepsilon_2 + 24\sqrt{3}\varepsilon_3}{14}}$

$\therefore \quad a \approx \left[49 + 14\eta\right]^{\frac{1}{2}} = 7\left[1 + \underbrace{\frac{14}{49}\eta}_{2\eta/7}\right]^{\frac{1}{2}}$

$\therefore$ using generalized Binomial Expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \dots \qquad (|x|<1)$$

$$a \approx 7\left(1 + \frac{1}{2} \times \frac{2\eta}{7}\right)$$

$\boxed{a \approx 7 + \eta}$ as required.

let $|\varepsilon_1| \leq 2\times10^{-3}$ $\quad |\varepsilon_2| \leq 4.9\times10^{-2}$ $\quad |\varepsilon_3| \leq \sqrt{3}\times10^{-3}$

Max η is $\dfrac{13\times2\times10^{-3} - 2(-4.9\times10^{-2}) + 24\sqrt{3}\,(\sqrt{3}\times10^{-3})}{14}$

$$= \frac{26\times10^{-3} + 9.8\times10^{-2} + 24\times3\times10^{-3}}{14}$$

$$= \frac{26+98+72}{14}\times10^{-3}$$

$$= \frac{196}{14}\times10^{-3}$$

$$= \frac{14^2}{14}\times10^{-3}$$

$$= 14\times10^{-3}$$

$$= \boxed{1.4\times10^{-2}}$$

Min value of η is

$$\frac{13(-2\times10^{-3}) - 2(4.9\times10^{-2}) + 24\sqrt{3}(-\sqrt{3}\times10^{-3})}{14}$$

$$= -\frac{10^{-3}}{14}\left(26 + 98 + 72\right)$$ Same as above but -ve

$$= -\frac{10^{-3}}{14}\times196$$

$$= \boxed{-1.4\times10^{-2}}$$

So $\boxed{-1.4\times10^{-2} < \eta < 1.4\times10^{-2}}$

4/ $(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$

$i^2 = -1$

$= \cos\theta\cos\phi + i\sin\theta\cos\phi + i\sin\phi\cos\theta - \sin\theta\sin\phi$

$= \cos\theta\cos\phi - \sin\theta\sin\phi + i\left(\sin\theta\cos\phi + \sin\phi\cos\theta\right)$

Now $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$
$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ {Addition formulae assumed}

So $\boxed{(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi)}$

as required

[Alternatively using De Moivre's Theorem : $e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$

$= \cos(\theta+\phi) + i\sin(\theta+\phi)$]

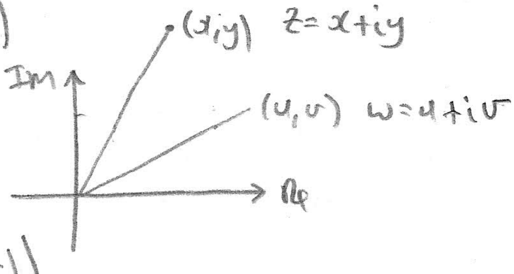
The latter is a very nice way of proving the next result

$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \boxed{\cos n\theta + i\sin n\theta}$

Now, consider two complex numbers $\boxed{\begin{array}{l} z = x + iy \\ w = u + iv \end{array}}$

$z = |z|e^{i\tan^{-1}(y/x)}$ $\quad w = |w|e^{i\tan^{-1}(v/u)}$

$|z| = \sqrt{x^2+y^2}$ $\quad |w| = \sqrt{u^2+v^2}$



So $zw = |z||w|e^{i\left(\tan^{-1}(y/x) + \tan^{-1}(v/u)\right)}$

i.e $\boxed{\arg(zw) = \arg(z) + \arg(w) + 2\pi N}$

Integer

let's take the case where $N=0$ i.e

we define $\tan^{-1}x$ in the range $-\pi < \tan^{-1}x < \pi$

The result is obviously extensible to

$\arg(zwk) = \arg(z) + \arg(w) + \arg(k)$

So if $\boxed{\arg(z) = \tan^{-1}\left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right)}$ ← ie restricting to range $[-\pi, \pi]$

then $\boxed{\tan^{-1}\frac{7}{17} + 2\tan^{-1}\frac{1}{5}}$ ← ie $\tan^{-1}\frac{7}{17} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{5}$

So $\boxed{n\tan^{-1}\left(\frac{y}{x}\right) = \arg\left((x+iy)^n\right)}$ (n integer $n \geq 1$)

$= \arg\left((17+7i)(5+i)^2\right)$

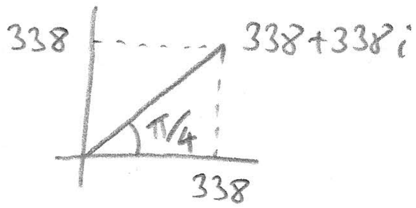
$= \arg\left((17+7i)(25+10i-1)\right)$

$= \arg\left((17+7i)(24+10i)\right)$

$= \arg\left(408+168i+170i-70\right)$

$= \arg\left(338+338i\right)$

$= \boxed{\pi/4}$ as required



By a similar method:

$3\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{20}\right) + \tan^{-1}\left(\frac{1}{1985}\right)$

$= \arg\left((4+i)^3(20+i)(1985+i)\right)$

$= \arg\left(\left(4^3 + 3(4)^2 i + 3(4)i^2 + i^3\right)(39700 + 2005i - 1)\right)$

$= \arg\left((64 + 48i - 12 - i)(39699 + 2005i)\right)$

$= \arg\left((52+47i)(39699+2005i)\right)$

$= \arg\left(2064348 + 1865853i + 104260i - 94235\right)$

$= \arg(197\,0113 + 197\,0113i)$

$= \boxed{\pi/4}$ as required

5/ $R^2 = \int_0^1 (f(x) - \lambda x)^2 \, dx$

λ is chosen to minimise

$R^2 = \int_0^1 (f(x))^2 \, dx - 2\lambda \int_0^1 x f(x) \, dx + \lambda^2 \int_0^1 x^2 \, dx$

$R^2 = \int_0^1 (f(x))^2 \, dx - 2\lambda \int_0^1 x f(x) \, dx + \lambda^2 \left[\frac{1}{3}x^3\right]_0^1$

$R^2 = \int_0^1 (f(x))^2 \, dx - 2\lambda \int_0^1 x f(x) \, dx + \lambda^2/3$

$\frac{\partial R^2}{\partial \lambda} = -2\int_0^1 x f(x) \, dx + \frac{2\lambda}{3}$

$\frac{\partial^2 R^2}{\partial \lambda^2} = 2/3$

At the minima of (λ), $\frac{\partial R^2}{\partial \lambda} = 0$ and $\frac{\partial^2 R^2}{\partial \lambda^2} > 0$

The latter is always true since $\frac{\partial^2 R^2}{\partial \lambda^2}$ is a +ve constant $(2/3)$ when $\frac{\partial R^2}{\partial \lambda} = 0$

(5.1) $\boxed{\lambda = 3\int_0^1 x f(x) \, dx}$ as required.

Define residual error $\boxed{R^2 = \int_0^1 (f(x) - \lambda x)^2 \, dx}$ (5.2)

$R^2 = \int_0^1 (f(x))^2 \, dx - 2\lambda \int_0^1 x f(x) \, dx + \lambda^2/3$ from above

Substituting for λ from (5.1)

$R^2 = \int_0^1 (f(x))^2 \, dx - 2\lambda\left(\frac{\lambda}{3}\right) + \lambda^2/3$

$$\therefore \quad \boxed{R^2 = \int_0^1 (f(x))^2\,dx - \lambda^2/3}$$ as required (Q3)

let $\boxed{f(x) = \sin(\pi x/n)}$

(i) Consider large n

$$\lambda = 3\int_0^1 x\sin(\pi x/n)\,dx$$

Note for small z

$\sin z \cong z - z^3/3!$

$\cos z \simeq 1 - z^2/2!$

Also $\boxed{\sin^2 z = \frac{1-\cos 2z}{2}}$

By parts:

$$\int x\sin(\pi x/n)\,dx$$

$$= x\left(-\frac{n}{\pi}\cos\left(\frac{\pi x}{n}\right)\right) - \int\left(-\frac{n}{\pi}\cos\left(\frac{\pi x}{n}\right)\right)dx$$

$$= -\frac{nx}{\pi}\cos\left(\frac{\pi x}{n}\right) + \frac{n}{\pi}\int\cos\left(\frac{\pi x}{n}\right)dx$$

$$= -\frac{nx}{\pi}\cos\left(\frac{\pi x}{n}\right) + \frac{n^2}{\pi^2}\sin\left(\frac{\pi x}{n}\right) + C$$

$$\text{So } \lambda = 3\left[\frac{n^2}{\pi^2}\sin\left(\frac{\pi x}{n}\right) - \frac{nx}{\pi}\cos\left(\frac{\pi x}{n}\right)\right]_0^1$$

$$\lambda = 3\left(\frac{n^2}{\pi^2}\sin\left(\frac{\pi}{n}\right) - \frac{n}{\pi}\cos\left(\frac{\pi}{n}\right)\right)$$

