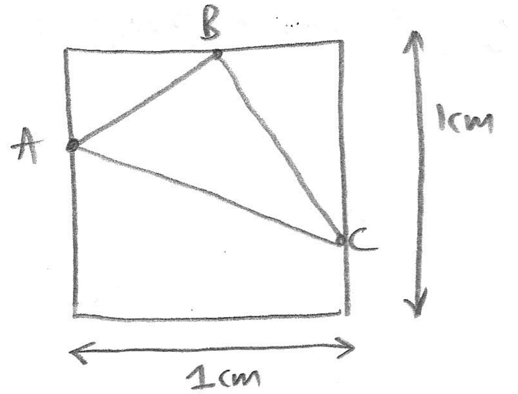
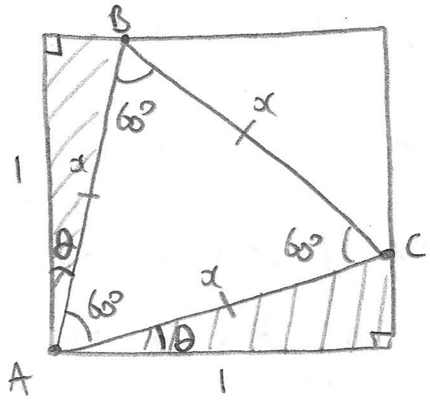
STEP I 2001

1/



Points A, B, C lie on <u>different</u> sides of a 1cm × 1cm square and form a triangle ABC.

The largest possible value of the <u>smallest</u> side is when all three sides of ABC are <u>equal</u> ie ABC is an equilateral triangle. Let the 'largest, smallest side' $= x$



Since AB = AC then shaded triangles must be congruent.

$\therefore 2\theta + 60° = 90°$

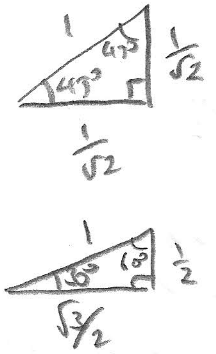
$\Rightarrow \boxed{\theta = 15°}$

Hence $x\cos 15° = 1 \quad \therefore x = \dfrac{1}{\cos 15°}$

Now $\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$



$$\therefore x = \frac{2\sqrt{2}}{1+\sqrt{3}} = \frac{2\sqrt{2}(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})} = \frac{2\sqrt{2} - 2\sqrt{6}}{1-3} = \frac{2\sqrt{6} - 2\sqrt{2}}{2}$$

$\therefore \boxed{x = \sqrt{6} - \sqrt{2}}$ as required.

2/ (i) $1+2x-x^2 > \frac{2}{x}$ $\quad (x \neq 0)$

Since $x^2 > 0$, multiplying both sides by x^2 doesn't change the sign of the inequality

$$x^2(1+2x-x^2) > 2x$$

$$x^2+2x^3-x^4 > 2x$$

$$0 > x^4-2x^3-x^2+2x$$

Consider $f(x) = x^4-2x^3-x^2+2x$ $\quad$ [$\therefore$ inequality above $\Rightarrow f(x)<0$]

$(*)$ $f(x) = x(x^3-2x^2-x+2)$ $\quad$ so $x=0$ is a factor of $f(x)$

$f(1) = 1(1-2-1+2) = 0$ $\quad \therefore$ $x-1$ is a factor of $f(x)$

$$\therefore \quad f(x) = x(x-1)(x^2+ax-2)$$

$$= x(x^3-x^2+ax^2-ax-2x+2)$$

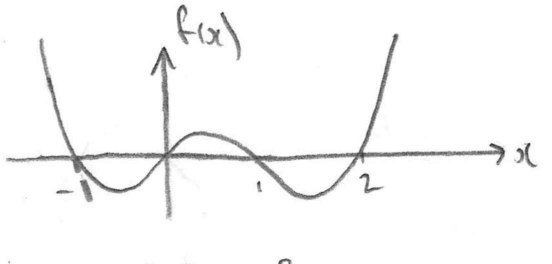
$$= x(x^3+x^2(a-1)-x(a+2)+2)$$

comparing coefficients with $(*)$ $\quad a-1=-2 \quad [x^2] \Rightarrow a=-1$

$\quad a+2 = 1 \quad [x] \Rightarrow a=-1$

$$\therefore \quad f(x) = x(x-1)(x^2-x-2)$$

$$\boxed{f(x) = x(x-1)(x+1)(x-2)}$$



$\therefore$ $f(x)<0$ means

$$\boxed{\begin{cases} -1<x<0 \\ 1<x<2 \end{cases}}$$

which solves the inequality

[Note $f(x)>0$ as $|x| \gg 1$ since $f(x) \propto x^4$ as $|x|$ large]

(ii) $\sqrt{3x+10} > 2+\sqrt{x+4}$ $\quad x \geq -10/3$

$$\therefore \quad 3x+10 > (2+\sqrt{x+4})^2$$

$$3x+10 > 4+4\sqrt{x+4}+x+4$$

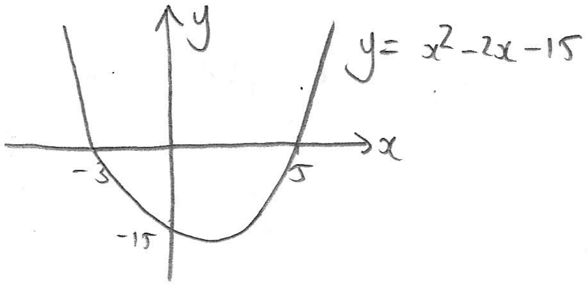
$$2x+2 > 4\sqrt{x+4}$$

$x+1 > 2\sqrt{x+4}$ ← Note both sides +ve if $x>-1$

$x^2+2x+1 > 4(x+4)$

$x^2-2x-15 > 0$

$(x+3)(x-5) > 0$



So from graph: $\boxed{\begin{matrix} x < -3 \\ x > 5 \end{matrix}}$

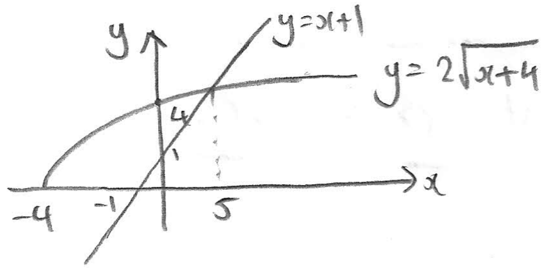
check lower limit of $x = -3$

$\sqrt{3x+10} = 1$

$2 + \sqrt{x+4} = 3$

So clearly $\sqrt{3x+10} > 2 + \sqrt{x+4}$ is not true here....

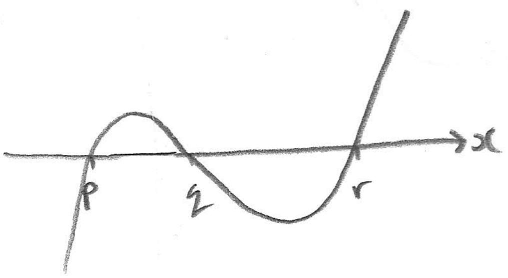
↳ go back to $x+1 > 2\sqrt{x+4}$



clearly $x > 0$ and only one intersection is $x = 5$, since $2\sqrt{x+4} \geq 0$

So overall solution to $\sqrt{3x+10} > 2 + \sqrt{x+4}$ is $\boxed{x > 5}$

3/ $f(x) = (x-p)(x-q)(x-r)$ $\quad p < q < r$



$f(x) = (x-p)(x^2 - qx - rx + qr)$

$f(x) = x^3 - px^2 - qx^2 + pqx - rx^2 + prx + qrx - pqr$

$f(x) = x^3 + x^2(-p-q-r) + x(pq + pr + qr) - pqr$

ie/ $f = 0$ at $x = p, q, r$

$\therefore f'(x) = 3x^2 - 2x(p+q+r) + pq + pr + qr$

if $f'(x) = 0$ (ie stationary points), by the quadratic formula

$$x = \frac{2(p+q+r) \pm \sqrt{4(p+q+r)^2 - 4(3)(pq+pr+qr)}}{6}$$

For there to be real solutions, and distinct stationary points (ie as described in the sketch of $f(x)$ above and required since $p < q < r$)

$\Rightarrow \boxed{(p+q+r)^2 > 3(pq + pr + qr)}$ ie discriminant of $f'(x) = 0$ quadratic is > 0

Now consider the cubic of the form $f(x) = (x^2 + gx + h)(x - k)$. Motivated by the first part of this question, let's see what conditions arise from $f'(x) = 0$

$f(x) = x^3 - kx^2 + gx^2 - kgx + hx - kh$

$f(x) = x^3 + x^2(g-k) + x(h-kg) - kh$

$f'(x) = 3x^2 + 2x(g-k) + h - kg$

$\therefore$ if $f'(x) = 0 \Rightarrow$ for real solutions $4(g-k)^2 \geq 4(3)(h-kg)$

⇒ $\boxed{(g-k)^2 \geq 3(h-gk)}$

So $f'(x) = 0$ having distinct real solutions ⇒ $(g-k)^2 > 3(h-gk)$

ie $f(x)$ has two, distinct, stationary points.

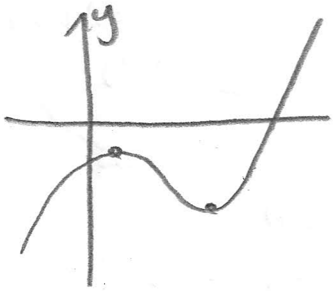
[one when $(g-k)^2 = 3(h-gk)$]

Now what about $g^2 > 4h$, well this is the condition for $x^2+gx+h=0$ to have two distinct real solutions.

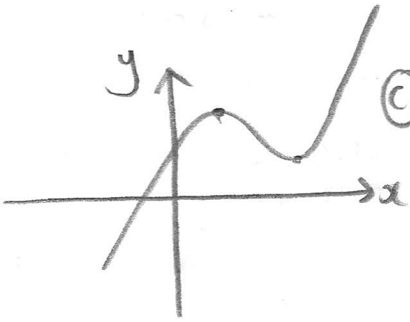
[ie $x = \dfrac{-g \pm \sqrt{g^2-4(1)h}}{2}$]

So, in practical terms $g^2 > 4h \Rightarrow f(x)$ crosses the x axis at k, and $\dfrac{-g \pm \sqrt{g^2-4h}}{2}$ ie two or three times depending on k.

But...

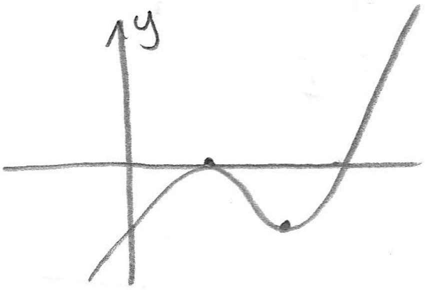


(A)

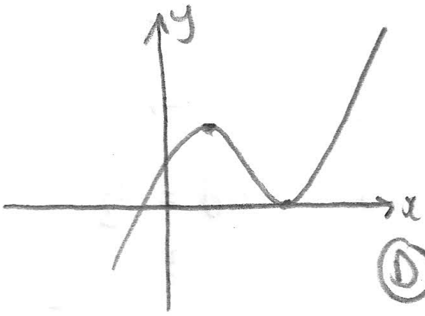


(C)

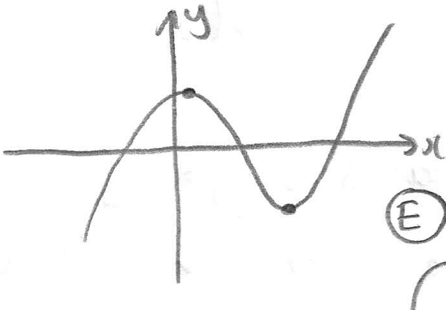
... are the possibilities for two distinct stationary points given $f(x) = (x^2+gx+h)(x-k)$



(B)



(D)



(E)

In cases (A), (C) there are two distinct stationary points but only one distinct root.

∴ $g^2 > 4h$ is not true in these cases, but $(g-k)^2 > 3(h-gk)$ is true.

So $g^2 - 4h$ is a sufficient BUT NOT NECESSARY condition for $(g-k)^2 > 3(h-gk)$

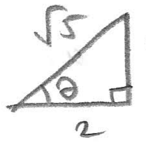
4/ $\tan 3\theta = \tan(2\theta + \theta) = \dfrac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}$

$\tan 2\theta = \tan(\theta + \theta) = \dfrac{\tan\theta + \tan\theta}{1 - \tan^2\theta} = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

$\therefore \tan 3\theta = \dfrac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \dfrac{2\tan^2\theta}{1-\tan^2\theta}}$

$\tan 3\theta = \dfrac{2\tan\theta + \tan\theta(1-\tan^2\theta)}{1 - \tan^2\theta - 2\tan^2\theta}$

$$\boxed{\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}}$$

[Note 

So $\tan\theta = \frac{1}{2}$ can be obtained more easily!]