So using $\sin\left(\frac{\pi}{n}\right) \simeq \frac{\pi}{n} - \frac{\pi^3}{6n^3}$

$\cos\left(\frac{\pi}{n}\right) \simeq 1 - \frac{\pi^2}{2n^2}$

$$\therefore \lambda \simeq 3\left(\frac{n^2}{\pi^2}\left(\frac{\pi}{n} - \frac{\pi^3}{6n^3}\right) - \frac{n}{\pi}\left(1 - \frac{\pi^2}{2n^2}\right)\right) \quad (n \gg 1)$$

$$= 3\left(\cancel{\frac{n}{\pi}} - \frac{\pi}{6n} - \cancel{\frac{n}{\pi}} + \frac{\pi}{2n}\right)$$

$$= \frac{3\pi}{n}\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{\pi}{n}\,\frac{3\times 2}{6} = \boxed{\frac{\pi}{n}}$$ as required (for large n)

(ii) $$R^2 = \int_0^1 \sin^2\left(\frac{\pi x}{n}\right)dx - \lambda^2/3$$

$$= \frac{1}{2}\int_0^1\left(1 - \cos\left(\frac{2\pi x}{n}\right)\right)dx - \lambda^2/3$$

$$R^2 = \frac{1}{2}\left[x - \frac{n}{2\pi}\sin\left(\frac{2\pi x}{n}\right)\right]_0^1 - \lambda^2/3$$

$$= \frac{1}{2}\left(1 - \frac{n}{2\pi}\sin\left(2\pi/n\right)\right) - \lambda^2/3$$

$\lambda \simeq \pi/n$ in this limit

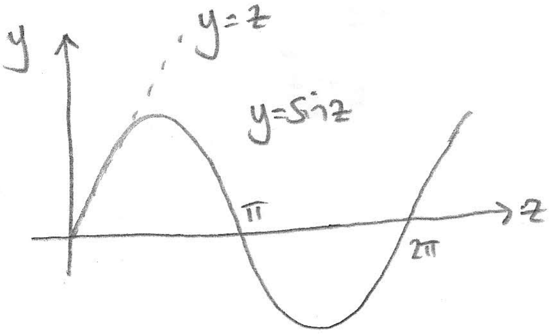
$\therefore$ when $n \to \infty$,

$$R^2 \to \frac{1}{2}\left(1 - \frac{n}{2\pi}\left(\frac{2\pi}{n} - \frac{8\pi^3}{6n^3}\right)\right) - \frac{\pi^2}{3n^2}$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{\pi^2}{3n^2} - \frac{\pi^2}{3n^2}$$

$$\therefore \boxed{\lim_{n\to\infty} R^2 = 0}$$

These results are to be expected since $\sin z \approx z$ when z is small, so difference between $\sin z$ and $z \to 0$ as z becomes smaller.



6/ $\boxed{t = \tan\frac{\theta}{2}}$

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\left(\tan\frac{\theta}{2}\right)^2} = \boxed{\frac{2t}{1-t^2}}$$

$$\boxed{\frac{1}{\cos^2\theta} = 1 + \tan^2\theta} \qquad \therefore \frac{1}{\cos^2\theta} = 1 + \frac{4t^2}{(1-t^2)^2}$$

$$\frac{1}{\cos^2\theta} = \frac{(1-t^2)^2 + 4t^2}{(1-t^2)^2}$$

$$= \frac{1 - 2t^2 + t^4 + 4t^2}{(1-t^2)^2}$$

$$= \frac{(1+t^2)^2}{(1-t^2)^2}$$

$$\therefore \boxed{\cos\theta = \frac{1-t^2}{1+t^2}}$$

Now $\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{1-t^2}{1+t^2}\right)^2$

$$= \frac{(1+t^2)^2 - (1-t^2)^2}{(1+t^2)^2}$$

$$= \frac{(1+t^2+1-t^2)(1+t^2-1+t^2)}{(1+t^2)^2}$$

$$= \frac{4t^2}{(1+t^2)^2}$$

$$\therefore \boxed{\sin\theta = \frac{2t}{1+t^2}}$$

[Alternatively $\boxed{\sin\theta = \tan\theta\cos\theta}$

$$\therefore \sin\theta = \frac{2t}{1-t^2} \times \frac{1-t^2}{1+t^2}$$

$$= \boxed{\frac{2t}{1+t^2}}$$

$$\frac{1+\cos\theta}{\sin\theta} = \frac{1+\frac{1-t^2}{1+t^2}}{2t/1+t^2} = \frac{\frac{1+t^2+1-t^2}{1+t^2}}{2t/1+t^2}$$

$$= \boxed{\frac{1}{t}}$$

Now $\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \dfrac{\tan\frac{\pi}{2} - \tan\frac{\theta}{2}}{1+\tan\frac{\pi}{2}\tan\theta/2}$

$\tan\pi/2 \to \infty$ So $\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right) \simeq \dfrac{\tan\frac{\pi}{2}}{\tan\frac{\pi}{2}\tan\theta/2} = \dfrac{1}{\tan\theta/2}$

$$= \boxed{1/t}$$

So $\boxed{\dfrac{1+\cos\theta}{\sin\theta} = \tan\left(\frac{\pi}{2}-\theta/2\right)}$

Now consider $\boxed{I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\alpha\sin\theta}\,d\theta}$

let $t = \tan\theta/2$ $\therefore$ $\dfrac{dt}{d\theta} = \dfrac{1}{2}\dfrac{1}{\cos^2\theta/2} = \dfrac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)$

$\therefore \dfrac{dt}{d\theta} = \frac{1}{2}(1+t^2)$ $\therefore$ $d\theta = \dfrac{2\,dt}{1+t^2}$

Now if $\sin\theta = \dfrac{2t}{1+t^2}$

$\therefore$ $I = \displaystyle\int_0^1 \frac{1}{1+\frac{2t}{1+t^2}\cos\alpha} \times \frac{2\,dt}{1+t^2} = 2\int_0^1 \frac{dt}{1+t^2+2t\cos\alpha}$

Note when $\theta = 0,\ t = 0$
$\theta = \frac{\pi}{2},\ t = \tan\frac{\pi}{4} = 1$

Now $(t+\cos\alpha)^2$
$= t^2 + 2t\cos\alpha + \cos^2\alpha$
So $1 + t^2 + 2t\cos\alpha$
$= (t+\cos\alpha)^2 + 1 - \cos^2\alpha$

$$\therefore \quad 1+t^2+2t\cos\alpha = (t+\cos\alpha)^2+\sin^2\alpha$$

So $\quad I = 2\displaystyle\int_0^1 \frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}$

To factor out $\sin^2\alpha$ let $\boxed{t+\cos\alpha = \sin\alpha\tan\phi}$

why $\tan\phi$? Since $1+\tan^2\phi = \frac{d}{d\phi}\tan\phi$ which could facilitate a useful substitution

$$\therefore \quad \frac{dt}{d\phi} = \sin\alpha\,(1+\tan^2\phi)$$

$$\therefore \quad I = 2\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\frac{\alpha}{2}} \frac{\sin\alpha\,(1+\tan^2\phi)\,d\phi}{\sin^2\alpha\tan^2\phi+\sin^2\alpha}\,d\phi$$

Aha! The $1+\tan^2\phi$ terms cancel

LIMITS

[when $t=0$, $\tan\phi = \dfrac{\cos\alpha}{\sin\alpha} = \dfrac{1}{\tan\alpha}$

$$\tan\left(\tfrac{\pi}{2}-\alpha\right) = \frac{\tan\frac{\pi}{2}-\tan\alpha}{1+\tan\frac{\pi}{2}\tan\alpha} = \frac{1}{\tan\alpha}$$

So $\tan\phi = \tan(\pi/2-\alpha)$

$\therefore \boxed{\phi_{t=0} = \tfrac{\pi}{2}-\alpha}$

when $t=1$, $\tan\phi = \dfrac{1+\cos\alpha}{\sin\alpha} = \tan\left(\dfrac{\pi}{2}-\dfrac{\alpha}{2}\right)$ from above

$\therefore \boxed{\phi_{t=1} = \tfrac{\pi}{2}-\alpha/2}$]

$$\therefore \quad I = \frac{2}{\sin\alpha}\Big[\phi\Big]_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}-\alpha/2} = \frac{2}{\sin\alpha}\left(\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\pi}{2}+\alpha\right)$$

$$= \boxed{\frac{\alpha}{\sin\alpha}} \quad \text{as required.}$$

Now let $\boxed{J = \displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\alpha\cos\theta}}$

Can we transform this into something which looks like I to avoid repetitive work?