Now let $\theta = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$ and $0 < \theta < \pi/2$

$\therefore \cos\theta = \frac{2}{\sqrt{5}}$ Now $1 + \tan^2\theta = \dfrac{1}{\cos^2\theta}$

$\Rightarrow \tan^2\theta = \frac{5}{4} - 1 = \frac{1}{4}$

$\Rightarrow \tan\theta = \pm\frac{1}{2}$

Now $\theta > 0$ so $\tan\theta = \frac{1}{2}$

$\therefore \tan 3\theta = \dfrac{3\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^3}{1 - 3\left(\frac{1}{2}\right)^2} = \dfrac{3/2 - 1/8}{1 - 3/4} = \dfrac{\frac{12-1}{8}}{\frac{1}{4}}$

$= \boxed{11/2}$

(i) Consider $\tan(3\cos^{-1}x) = 11/2$

$\therefore$ if $\cos^{-1}x = \theta \Rightarrow \tan 3\theta = 11/2 \Rightarrow 3\theta = \tan^{-1}(11/2) + N\pi$ (Integer)

Now from above, $\tan^{-1}(11/2) = 3\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$ ← in range $[0, \pi]$

So $\quad 3\cos^{-1}(x) = 3\cos^{-1}\left(\frac{2}{\sqrt{5}}\right) + N\pi$

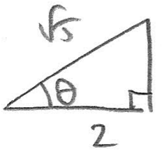
[Alternatively write $11/2$ as
$\tan^{-1}(11/2) = 3\cos^{-1}\left(\frac{2}{\sqrt{5}}\right) + N\pi$
i.e. write down this line directly....]

$$\cos^{-1}(x) = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) + \frac{N\pi}{3}$$

$$x = \cos\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right) + \frac{N\pi}{3}\right)$$

$$= \cos\left(\cos^{-1}\frac{2}{\sqrt{5}}\right)\cos\left(\frac{N\pi}{3}\right) - \sin\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)\sin\frac{N\pi}{3}$$

Now :



so $\sqrt{5}\sin\theta = 1$

$$\Rightarrow \sin\left(\cos^{-1}\frac{2}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}$$

$\Rightarrow \quad \sqrt{5}\cos\theta = 2$

$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$

$$\therefore \quad \boxed{x = \frac{2}{\sqrt{5}}\cos\frac{N\pi}{3} - \frac{1}{\sqrt{5}}\sin\frac{N\pi}{3}}$$

From unit circle possibilities of $\left(\cos\frac{N\pi}{3}, \sin\frac{N\pi}{3}\right)$ are

$(1,0)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

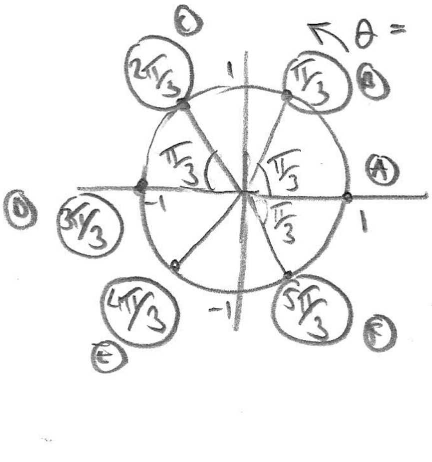
So $x = \pm\frac{2}{\sqrt{5}}$ Ⓐ, Ⓓ

$\frac{1}{\sqrt{5}} - \frac{\sqrt{3}}{2\sqrt{5}}$ Ⓑ

$-\frac{1}{\sqrt{5}} - \frac{\sqrt{3}}{2\sqrt{5}}$ Ⓒ

$-\frac{1}{\sqrt{5}} + \frac{\sqrt{3}}{2\sqrt{5}}$ Ⓔ

$\frac{1}{\sqrt{5}} + \frac{\sqrt{3}}{2\sqrt{5}}$ Ⓕ



So solutions are $\boxed{x = \pm\frac{2}{\sqrt{5}}, \quad \frac{\pm 1 \pm \sqrt{3}}{2\sqrt{5}}}$

(Note solutions don't allow all of these ... seemingly imposed constraint on $0 < \frac{N\pi}{3} < \pi$?)

$\frac{2}{\sqrt{5}}, \quad \frac{\pm 2 - \sqrt{3}}{2\sqrt{5}}$ are the answers. So Ⓓ, Ⓔ, Ⓕ excluded.

(ii) $\cos\left(\frac{1}{3}\tan^{-1} y\right) = \frac{2}{\sqrt{5}}$

if $\theta = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) \Rightarrow \cos\left(\frac{1}{3}\tan^{-1} y\right) = \cos\theta$

So given $\cos x$ repeats every 2π and $\cos(-x) = \cos x$

$\Rightarrow \frac{1}{3}\tan^{-1} y = 2\pi N \pm \theta$ (N integer)

$\Rightarrow \tan^{-1} y = 6\pi N \pm 3\theta$

$y = \tan(6\pi N \pm 3\theta)$

Now $\tan x$ repeats every π radians so $y = \tan(\pm 3\theta)$

and $\tan(-x) = -\tan x$

$\therefore \quad y = \pm\tan 3\theta$

$\therefore$ using first part $\boxed{y = \pm 11/2}$

5/ To answer this question, let us consider a more general integral

$$I_n = \int \frac{x^n}{(1+tx)^{n+2}}\, dx$$

let $u = \dfrac{x}{1+tx}$ $\therefore$ $\dfrac{du}{dx} = \dfrac{(1+tx)(1) - x(t)}{(1+tx)^2}$

$\dfrac{du}{dx} = \dfrac{1}{(1+tx)^2}$

$\Rightarrow dx = (1+tx)^2\, du$

$$\therefore I_n = \int \frac{u^n}{(1+tx)^2} \times (1+tx)^2\, du$$

$$I_n = \int u^n\, du$$

$$I_n = \frac{u^{n+1}}{n+1} + C$$

$$\therefore \boxed{I_n = \frac{\left(\frac{x}{1+tx}\right)^{n+1}}{n+1} + C}$$

$$\therefore \int_0^1 \frac{x^n}{(1+tx)^{n+2}}\, dx = \left[\left(\frac{x}{1+tx}\right)^{n+1} \Big/ n+1 \right]_0^1$$

$$= \boxed{\frac{1}{n+1} \frac{1}{(1+t)^{n+1}}}$$

(i) $\therefore \displaystyle\int_0^1 \frac{1}{(1+tx)^2}\, dx = \boxed{\frac{1}{1+t}}$ ie $n=0$