Yes! $\cos\left(\tfrac{\pi}{2}-\alpha\right) = \cos\tfrac{\pi}{2}\cos\alpha + \sin\tfrac{\pi}{2}\sin\alpha = \boxed{\sin\alpha}$

$\sin\left(\tfrac{\pi}{2}-\theta\right) = \sin\tfrac{\pi}{2}\cos\theta + \cos\tfrac{\pi}{2}(-\sin\theta) = \boxed{\cos\theta}$

$$\therefore \quad J = \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\cos(\frac{\pi}{2}-\alpha)\sin(\frac{\pi}{2}-\theta)}$$

Now $\boxed{I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\cos\alpha\sin\theta} = \frac{\alpha}{\sin\alpha}}$

let $\beta = \frac{\pi}{2} - \alpha$

$\phi = \frac{\pi}{2} - \theta \quad \therefore \quad \frac{d\phi}{d\theta} = -1$

when $\theta = 0, \ \phi = \frac{\pi}{2}$

$\theta = \frac{\pi}{2} \ \ \phi = 0$

$$\therefore \quad J = \int_{\frac{\pi}{2}}^{0} \frac{-d\phi}{1+\cos\beta\sin\phi} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{1+\cos\beta\sin\phi}$$

Since the latter has the same form as I, but $\theta \Leftrightarrow \phi$, $\alpha \Leftrightarrow \beta$

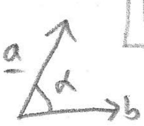
$$\therefore \quad J = \frac{\beta}{\sin\beta}$$

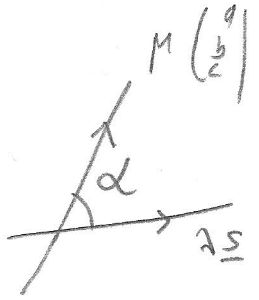
$$\therefore \quad J = \frac{\frac{\pi}{2}-\alpha}{\sin(\frac{\pi}{2}-\alpha)} = \boxed{\frac{\frac{\pi}{2}-\alpha}{\cos\alpha}}$$

7/ line l has vector equation $\underline{r} = \lambda \underline{s}$

$$\underline{s} = \begin{pmatrix} \cos\theta + \sqrt{3} \\ \sqrt{2}\sin\theta \\ \cos\theta - \sqrt{3} \end{pmatrix}$$

$\boxed{|\underline{a}||\underline{b}|\cos\alpha = \underline{a}.\underline{b}}$





Angle between l and $\underline{r} = M\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is α

i.e. $$M\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \lambda \begin{pmatrix} \cos\theta + \sqrt{3} \\ \sqrt{2}\sin\theta \\ \cos\theta - \sqrt{3} \end{pmatrix} = M\sqrt{a^2+b^2+c^2}\,\lambda\sqrt{(\cos\theta+\sqrt{3})^2 + 2\sin^2\theta + (\cos\theta - \sqrt{3})^2} \times \cos\alpha$$

$$\therefore a(\cos\theta + \sqrt{3}) + b\sqrt{2}\sin\theta + c(\cos\theta - \sqrt{3})$$

$$= \sqrt{a^2+b^2+c^2}\sqrt{\cos^2\theta + 2\sqrt{3}\cos\theta + 3 + 2\sin^2\theta + \cos^2\theta - 2\sqrt{3}\cos\theta + 3} \times \cos\alpha$$

$$\therefore \frac{\cos\theta(a+c) + b\sqrt{2}\sin\theta + (a-c)\sqrt{3}}{\sqrt{a^2+b^2+c^2}\sqrt{2(\cos^2\theta + \sin^2\theta) + 6}} = \cos\alpha$$

$$\therefore \boxed{\alpha = \cos^{-1}\left(\frac{(a+c)\cos\theta + b\sqrt{2}\sin\theta + (a-c)\sqrt{3}}{\sqrt{8}\sqrt{a^2+b^2+c^2}}\right)}$$

So when $a + c = 0$ and $b = 0$; $a - c = 2a$

$a^2+b^2+c^2 = 2a^2$

$$\therefore \alpha = \cos^{-1}\left(\frac{2a\sqrt{3}}{\sqrt{8}\sqrt{8}a}\right)$$

$[\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}]$

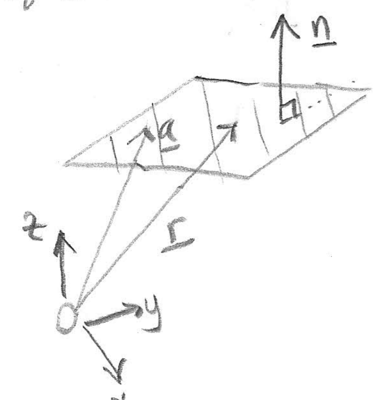
$$\alpha = \cos^{-1}\left(\frac{2\sqrt{3}}{2\sqrt{2}\sqrt{2}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi}{6}}$$

as required.

Now consider a plane with equation $\boxed{x - z = 4\sqrt{3}}$ line ℓ meets the plane at P.

Now this plane has normal // to $\begin{pmatrix}1\\0\\-1\end{pmatrix}$

[Plane equation:



$(\underline{r} - \underline{a}) \cdot \underline{n} = 0$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \underline{n} = \underline{a} \cdot \underline{n}$

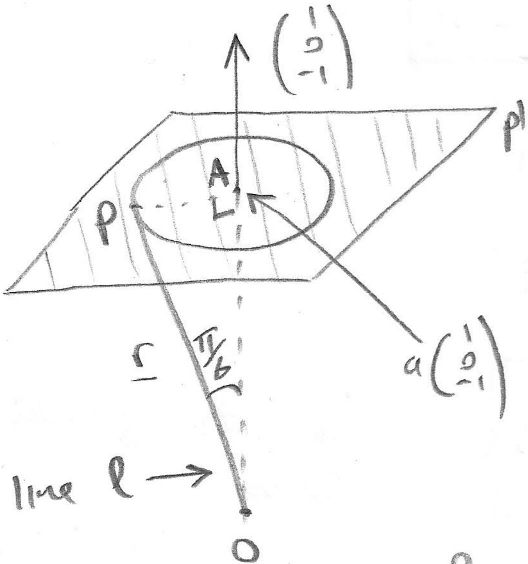
ie, $\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \underline{n} = \text{constant}$

So in our case $x - z = \begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}1\\0\\-1\end{pmatrix}$]

Now in previous section line $\mu \begin{pmatrix}a\\b\\c\end{pmatrix}$ met $\lambda \begin{pmatrix}\cos\theta + \sqrt{3}\\ \sqrt{2}\sin\theta \\ \cos\theta - \sqrt{3}\end{pmatrix}$

at angle $\pi/6$ when $\begin{pmatrix}a\\b\\c\end{pmatrix} = a \begin{pmatrix}1\\0\\-1\end{pmatrix}$

So in this case $\mu \begin{pmatrix}a\\b\\c\end{pmatrix} \perp$ to plane and point P is described via the diagram below