(ii) $\therefore \displaystyle\int_0^1 \frac{x}{(1+tx)^3}\, dx = \frac{1}{2} \frac{1}{(1+t)^2}$ ie $n=1$

$$\therefore \int_0^1 \frac{-2x}{(1+tx)^3}\,dx = -\frac{1}{(1+t)^2}$$

$$\therefore \int_0^1 \frac{6x^2}{(1+x)^4}\,dx = 6 \times \frac{1}{3} \times \frac{1}{(1+1)^3}$$

ie $t = 1$

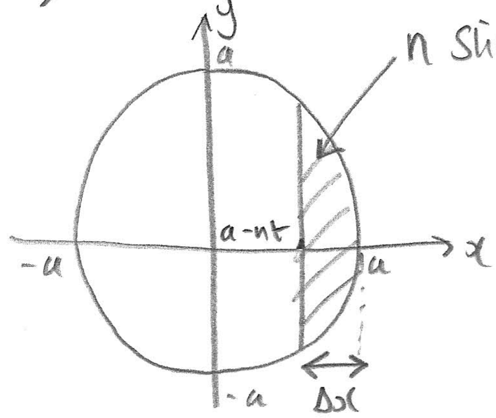
$$= \frac{2}{8}$$

$$= \boxed{\frac{1}{4}}$$

From this question we can note a nice standard integral

$$\boxed{\int \frac{x^n}{(1+tx)^{n+2}}\,dx = \frac{\left(\frac{x}{1+tx}\right)^{n+1}}{n+1} + c}$$

6/



↑
x,y section of a spherical loaf of bread

n slices of thickness t removed $\therefore$ # remaining slices is $\boxed{2a/t - n}$

Surface area of crust remaining (after n slices removed) is

$$A = 2\pi \int_{-a}^{-a+2a-nt} y\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx$$

[Since $\Delta x = nt$]

Now $x^2+y^2 = a^2$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -x/y$$

$$\therefore \quad y\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} = y\left(1+\frac{x^2}{y^2}\right)^{\frac{1}{2}}$$

$$= \sqrt{y^2\left(1+x^2/y^2\right)}$$

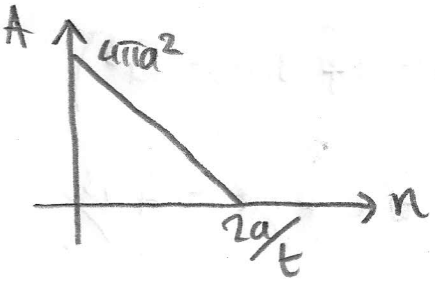
$$= \sqrt{y^2+x^2} = a.$$

So $$A = 2\pi a\int_{-a}^{a-nt} dt = 2\pi a\left[t\right]_{-a}^{a-nt}$$

$$= 2\pi a\left(2a-nt\right)$$

$$\therefore \quad \boxed{A = 4\pi a^2 - 2\pi a n t}$$

ie



So crust area remaining $\propto$ # slices taken and $\therefore$ if m is the # of slices remaining $= \frac{2a}{t} - n$

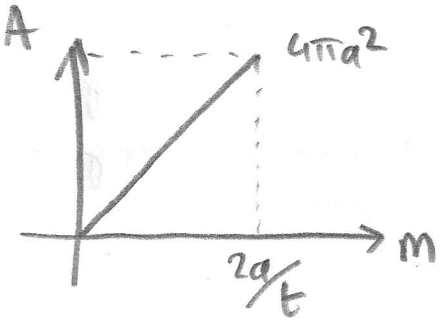
$$m = \frac{2a}{t} - n \quad \Rightarrow \quad n = \frac{2a}{t} - m$$

$$\therefore \quad A = 4\pi a^2 - 2\pi a t\left(\frac{2a}{t} - m\right)$$

$$A = 4\pi a^2 - 4\pi a^2 + 2\pi a t m$$

$$\boxed{A = 2\pi a t\, m}$$

So $A \propto$ # slices remaining



crushy baf if $V/A < 1$ where V is volume of bread remaining after n slices have been taken.

$$V = \pi\int_{-a}^{-a+2a-nt} \underbrace{(a^2 - x^2)}_{y^2}\,dx$$

$$= \pi\left[a^2x - \tfrac{1}{3}x^3\right]_{-a}^{a-nt}$$

Now lets evaluate in terms of # slices remaining m

ie $mt = 2a - nt$

So $mt - a = a - nt$

$$\therefore \quad V = \pi\left[a^2x - \tfrac{1}{3}x^3\right]_{-a}^{mt-a}$$

$$V = \pi\left\{\left(a^2(mt-a) - \tfrac{1}{3}(mt-a)^3\right) - \left(-a^3 + \tfrac{1}{3}a^3\right)\right\}$$

$$= \pi\left\{a^2mt - a^3 - \tfrac{1}{3}\left(m^3t^3 - 3m^2t^2a + 3mta^2 - a^3\right) + a^3 - \tfrac{1}{3}a^3\right\}$$

$$= \pi\left\{\cancel{a^2mt} - a^3 - \tfrac{1}{3}m^3t^3 + m^2t^2a - \cancel{mta^2} + \cancel{\tfrac{a^3}{3}} + a^3 - \cancel{\tfrac{a^3}{3}}\right\}$$

$$= \pi\left(m^2t^2a - \tfrac{1}{3}m^3t^3\right)$$

$\therefore \boxed{V = \frac{1}{3}\pi m^2 t^2 (3a - mt)}$

Hence $V/A = \dfrac{\frac{1}{3}\pi m^2 t^2 (3a - mt)}{2\pi a t M}$

$= \frac{1}{6a} mt (3a - mt)$

So if $V/A < 1$

$\Rightarrow \quad mt(3a - mt) < 6a$

$0 < 6a - 3amt + m^2t^2$

Now # remaining slices m and slice thickness t must both > 0

let $z = mt \qquad \therefore z^2 - 3az + 6a > 0$

$\left(z - \frac{3}{2}a\right)^2 - \frac{9}{4}a^2 + 6a > 0$

$\left(z - \frac{3}{2}a\right)^2 + a\left(6 - \frac{9a}{4}\right) > 0$

$\left(z - \frac{3}{2}a\right)^2 + \frac{a}{4}(24 - 9a) > 0$

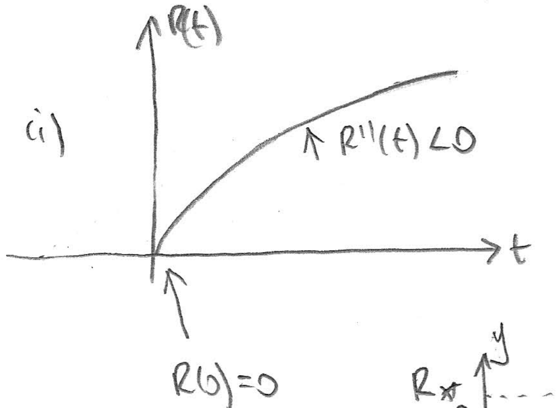
$\left(z - \frac{3}{2}a\right)^2 + \frac{3a}{4}(8 - 3a) > 0 \quad (*)$