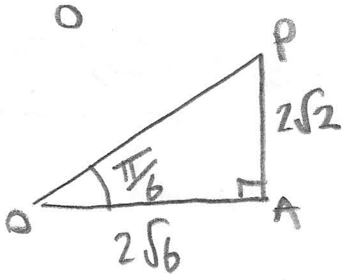


So P is the base of a cone from O ie a circle on the plane $x - z = 4\sqrt{3}$

Now point A is in the plane

So $x - z = 4\sqrt{3}$

$\therefore a + a = 4\sqrt{3} \Rightarrow \boxed{a = 2\sqrt{3}}$



$|\overrightarrow{OA}| = a\sqrt{2} = 2\sqrt{6}$

Radius of circle is $|\overrightarrow{AP}| = 2\sqrt{6}\tan\frac{\pi}{6}$

Now $\tan\pi/6 = \frac{1}{\sqrt{3}}$ $\therefore$ $\boxed{|\overrightarrow{AP}| = 2\sqrt{2}}$

8/ (i) consider the differential equation

$$\boxed{\frac{dy}{dx} + 4x e^{-x^2}(y+3)^{\frac{1}{2}} = 0 \qquad x \geq 0}$$

when $x = 0$
$y = 6$

The ODE is <u>seperable</u>

ie $$\int \frac{dy}{(y+3)^{\frac{1}{2}}} = -4\int x e^{-x^2}\,dx$$

Now $$\frac{d}{dx} e^{-x^2} = -2x e^{-x^2}$$

$$\therefore \quad e^{-x^2} + C = -2\cdot\int x e^{-x^2}\,dx$$

$$\therefore \quad \int x e^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + k$$

$$\therefore \quad 2(y+3)^{\frac{1}{2}} = 2e^{-x^2} + K$$

using initial conditions: $2(6+3)^{\frac{1}{2}} = 2 + K$

$$\Rightarrow \quad \boxed{4 = K}$$

$$\therefore \quad 2(y+3)^{\frac{1}{2}} = 2e^{-x^2} + 4$$

$$(y+3)^{\frac{1}{2}} = e^{-x^2} + 2$$

$$\Rightarrow \quad \boxed{y = \left(e^{-x^2} + 2\right)^2 - 3}$$ is the general solution

when $x \to \infty$, $e^{-x^2} \to 0$ So $\boxed{y \to 1}$

as required.

(ii) Now consider the following ODE

$$\boxed{\frac{dy}{dx} - xe^{6x^2}(y+3)^{1-k} = 0} \qquad (x \geq 0)$$

This is also seperable:

$$\int \frac{dy}{(y+3)^{1-k}} = \int xe^{6x^2}\,dx$$

$$\int (y+3)^{k-1}\,dy = \frac{1}{12}e^{6x^2} + c$$

$$\left[\frac{d}{dx} e^{6x^2} = 12xe^{6x^2} \quad \text{So} \quad \int xe^{6x^2}\,dx = \frac{1}{12}e^{6x^2} + c \right]$$

$$\frac{1}{k}(y+3)^k = \frac{1}{12}e^{6x^2} + c$$

$$\boxed{y = \left(\frac{k}{12}e^{6x^2} + c\right)^{1/k} - 3}$$

Now $$e^{-3x^2}y = \left(e^{-3kx^2}\right)^{1/k} y$$

$$= \left(\frac{k}{12}e^{-3kx^2}e^{6x^2} + ce^{-3kx^2}\right)^{1/k} - 3e^{-3x^2}$$

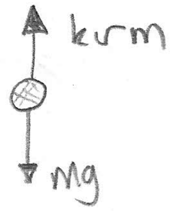
let $\boxed{k=2}$, which will cancel the $e^{-3kx^2}e^{6x^2}$ term

$$\therefore e^{-3x^2}y = \left(\frac{1}{6} + ce^{-6x^2}\right)^{1/2} - 3e^{-3x^2}$$

So when $x \to \infty$, $e^{-3x^2} \to \boxed{\frac{1}{\sqrt{6}}}$

9/

Jane:



$\downarrow x$ $\quad \downarrow g = 9.81\,\text{ms}^{-2}$ $\quad (k>0)$

$\downarrow v$

$\downarrow a$

[Note $a = \frac{dv}{dt}$]

Newton II: $\boxed{m\frac{dv}{dt} = mg - kvm}$

$$\therefore \int_0^v \frac{dv}{g-kv} = \int_0^t dt$$

$$\therefore \frac{1}{-k}\int_0^v \frac{-k\,dv}{g-kv} = t$$

$$-\frac{1}{k}\Big[\ln|g-kv|\Big]_0^v = t$$

$$-\frac{1}{k}\ln\left|\frac{g-kv}{g}\right| = t$$

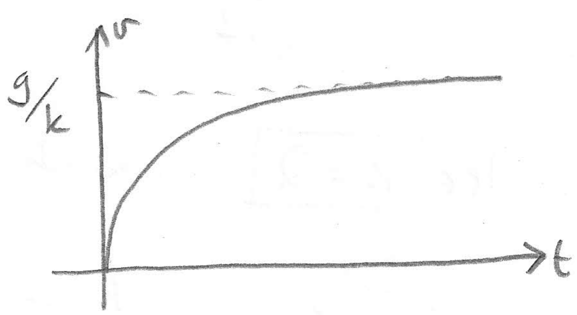
$$\left|\frac{g-kv}{g}\right| = e^{-kt}$$

$$|g-kv| = ge^{-kt}$$

Now as $t \to \infty$, $|g-kv| \to 0$

If $kv < g$ we can ignore the $|\ldots|$ sign

ie $\boxed{v = \frac{g}{k}\left(1-e^{-kt}\right)}$



This seems to be a sensible model of air resistance, with Jane allowing a terminal velocity $v \to \frac{g}{k}$ as $t \to \infty$.

When $t=0$, $v=0$ which is a sensible starting point if $kv > g$ $\quad kv - g = ge^{-kt}$

$$\Rightarrow \quad v = \frac{g(1+e^{-kt})}{k}$$

when $t \to \infty$, $v \to g/k$ ie Same behaviour

But when $t=0$, $v = 2g/k$ which is not a sensible starting point.

So in general, $\boxed{v = \frac{g}{k}(1-e^{-kt})}$

Karen

Newton II: $m\frac{dv}{dt} = mg - kvm - \frac{2k^2v^2m}{g}$

Extra term due to 'spreadeagling'!

So $$\int_0^v \frac{dv}{g - kv - 2k^2v^2/g} = t$$

$$g - kv - 2k^2v^2/g = g\left(1 - \frac{kv}{g} - \frac{2k^2v^2}{g^2}\right)$$

$$= g\left(1 + kv/g\right)\left(1 - 2kv/g\right)$$

[$1 - x - 2x^2 = (1+x)(1-2x)$ is a useful factorization]

Now let $$\frac{A}{1+kv/g} + \frac{B}{1-2kv/g} = \frac{1}{(1+kv/g)(1-2kv/g)}$$

$$A(1-2kv/g) + B(1+kv/g) = 1$$

let $v = -g/k$: $\quad 3A = 1 \quad \Rightarrow \quad \boxed{A = 1/3}$

let $v = g/2k$: $\quad \frac{3}{2}B = 1 \quad \Rightarrow \quad \boxed{B = 2/3}$

So $$t = \frac{1}{3g}\int_0^v \left(\frac{1}{1+kv/g} + \frac{2}{1-2kv/g}\right) dv$$

$$3gt = \frac{g}{k}\int_0^v \left(\frac{k/g}{1+kv/g} - \frac{-2k/g}{1-2kv/g} \right) dv$$

$$3gt = \frac{g}{k}\left[\ln(1+kv/g) - \ln(1-2kv/g) \right]_0^v$$

$$3kt = \ln\left(\frac{1+kv/g}{1-2kv/g} \right)$$

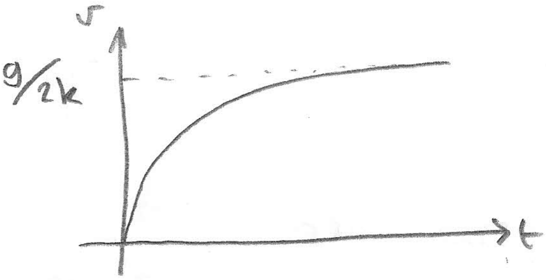
$$e^{3kt} = \frac{1+kv/g}{1-2kv/g}$$

$$(1-2kv/g)e^{3kt} = 1 + kv/g$$

$$kv/g\left(1+2e^{3kt}\right) = e^{3kt} - 1$$

$$v = \frac{\frac{g}{k}\left(e^{3kt}-1\right)}{1+2e^{3kt}}$$

$$\boxed{v = \frac{\frac{g}{k}\left(1-e^{-3kt}\right)}{2+e^{-3kt}}}$$



when $t \to \infty$, $\boxed{v \to \frac{g}{2k}}$

when $t=0, v=0$
so ✓

Jane opens her parachute when $v = \frac{g}{3k}$

$$v = \frac{g}{k}(1 - e^{-kt})$$

So $$\frac{g}{3k} = \frac{g}{k}(1 - e^{-kt})$$

$$\frac{1}{3} = 1 - e^{-kt}$$

$$e^{-kt} = 2/3$$

$$-kt = \ln(2/3)$$

$$kt = \ln(3/2)$$

$$\therefore \boxed{t = \frac{1}{k}\ln(3/2)}$$

Karen does the same, ie opens her chute when $v = g/3k$

$$\therefore \frac{g}{3k} = \frac{\frac{g}{k}(1 - e^{-3kt})}{2 + e^{-3kt}}$$

$$2 + e^{-3kt} = 3 - 3e^{-3kt}$$

$$3e^{-3kt} + e^{-3kt} = 1$$

$$e^{-3kt} = 1/4$$

$$e^{3kt} = 4$$

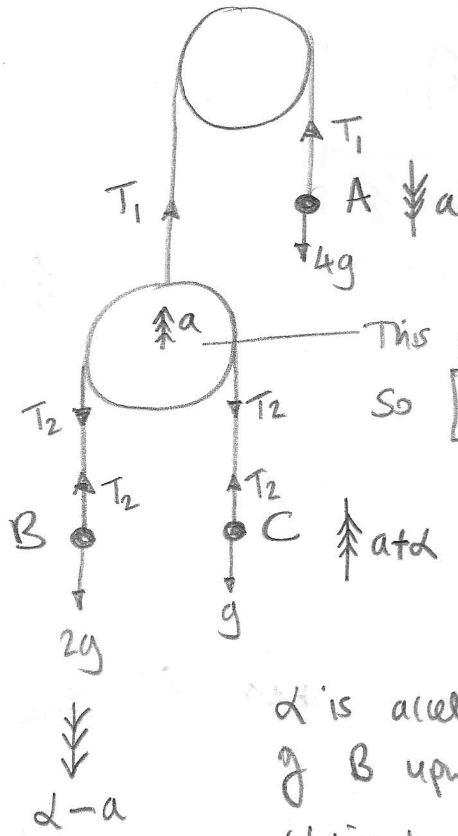
$$3kt = \ln 4$$

$$\boxed{t = \frac{1}{3k}\ln 4}$$

10/ [This is the 'second phase'!]

$g = 9.81\ ms^{-2}$

[All strings are light and inextensible
Pulleys are frictionless]



This pulley is light
So $\boxed{T_1 = 2T_2}$ (10.1)

α is acceleration of B upwards relative to pulley BC

[All accelerations are relative to 'lab frame']

Newton II! (for each mass)

$$4a = 4g - T_1 \quad (10.2)\ (A)$$

$$a + \alpha = T_2 - g \quad (10.3)\ (C)$$

$$2(\alpha - a) = 2g - T_2 \quad (10.4)\ (B)$$

(10.3) + (10.4)

$$3\alpha - a = g \quad (10.5)$$

(10.1) in (10.2)

$$4a = 4g - 2T_2 \quad (10.6)$$

From (10.3): $T_2 = a + \alpha + g$

$$\therefore\ 4a = 4g - 2a - 2\alpha - 2g$$

$$6a = 2g - 2\alpha$$

$$3a = g - \alpha \quad (10.7)$$

3 (10.5) + (10.7):

$$9\alpha = 3g + g - \alpha$$

$$10\alpha = 4g$$

$$\boxed{\alpha = \frac{2}{5}g}$$

In (10.7) $a = \dfrac{g-\alpha}{3} = \dfrac{g - \frac{2}{5}g}{3} = \boxed{\frac{1}{5}g}$ Downwards

So acceleration of A is $\boxed{a = g/5}$

" of B is $\alpha - a = \boxed{\frac{g}{5}}$ Downwards

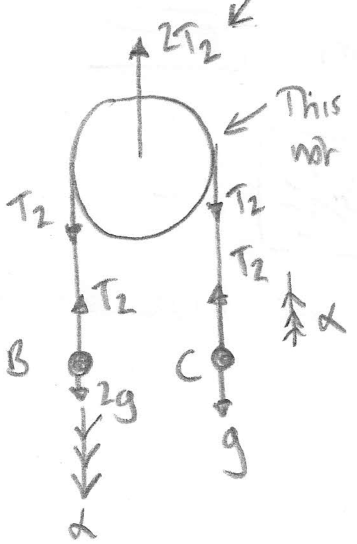
" of C is $a+\alpha = \boxed{\frac{3}{5}g}$ upwards

[For completeness, in (10.2): $T_1 = 4g - 4a = 4g(1-\frac{1}{5})$

$$T_1 = \frac{16}{5}g = \boxed{3\tfrac{1}{5}g}$$

Since $T_2 = T_1/2 \Rightarrow T_2 = \frac{8}{5}g = \boxed{1\tfrac{3}{5}g}$]

Now let's return to the first part. " The system is held in equilibrium and then BC is released "



<u>Newton II</u>

$$B: \quad 2\alpha = 2g - T_2 \qquad (10.8)$$

$$C: \quad \alpha = T_2 - g \qquad (10.9)$$

$$(10.8)+(10.9) \qquad 3\alpha = g$$

$$\boxed{\alpha = g/3}$$

Now we want to know the speed of A when B has moved a total distance of $0.6gT^2$ metres. T is the time A is held in equilibrium.

So for $0 < t < T$: B moves downwards by $\frac{1}{2}\alpha T^2$

ie $\boxed{\frac{1}{6}gT^2}$

After T seconds it has reached speed $\frac{g}{3}T$ downwards

After another τ seconds, B has moved $0.6gT^2$

$$\therefore \quad 0.6gT^2 = \frac{1}{6}gT^2 + \frac{g}{3}T\tau + \frac{1}{2}\frac{g}{5}\tau^2$$ [constant acceleration motion of B]

$\therefore \quad \frac{6}{10}T^2 = \frac{1}{6}T^2 + \frac{T\tau}{3} + \frac{\tau^2}{10}$

$\frac{13}{30}T^2 = \frac{T\tau}{3} + \frac{\tau^2}{10}$

$13T^2 = 10T\tau + 3\tau^2$

$\therefore \quad 3\tau^2 + 10\tau T - 13T^2 = 0$

$(3\tau + 13T)(\tau - T) = 0$

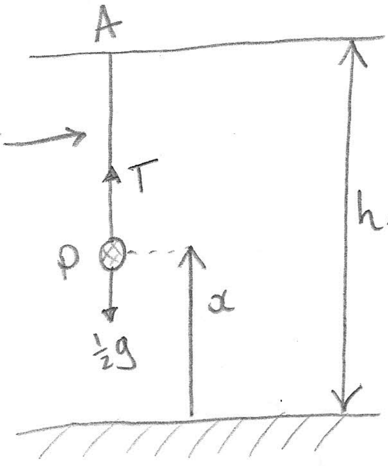
So $\tau = T$ or $-\frac{13}{3}T$

Since $\tau > 0$ then use solution $\boxed{\tau = T}$

Hence A reaches speed $\boxed{gT/5}$ ms^{-1}

(Since it starts from rest and its acceleration is $g/5$ downwards).

11/



$g = 9.81\ \text{ms}^{-2}$

* length of elastic string is light $\boxed{l = 1.5\text{m}}$
* Modulus of elasticity $\boxed{\lambda = 5g}$
* End of string, with mass of 0.5kg is released from rest at $\boxed{x = 0.5\text{ m}}$

(i) Motion from rest

Particle at P moves upward due to tension in string

$$T = \underbrace{\frac{5g}{3/2}}_{\lambda/l}\underbrace{\left(\frac{5}{2} - x - \frac{3}{2}\right)}_{\text{extension}} = \frac{10g}{3}(1-x)$$

ie assume a linear 'Hooken' string

String will become slack if $\frac{5}{2} - x - \frac{3}{2} = 0$

ie $\boxed{x = 1}$ metre. Note $\boxed{x = \frac{1}{2}}$ when $t=0$.