Now $0 \leq mt \leq 2a$ so it is possible for $z = mt = \frac{3}{2}a$ $\therefore$ for $(*)$ to be always true $\forall$ allowed values of z, $\Rightarrow 8 - 3a > 0$

$\Rightarrow a < 8/3 \Rightarrow \boxed{a = 2\frac{2}{3}}$

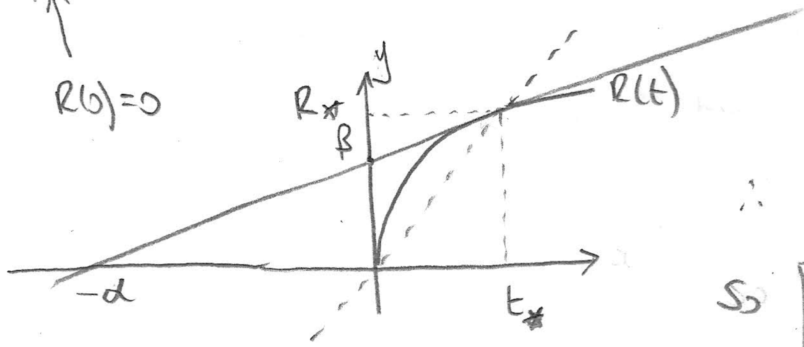
7/ $R(t)$ — radius of the universe at time t $(t>0)$

(i)



$R'(t) > 0$, $t>0$ ie R increases as $t\uparrow$

$R''(t) < 0$, $t>0$ ie gradient reduces as $t\uparrow$



$y_T = R't + \beta$

$R_* = R'(t_*)t_* + \beta$

$\therefore \beta = R_* - R'(t_*)t_*$

So $\boxed{y_T = R'(t_*)(t - t_*) + R_*}$

Define $H(t) = \dfrac{R'(t)}{R(t)}$

if $t < 1/H(t) \Rightarrow t < R/R' \Rightarrow R > R't$

Now $R = R't + \beta$, and from convexity of $R(t)$, $\beta > 0$

$\therefore R > R't$ So $\boxed{t < \dfrac{1}{H(t)}}$ as required

(ii) $H(t) = a/t$ a = constant

$\therefore \dfrac{R'}{R} = a/t \Rightarrow \displaystyle\int \frac{1}{R}\,dR = \int \frac{a\,dt}{t}$

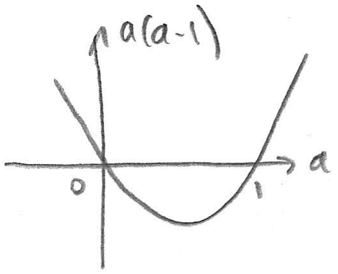
$\Rightarrow \ln R = a\ln t + c$

$\Rightarrow R = t^a e^c$

$\Rightarrow \boxed{R = At^a}$ Since $A = e^c \Rightarrow \boxed{A>0}$

Now $R'(t) = aAt^{a-1}$ so if $R' > 0$, $\Rightarrow \boxed{a>0}$

$R''(t) = a(a-1)At^{a-2}$ so if $R'' < 0 \Rightarrow \boxed{a(a-1) < 0}$



$\therefore \quad \boxed{0 < a < 1}$

So in summary, if $H(t) = a/t$

$$\Rightarrow \quad \boxed{R(t) = At^a \qquad A > 0 \quad 0 < a < 1}$$

(iii) $H(t) = b/t^2$

$\therefore \quad \frac{R'}{R} = \frac{b}{t^2}$

$$\int \frac{1}{R}\,dR = \int \frac{b\,dt}{t^2}$$

$\ln R = -b/t + c$

$\boxed{R = Ae^{-b/t}}$ where $A = e^c$

$\therefore \quad \boxed{A > 0}$

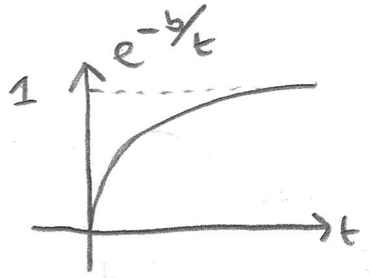
$$\therefore \quad R' = Ae^{-b/t}\left(\frac{b}{t^2}\right) = \boxed{\frac{Abe^{-b/t}}{t^2}}$$

$$R'' = Abe^{-b/t}\left(-\frac{2}{t^3}\right) + \frac{Ab}{t^2}e^{-b/t}\left(+\frac{b}{t^2}\right)$$

$$\boxed{R'' = \frac{Abe^{-b/t}}{t^4}\left(-2t + b\right)}$$

So in summary:
(A>0)

$$R(t) = Ae^{-b/t}$$
$$R'(t) = Abe^{-b/t}$$
$$R''(t) = \frac{Abe^{-b/t}}{t^4}(b - 2t)$$



when $t = 0$, $R(0) = 0$ if $\boxed{b > 0}$ $\quad R'(t) > 0$ is also true if $b > 0$

but $R''(t) < 0$ if $t, A > 0$ if $b < 2t$

ie $\boxed{t > b/2}$ so model $H(t) = b/t^2$ cannot be consistent with $R(0)$, $R'(t) > 0$, $R''(t) < 0$ for $t > 0$ regardless of what b is chosen $(b > 0)$

8/ $\dfrac{d^2y}{dx^2} + p(x)\dfrac{dy}{dx} + q(x)y = 0$ (*)

Assume $y = x$ and $y = 1 - x^2$ satisfy (*)

$y = x$: $0 + p + qx = 0 \Rightarrow \boxed{p + qx = 0}$ ①

$y = 1 - x^2$: $-2 - 2px + q - qx^2 = 0 \Rightarrow \boxed{qx^2 + 2px + 2 - q = 0}$ ②

$\therefore$ using $p = -qx$ and substituting into ②

$qx^2 + 2(-qx)x + 2 - q = 0$

$qx^2 - 2qx^2 + 2 - q = 0$

$-qx^2 - q + 2 = 0$

$-q(1 + x^2) + 2 = 0$

$\boxed{q(x) = \dfrac{2}{1+x^2}}$

$\therefore \boxed{p = -\dfrac{2x}{1+x^2}}$

using $p = -qx$

Now consider $y = ax + b(1 - x^2)$

$\dfrac{dy}{dx} = a - 2bx$

$\dfrac{d^2y}{dx^2} = -2b$

$\therefore$ in (*), given $q(x) = \dfrac{2}{1+x^2}$ and $-\dfrac{2x}{1+x^2}$

$-2b - \dfrac{2x}{1+x^2}(a - 2bx) + \dfrac{2}{1+x^2}(ax + b - bx^2) = 0$

$\dfrac{1}{1+x^2}\left\{ \cancel{-2b} - \cancel{2bx^2} - \cancel{2xa} + \cancel{4bx^2} + \cancel{2ax} + \cancel{2b} - \cancel{2bx^2} \right\} = 0$

ie $\{\ldots\} = 0$ for any a, b so $y = ax + b(1 - x^2)$

satisfies (*) for any a, b.

Now consider $y = \cos^2(x^2/2)$

To make the algebra easier use identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$\therefore \cos^2(x^2/2) = \frac{1}{2}(1+\cos x^2)$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(-\sin x^2)\times 2x = -x\sin x^2$$

$$\frac{d^2y}{dx^2} = -x\cos x^2 \times 2x - \sin x^2 = -2x^2\cos x^2 - \sin x^2$$

$$\therefore \text{ in } (*) \quad -2x^2\cos x^2 - \sin x^2 - px\sin x^2 + q/2 + \tfrac{q}{2}\cos x^2 = 0$$

$$\Rightarrow \quad \cos x^2\left(-2x^2 + q/2\right) + \sin x^2\left(-1-px\right) + q/2 = 0 \quad ③$$

Now consider $y = \sin^2(x^2/2) = \frac{1}{2}(1-\cos x^2)$

$$\therefore \frac{dy}{dx} = x\sin x^2$$

$$\frac{d^2y}{dx^2} = 2x^2\cos x^2 + \sin x^2$$

$\}$ $-$ y above.

$$\therefore \text{ in } (*) \quad 2x^2\cos x^2 + \sin x^2 + px\sin x^2 + q/2 - q/2\cos x^2 = 0$$

$$\Rightarrow \cos x^2\left(2x^2 - q/2\right) + \sin x^2\left(1+px\right) + q/2 = 0 \quad ④$$

③ + ④ : $\boxed{q = 0}$

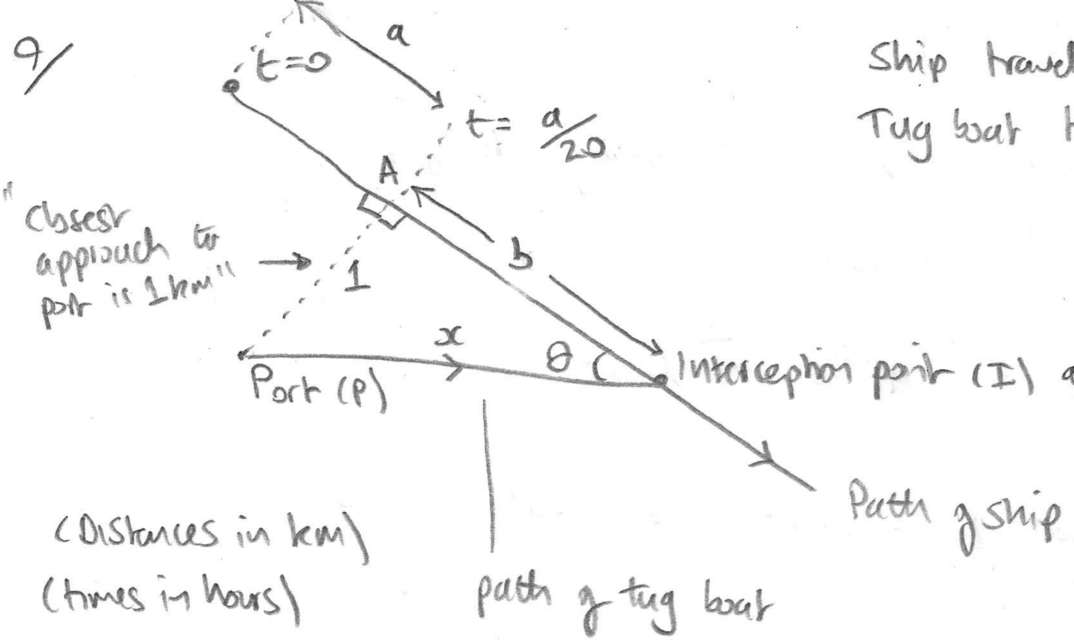
$$\therefore \quad 2x^2\cos x^2 + \sin x^2 + px\sin x^2 = 0$$

$$p = \frac{-\sin x^2 - 2x^2\cos x^2}{x\sin x^2}$$

$$\Rightarrow \quad \boxed{p = -\frac{1}{x} - 2x\cot x^2}$$

9/

Ship travels at speed 20 km/h
Tug boat travels at speed 12 km/h



$$\boxed{t = \frac{a+b}{20}} \quad ①$$

$$\boxed{\begin{aligned} x\cos\theta &= b \\ x\sin\theta &= 1 \end{aligned}} \text{ (Trig)}$$

Now from geometry of diagram $\boxed{x^2 = 1 + b^2}$ (pythagorus)

let tug boat depart at time T (ie T hours after ship is sighted a distance a from point of closest approach to port at A). AIM IS TO MAXIMIZE T

$\therefore$ $\boxed{x = 12(t-T)}$ ② Since $t-T$ is the time taken to intercept once the tug has departed.

θ appears to characterize the trajectory of the tug (it is the only variable available apart from T) $\therefore$ consider $T(\theta, a)$ and find T s.t. $\frac{dT}{d\theta} = 0$

① ↘

$$\therefore t = \frac{a}{20} + \frac{x\cos\theta}{20} = \frac{a}{20} + \frac{\cos\theta}{20\sin\theta} \quad \text{Since } x = \frac{1}{\sin\theta}$$

$$\therefore \frac{1}{\sin\theta} = 12\left(\frac{a}{20} + \frac{\cos\theta}{20} - T\right)$$

② ↗

$$\therefore \frac{1}{12\sin\theta} = \frac{a}{20} + \frac{\cos\theta}{20} - T$$

$$\boxed{T = \frac{a}{20} + \frac{\cos\theta}{20} - \frac{1}{12\sin\theta}}$$

$$\therefore \quad \frac{dT}{d\theta} = -\frac{1}{20}\frac{1}{\sin^2\theta} + \frac{1}{12\sin^2\theta}\cos\theta = \frac{\frac{1}{12}\cos\theta - \frac{1}{20}}{\sin^2\theta}$$