Newton II, assuming $x \leq 1$ and ignoring air resistance

$$\frac{1}{2}\ddot{x} = \frac{10g}{3}(1-x) - \frac{1}{2}g$$

$$\ddot{x} = \frac{20g}{3} - \frac{20g}{3}x - g \Rightarrow \boxed{\ddot{x} = -\frac{20g}{3}x + \frac{17}{3}g}$$

(11.1)

Now mass in equilibrium when $\ddot{x} = 0$

ie $\boxed{x = \frac{17}{20} = 0.85\text{m}}$

Define $z = x - \frac{17}{20}$ $\therefore$ $x = z + \frac{17}{20}$

$\therefore$ (11.1) $\boxed{\ddot{z} = -\frac{20g}{3}z}$ SHM where $\omega^2 = \frac{20g}{3}$

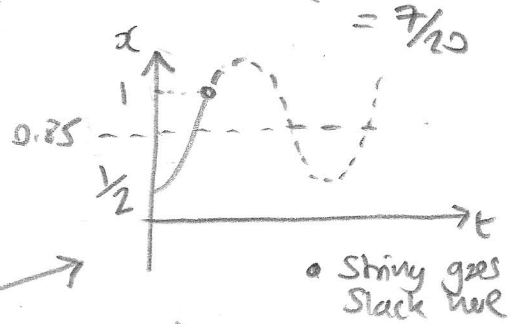
$\omega = \frac{2\pi}{P}$ ↑ period

$$\therefore \left(\frac{2\pi}{P}\right)^2 = \frac{20g}{3} \Rightarrow \boxed{P = 2\pi\sqrt{\frac{3}{20g}}}$$

Now $z_0 = -(0.85 - 0.5) = -0.35$ So amplitude of SHM is 0.35 m

If $z = -0.35 \cos\left(t\sqrt{20g/3}\right)$ (0.35 = 7/20)

$\therefore$ $$\boxed{x = \frac{17}{20} - \frac{7}{20}\cos\left(t\sqrt{20g/3}\right)}$$



If the string did not go slack, the maximum value of x would be $\frac{17}{20} + \frac{7}{20} = \boxed{1.2\text{ m}}$ $(1\frac{1}{5})$

So the tension in the string is sufficient to make the 0.5 kg particle rise to $x = 1$, whereupon the string goes slack

This takes t seconds so

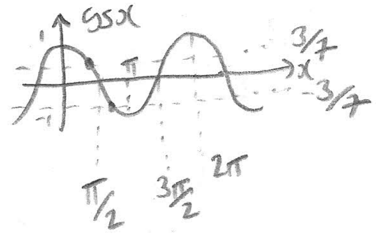
$$1 = \frac{17}{20} - \frac{7}{20}\cos\left(t\sqrt{20g/3}\right)$$

$$\frac{7}{20}\cos\left(t\sqrt{20g/3}\right) = -\frac{3}{20}$$

$$\cos\left(t\sqrt{20g/3}\right) = -\frac{3}{7}$$

$$t = \sqrt{\frac{3}{20g}}\cos^{-1}\left(-\frac{3}{7}\right)$$

$$\boxed{t = \frac{1}{2}\sqrt{\frac{3}{5g}}\left(\pi - \cos^{-1}\left(\frac{3}{7}\right)\right)}$$



At t seconds mass at P is moving upwards at

$$\dot{x} = \sqrt{\frac{20g}{3}} \times \frac{7}{20}\sin\left(t\sqrt{\frac{20g}{3}}\right)$$

However, this expression is a little unwieldy

Instead we can use the useful result for SHM $\boxed{\ddot{z} = -\omega^2 z}$

$$\boxed{\dot{z}^2 = \omega^2(a^2 - z^2)}$$

where a is the amplitude of the oscillations

$\boxed{z = a\cos(\omega t - \phi)}$ [ϕ phase]

[Quick proof: $\dot{z} = -\omega a \sin(\omega t - \phi)$
$\ddot{z} = -\omega^2 a \cos(\omega t - \phi)$
$\ddot{z} = -\omega^2 z$

$$\dot{z}^2 = \omega^2 a^2 \sin^2(\omega t - \phi)$$

$$\dot{z}^2 + \omega^2 z^2 = \omega^2 a^2 \left(\sin^2(\omega t - \phi) + \cos^2(\omega t - \phi) \right) = \omega^2 a^2$$

$$\therefore \boxed{\dot{z}^2 = \omega^2 (a^2 - z^2)} \quad]$$

Now $\dot{x} = \dot{z}$ so $\dot{x}^2 = \left(\sqrt{\frac{20g}{3}}\right)^2 \left(0.35^2 - 0.15^2\right) = \frac{1}{10} \times \frac{20g}{3}$

$z = 1 - 0.85 = 0.15$

$\therefore$ $\boxed{\dot{x} = \sqrt{\frac{2g}{3}}}$ ms^{-1} at t.

Since the string is slack, the mass will then rise as a projectile, until the string becomes taut again.

The mass has initial kinetic energy of $\frac{1}{2} \frac{1}{2} \dot{x}^2 = \frac{1}{4} \times \frac{2g}{3}$

$= \boxed{\frac{g}{6}}$ J

It can therefore rise to a maximum height $x = 1 + X$

Starting height

where $\frac{1}{2} g X = \frac{g}{6} \Rightarrow \boxed{X = \frac{1}{3}}$

So maximum height reached by P is $\boxed{1\frac{1}{3} \text{ m}}$. Since this is less than 2.5m, the particle does not reach the ceiling.

Now the particle spends 2τ seconds as a projectile, τ seconds from $x = 1$ till apogee at $x = 1\frac{1}{3}$

At apogee $\dot{x} = 0$

$$0 = \underbrace{\sqrt{2g/3}}_{\dot{x} \,@\, x=1} - g\tau \qquad \text{" } v = u + at \text{"}$$

$$\therefore \quad \boxed{\tau = \sqrt{2/3g}}$$

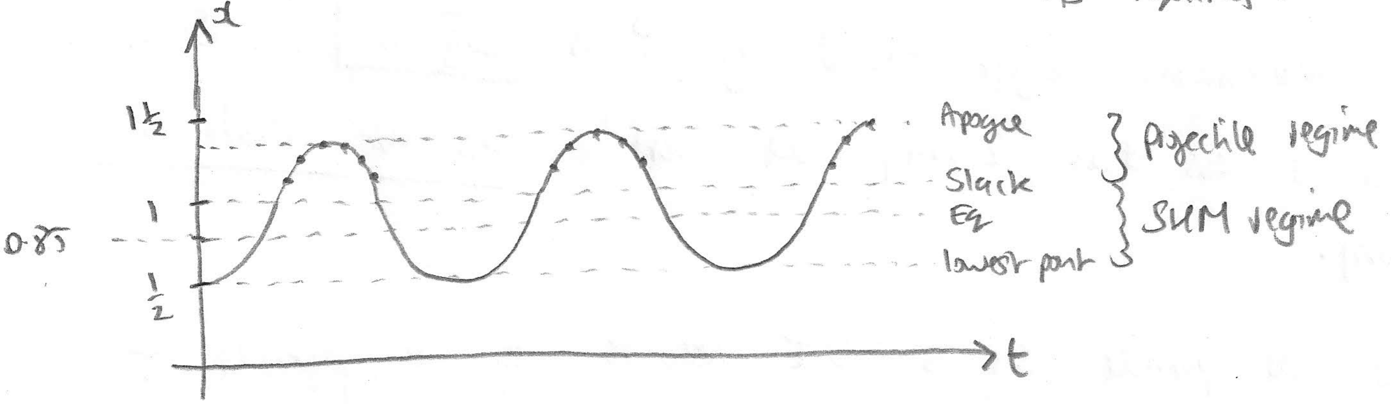
Since we assume no energy loss in the system, one assumes the time taken for the particle to reach the starting position at $x = \frac{1}{2}$, after it has returned to $x = 1$, is the same as t going the other way. Hence the total time between successive $x = \frac{1}{2}$ positions is

$$t_{tot} = 2\tau + 2t$$

$$= 2\sqrt{\frac{2}{3g}} + \frac{2}{2}\sqrt{\frac{3}{5g}}\left(\pi - \cos^{-1}\left(\frac{3}{7}\right)\right)$$

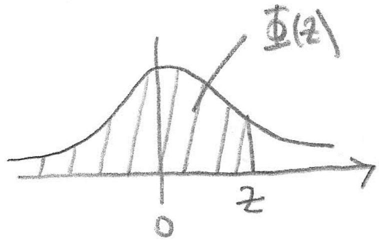
$$= \boxed{\left(\frac{8}{3g}\right)^{\frac{1}{2}} + \left(\frac{3}{5g}\right)^{\frac{1}{2}}\left(\pi - \cos^{-1}\left(\frac{3}{7}\right)\right)}$$

As required.