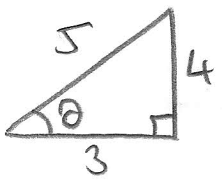
$$\left[\frac{d}{d\theta}(\cot\theta) = \frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta}\right) = \frac{\sin\theta(-\sin\theta) - \cos\theta\cos\theta}{\sin^2\theta} = -\frac{1}{\sin^2\theta} \right]$$

Since $\sin^2\theta + \cos^2\theta = 1$

$$\therefore \quad \boxed{\frac{dT}{d\theta} = \frac{\frac{1}{3}\cos\theta - \frac{1}{5}}{4\sin^2\theta}}$$

$\therefore \quad \frac{dT}{d\theta} = 0$ when $\frac{1}{3}\cos\theta = \frac{1}{5} \Rightarrow \boxed{\cos\theta = 3/5}$

$\Rightarrow \theta \approx 53.13°$



Using the 3,4,5 triangle, $\boxed{\sin\theta = \frac{4}{5}}$

$\therefore \quad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{3}{4}$

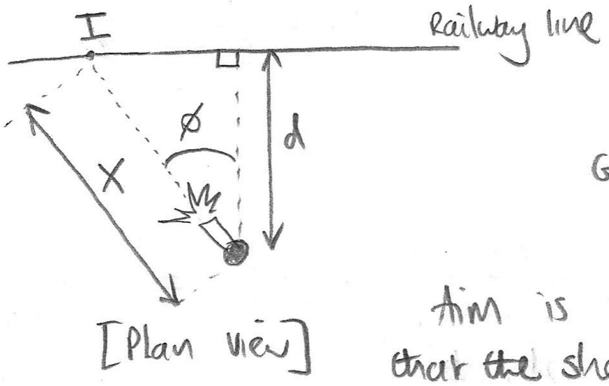
Hence $\quad T_{max} = \frac{a}{20} + \frac{3}{80} - \frac{5}{48} = \boxed{\frac{a}{20} - \frac{1}{15}}$

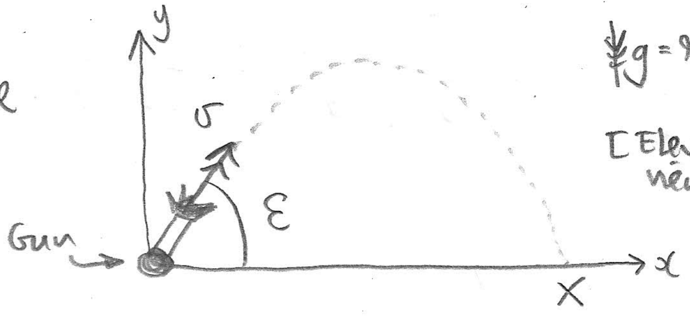
and distance travelled by the tug boat is $x = 1/\sin\theta = \boxed{1\tfrac{1}{4} \text{ km}}$ (ie 1.25 km)

So tug boat must leave $\frac{1}{15}$ of an hour before the ship passes point A ie $\frac{60}{15} = 4$ minutes before.

[This sounds like a nice entrée for a more general interception problem with target velocity $\underline{u}$ and interceptor velocity $\underline{v}$. Assume $|\underline{v}|$ and $\underline{u}$ are fixed, but bearing and delay before interception are variable. Let $t=0$ correspond to initial target position $\underline{a}$]

10/





$g = 9{\cdot}81\,ms^{-2}$

[Elevation view]

Aim is to maximize time of flight such that the shell fired from the gun hits the railway line at I. Assume only gravity acts upon the shell (i.e. ignore air resistance) and the shell leaves the gun with speed v. Let time of flight be T.

Since constant acceleration motion

$X = (v\cos\varepsilon)T$ ie $T = \dfrac{X}{v\cos\varepsilon}$

From plan view: $X\cos\phi = d$ $\therefore$ $\boxed{T = \dfrac{d}{v\cos\varepsilon\cos\phi}}$

Trajectory equation states:

$x = vt\cos\varepsilon$

$y = vt\sin\varepsilon - \frac{1}{2}gt^2$ $\qquad y = t\left(v\sin\varepsilon - \frac{gt}{2}\right)$

So when $y=0$, $t=T$: $v\sin\varepsilon - \frac{gT}{2} = 0$ $\therefore$ $\boxed{\sin\varepsilon = \frac{gT}{2v}}$

Want to eliminate ε so use $\cos\varepsilon = \sqrt{1-\sin^2\varepsilon}$

ie consider $T(\phi)$ rather than $T(\phi,\varepsilon)$

↑ we expect this as we have the constraint that the shell hits the line.

$$\Rightarrow \quad T = \frac{d}{v\sqrt{1-\frac{g^2T^2}{4v^2}}\cos\phi}$$

$$\Rightarrow \quad T^2 = \frac{d^2}{v^2\cos^2\phi\left(1-\frac{g^2T^2}{4v^2}\right)}$$

$$T^2 - \frac{g^2T^4}{4v^2} = \frac{d^2}{v^2\cos^2\phi}$$

$$\therefore \quad \frac{4v^2}{g^2}T^2 - T^4 = \frac{4d^2}{g^2\cos^2\phi}$$

$$\therefore \quad T^4 - \frac{4v^2}{g^2}T^2 + \frac{4d^2}{g^2\cos^2\phi} = 0$$

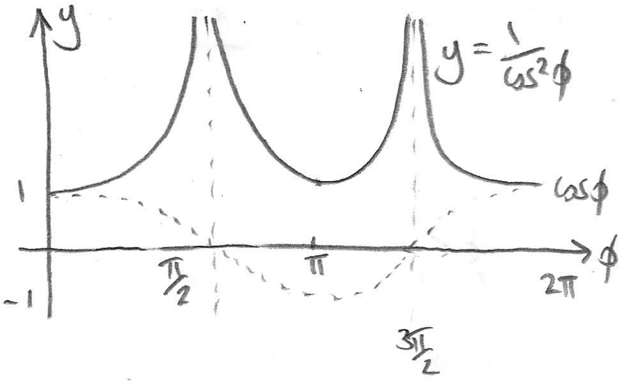
$$\therefore \quad T^2 = \frac{\frac{4v^2}{g^2} \pm \sqrt{\frac{16v^4}{g^4} - \frac{16d^2}{g^2\cos^2\phi}}}{2}$$

$$g^2T^2 = 2v^2 \pm 2\sqrt{v^4 - \frac{g^2d^2}{\cos^2\phi}}$$

Now aim is to maximize T so take the +ve solution

ie $g^2T^2 = 2v^2 + 2\sqrt{v^4 - g^2d^2/\cos^2\phi}$

Furthermore, g^2T^2 (and hence T) is maximized when $\frac{1}{\cos^2\phi}$ is minimized ie when $\phi = 0$ (since $\phi = \pm\pi$ will not hit the railway line!)