[In reality frictional losses will cause oscillation to decay]

12/ [Recap of Normal Distribution ideas



- z is a random variable with mean 0
- $z \in [-\infty, \infty]$
- $z \sim N(0,1)$ "Standard normal distribution"

i.e. probability of z being between z and $z+dz$ is

$$\boxed{p(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}}$$

Can generalize to a continuous random variable $x \sim N(\mu, \sigma^2)$ via:

μ = mean, σ = standard deviation (so σ^2 = variance)

$$\boxed{z = \frac{x-\mu}{\sigma}} \qquad \boxed{dz = \frac{dx}{\sigma}}$$

$$\boxed{x = \mu + \sigma z}$$

So if $\boxed{p(x)\,dx = p(z)\,dz}$

$$\boxed{p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}}$$

Define $$\boxed{\Phi(z) = \int_{-\infty}^{z} p(z)\,dz}$$

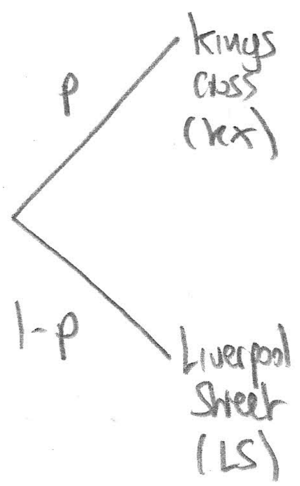
i.e. "probability of z or less"

This is typically tabulated but many calculators and computer programs can evaluate this integral]

Note: If $x \sim N(\mu_x, \sigma_x^2)$ and $y \sim N(\mu_y, \sigma_y^2)$
$x + y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

Henry's journey to work:

let t = total journey time in minutes



Journey times: $t_1 \sim N(55, 25)$ (Cambridge to KX)
$t_2 \sim N(30, 144)$ (KX to office)

$$\therefore \boxed{t = t_1 + t_2 \sim N(85, 13^2)}$$

Journey times: $t_1 \sim N(65, 16)$ (Cambridge to LS)
$t_2 \sim N(25, 9)$

$$\therefore \boxed{t = t_1 + t_2 \sim N(90, 5^2)}$$

Henry catches the KX train at 0715
LS train at 0720

He must be at work no later than 0900.

$\therefore$ "being late" means $\begin{cases} KX & t > 105 \\ LS & t > 100 \end{cases}$

$\therefore$ $P(\text{late via KX}) = 1 - \underbrace{\Phi\left(\frac{105-85}{13}\right)}$

probability of t taking 105 minutes or less via KX route

$= \boxed{1 - \Phi(20/13)}$

$= A$

$\therefore$ $P(\text{late via LS}) = 1 - \Phi\left(\frac{100-90}{5}\right)$

$= \boxed{1 - \Phi(2)}$

$= B$

[Note since $2 > 20/13$
$A > B$]

The overall probability of being late is therefore

$P(\text{late}) = p\,P(\text{late via KX}) + (1-p)\,P(\text{late via LS})$

$= pA + (1-p)B$

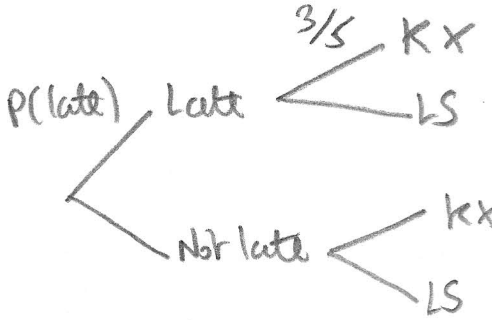
$\therefore$ If Henry takes M journeys, the expected # of late arrivals is

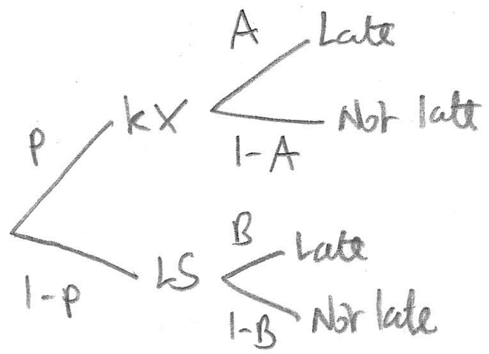
$$\boxed{L = MpA + (1-p)BM}$$

when $p = 0$, $L = BM$ $\therefore$ $\boxed{BM \leq L \leq AM}$
$p = 1$, $L = AM$

Now <u>when he has been late</u>, Henry caught the KX train

3 in 5 occasions.





From the equivalence of the tree diagrams above (i.e. we are essentially using Bayes' Theorem)

$$P(\text{late}) \times \tfrac{3}{5} = pA$$

$$\therefore \left(pA + (1-p)B\right) \times \tfrac{3}{5} = pA$$

$$p\left(\frac{3A}{5} - \frac{3B}{5} - A\right) = -\frac{3B}{5}$$

$$p\left(3A - 3B - 5A\right) = -3B$$

$$p\left(-2A - 3B\right) = -3B$$

$$\therefore \boxed{p = \frac{3B}{2A+3B}}$$

[$A \approx \boxed{0.062}$

$B \approx \boxed{0.02278}$

$\therefore p \approx \boxed{0.3553}$]

Also $P(\text{late}) \approx \boxed{0.0367}$

13/ Biologists capture a random sample of 200 voles and mark them. They are then released.

A second random sample of 200 voles reveals 11 of them marked.

If the number of voles is N, the probability of a marked vole before the second sample is $\boxed{p = \frac{200}{N}}$

∴ probability of 11 marked voles in next 200 is

$$P_N = \underbrace{P(\underbrace{MM \ldots\ldots M}_{11 \text{ marked}}\ \underbrace{\cancel{M}\cancel{M} \ldots \cancel{M}}_{189 \text{ not marked}})}_{\text{probability of 11 marked then 389 not marked IN THAT ORDER}} \times \underbrace{\frac{200!}{11!\,189!}}_{\text{\# permutations of 11 marked and 189 not marked voles}}$$

$$P_N = \underbrace{\left(\frac{200}{N}\right)\left(\frac{199}{N-1}\right)\left(\frac{198}{N-2}\right) \cdots\cdots \left(\frac{190}{N-10}\right)}_{\text{Marked voles}} \underbrace{\left(\frac{N-200}{N-11}\right)\left(\frac{N-201}{N-12}\right) \cdots\cdots \left(\frac{N-388}{N-199}\right)}_{\text{unmarked voles}}$$

$$\times\ {}^{200}C_{11} \text{ ie } \frac{200!}{11!\,189!}$$

Now: $200 \times 199 \times 198 \times \ldots\ldots 190 = \frac{200!}{189!}$

and $(N-200)(N-201) \ldots (N-388) = (N-200)!/(N-389)!$

and $N(N-1) \ldots N-199 = N!/(N-200)!$

$$\therefore\ P_N = \frac{200!}{11!\,189!} \times \frac{200!}{189!} \times \frac{(N-200)!}{(N-389)!} \times \frac{(N-200)!}{N!}$$

$$\boxed{P_N = \frac{k\,[(N-200)!]^2}{N!\,(N-389)!}}$$ as required

where $\boxed{k = \left(\frac{200!}{189!}\right)^2 \frac{1}{11!}}$

[An easier way perhaps (!)

$$P_N = \begin{array}{c}\#\text{ permutations}\\ \text{of 189 votes in}\\ 200\\ \text{un-marked}\end{array} \times \frac{\#\text{ permutations of unmarked votes in unmarked population}}{\#\text{ permutations of chosen votes from } N}$$

ie $${}^{200}C_{189} \times \frac{{}^{N-200}C_{189}}{{}^{N}C_{200}}$$

$$= \frac{200!}{189!\,11!} \times \frac{\dfrac{(N-200)!}{(N-189-200)!\,189!}}{\dfrac{N!}{(N-200)!\,200!}}$$

$$= \boxed{\left(\frac{200!}{189!}\right)^2 \frac{1}{11!} \quad \frac{[(N-200)!]^2}{(N-389)!\,N!}} \quad]$$

Now desire N that maximises P_N. Since N is a +ve integer $\frac{dP_N}{dN} = 0$ is not a good method!

Consider instead the inequality $\boxed{P_{N+1} < P_N}$

$$P_{N+1} < P_N$$

$$\frac{[(N-199)!]^2}{(N+1)!\,(N-388)!} < \frac{[(N-200)!]^2}{N!\,(N-389)!}$$

$$\frac{(N-199)^2\,[(N-200)!]^2}{(N+1)\,N!\,(N-388)(N-389)!} < \frac{[(N-200)!]^2}{N!\,(N-389)!}$$

$$(N-199)^2 < (N+1)(N-388)$$

$$\cancel{N^2} - 398N + 199^2 < \cancel{N^2} - 387N - 388$$

$$39989 < 11N$$

$$\therefore \quad N > \frac{39989}{11}$$

$$N > 3635\tfrac{4}{11}$$

$$\boxed{N \geq 3636}$$

So P_N is maximized when $\boxed{N = 3636}$

Now if the 189 unmarked voles in the second sample are marked and subsequently released, a third random sample* of the N voles will have j marked voles with probability P_j

This scenario is identical to the second sample, except there are j marked voles and now $200 + 189 = \boxed{389}$ marked voles out of the population of N.

* Another 200 voles chosen

Now when $j = 11$ and 200 marked voles

$$P_N = \frac{{}^{200}C_{200-11} \times {}^{N-200}C_{200-11}}{{}^{N}C_{200}}$$

[Note ${}^{200}C_{200-11} = {}^{200}C_{11}$]

Hence in the third sample $11 \rightarrow j$
$200 \rightarrow 389$

$$\boxed{P_j = \frac{{}^{389}C_j \times {}^{N-389}C_{200-j}}{{}^{N}C_{200}}}$$

This is a more efficient explanation of P_N

i.e ${}^{389}C_j$: # combinations of j marked voles from 389

${}^{N-389}C_{200-j}$: # combinations of $200-j$ unmarked voles from $N-389$ unmarked population

${}^{N}C_{200}$: # combinations of 200 distinct voles from population of N.

ie P_j is the total # of combinations of the j marked and $200-j$ unmarked voles / combinations of 200 voles from N

Now $\boxed{\sum_{j=1}^{200} P_j = 1}$

$$\therefore \sum_{j=1}^{200} {}^{389}C_j \times {}^{N-389}C_{200-j} = {}^{N}C_{200}$$

If $\boxed{N = 3636}$ (ie # voles which maximize P_N)

$$\boxed{\sum_{j=1}^{200} \binom{389}{j}\binom{3247}{200-j} = \binom{3636}{200}}$$

as required $\left[\binom{n}{r} \equiv {}^{n}C_{r}\right]$

14. The random variables $X_1, X_2, \ldots X_{2n+1}$ are all $\sim U(0,1)$ i.e. uniformly distributed within the interval $[0,1]$.

Let Y be the median of $\{X_1, X_2, \ldots X_{2n+1}\}$

The PDF of Y is $\boxed{g(y) = \begin{cases} ky^n(1-y)^n & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}}$

Now $\displaystyle\int_{-\infty}^{\infty} g(y)\,dy = 1$

$$\Rightarrow \int_0^1 ky^n(1-y)^n\,dy = 1$$

Now $\boxed{\displaystyle\int_0^1 y^r(1-y)^s\,dy = \frac{r!\,s!}{(r+s+1)!}}$ [Standard integral, supplied]

$$\therefore \int_0^1 y^n(1-y)^n\,dy = \frac{(n!)^2}{(2n+1)!}$$

$$\boxed{k = \frac{(2n+1)!}{(n!)^2}}$$ as required

Now $\displaystyle E[Y] = \int_0^1 y\,g(y)\,dy$

$$= k\int_0^1 y^{n+1}(1-y)^n\,dy$$

$$= k\frac{(n+1)!\,n!}{(2n+2)!} = \frac{(2n+1)!\,(n+1)!\,n!}{(n!)^2(2n+2)!}$$

$$= \frac{(2n+1)!\,(n+1)\,n!\,n!}{(n!)^2(2n+2)(2n+1)!} = \frac{n+1}{2n+2} = \boxed{\frac{1}{2}}$$

$$\mathrm{Var}[y] = E[Y^2] - (E[Y])^2 = \int_0^1 y^2 g(y)\,dy - \frac{1}{4}$$

$$= \frac{(2n+1)!}{(n!)^2} \int_0^1 y^{n+2} (1-y)^n \, dy - \frac{1}{4}$$

$$= \frac{(2n+1)!}{(n!)^2} \frac{(n+2)!\, n!}{(2n+3)!}$$

$$= \frac{(2n+1)!\ (n+2)(n+1)\ (n!)^2}{(n!)^2\ (2n+3)(2n+2)\ (2n+1)!} - \frac{1}{4}$$

$$= \frac{(n+2)(n+1)}{(2n+3)(2n+2)} - \frac{1}{4}$$

$$= \frac{1}{2} \frac{n+2}{2n+3} - \frac{1}{4}$$

$$= \frac{2(n+2)}{4(2n+3)} - \frac{1}{4} \frac{(2n+3)}{2n+3}$$

$$= \frac{2n+4-2n-3}{4(2n+3)}$$

$$= \boxed{\frac{1}{4(2n+3)}}$$

Now $E[X] = \frac{1}{2}$

[PDF is $f(x) = \begin{cases} 1/q & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$\int_0^1 f(x)\,dx = 1 \Rightarrow \boxed{q = 1}$$

$\therefore \quad E[X] = \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$ ✓]

$$V[X] = E[X^2] - (E[X])^2$$
$$= \int_0^1 x^2\,dx - \frac{1}{4}$$
$$= \left[\frac{1}{3}x^3\right]_0^1 - \frac{1}{4}$$
$$= \frac{1}{3} - \frac{1}{4}$$
$$= \frac{4-3}{12}$$
$$= \boxed{\frac{1}{12}}$$

$$\therefore \quad E[\bar{X}] = E\left[\frac{X_1 + X_2 + X_3 + \dots X_{2n+1}}{2n+1}\right]$$
$$= \boxed{\frac{1}{2}}$$

$[\, E[aX \pm bY] = aE[X] \pm bE[Y] \,]$

$$V[\bar{X}] = V\left[\frac{X_1 + X_2 + X_3 + \dots X_{2n+1}}{2n+1}\right]$$
$$= \frac{1}{(2n+1)^2} V[X_1] + \frac{1}{(2n+1)^2} V[X_2] + \dots\dots$$

$[\; V[aX \pm bY] = a^2 V[X] + b^2 V[Y] \,]$

$$= \boxed{\frac{1}{12(2n+1)}}$$

$\therefore$ If n is large Y and $\bar{X}$ are normally distributed
[This is stated]

$$\therefore \quad \boxed{\bar{X} \sim N\Big(\underset{\mu}{\tfrac{1}{2}},\ \underset{\sigma^2}{\tfrac{1}{12(2n+1)}}\Big)}$$

$$\boxed{Y \sim N\Big(\tfrac{1}{2},\ \tfrac{1}{4(2n+3)}\Big)}$$

So clearly $\bar{X}$ has a smaller variance than Y
$\therefore$ probability of of $\bar{X}$ being within a given interval from the mean is greater than Y.

$\longrightarrow$ which justifies the inequality

$$P\left(|Y-\tfrac{1}{2}| < d/\sqrt{n}\right) < P\left(|\bar{X}-\tfrac{1}{2}| < d/\sqrt{n}\right)$$

[d is any +ve number]

"Given interval from the mean"