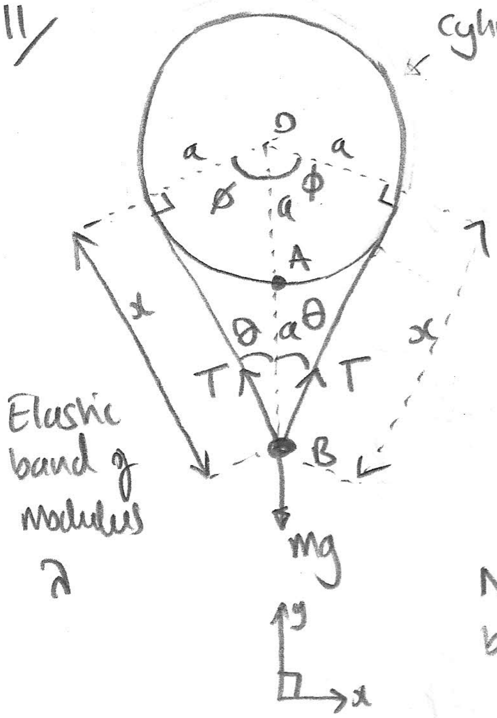


$\therefore$ the maximum time of flight satisfies the equation

$$\boxed{g^2T^2 = 2v^2 + 2\left(v^4 - g^2d^2\right)^{\frac{1}{2}}}$$

as required.

11/



Mass m is released from rest at point A

Elastic band has unstretched length $\boxed{L = 2\pi a}$

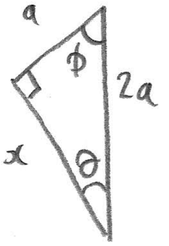
$\downarrow g = 9.81\,ms^{-2}$

$\therefore$ $\boxed{\begin{array}{l} T = \dfrac{\lambda}{2\pi a}\,\Delta x \\ \\ E = \dfrac{1}{2}\,\dfrac{\lambda}{2\pi a}\,\Delta x^2 \end{array}}$

(Tension is Hooke's law)

(Elastic potential energy)

Now Δx is extension of the elastic band.



$2a\sin\theta = a \quad \therefore \sin\theta = \frac{1}{2} \quad \therefore \theta = 30^\circ$ or $\boxed{\pi/6}$

$\therefore \phi = 60^\circ = \pi/3$

Hence $\Delta x = a\left(2\pi - 2\phi\right) + 2x - 2\pi a$

$\Delta x = 2x - 2\pi/3\, a$

Now $4a^2 = a^2 + x^2 \Rightarrow x^2 = 3a^2 \Rightarrow \boxed{x = a\sqrt{3}}$

$\therefore$ $\boxed{\Delta x = a\left(2\sqrt{3} - 2\pi/3\right)}$

(i) Scenario is that mass m is stationary at point B

This means gain in Elastic potential energy = loss of gravitational potential energy.

ie $mga = \frac{1}{2}\,\frac{\lambda a^2}{2\pi a}\left(2\sqrt{3} - 2\pi/3\right)^2$

$\frac{4mg\pi}{\left(2\sqrt{3} - 2\pi/3\right)^2} = \lambda \Rightarrow \boxed{\lambda = \frac{9\pi mg}{\left(3\sqrt{3} - \pi\right)^2}}$ as required.

(ii) In this case mass M reaches maximum velocity at point B. ie net force is zero.

$\therefore$ Newton II // y direction : $0 = 2T\cos\theta - Mg$

Now $\cos\pi/6 = \sqrt{3}/2$ and $T = \frac{\lambda}{2\pi a} \times a\left(2\sqrt{3} - 2\pi/3\right)$

$$\Rightarrow T = \frac{\lambda}{\pi}\left(\sqrt{3} - \pi/3\right)$$

$$\therefore \frac{2\lambda}{\pi}\left(\sqrt{3} - \frac{\pi}{3}\right)\frac{\sqrt{3}}{2} = Mg$$

$$\Rightarrow \lambda = \frac{\pi Mg}{\left(\sqrt{3} - \frac{\pi}{3}\right)\sqrt{3}}$$

$$\Rightarrow \boxed{\lambda = \frac{3\pi Mg}{9 - \pi\sqrt{3}}}$$

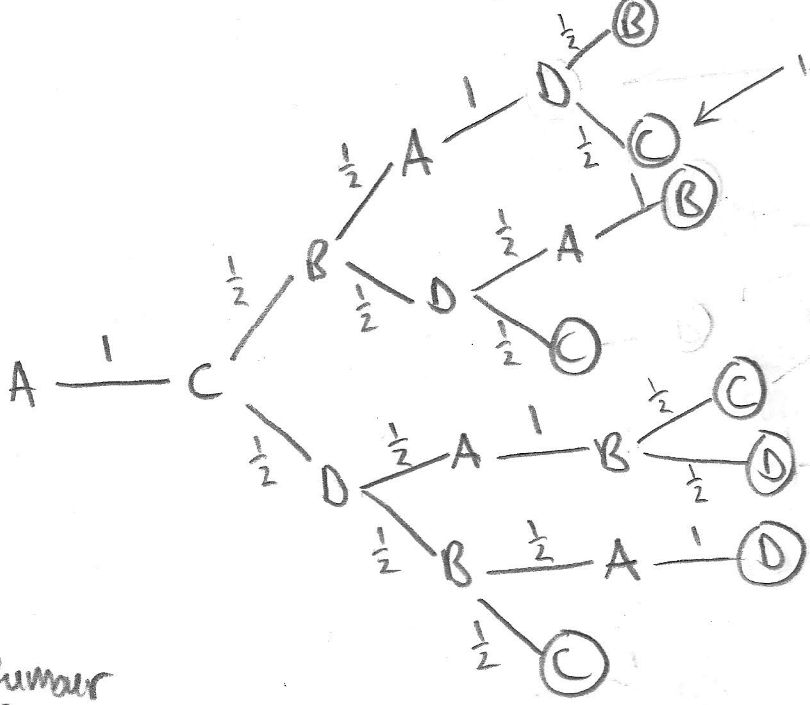
12/ Arthur, Bertha, Chandra & Delilah exchange gossip
A B C D

Rules for passing a rumour are:

"Tell all via random choice unless you know they have already heard it."

↑ ie because they have spoken to you or you to them

Hence if a rumour starts with $A \to C$, tree diagram of possible rumour communications is:



ie C has spoken with D, A, B
So rumour stops

Rumour
(X) Stops, since person X knows everyone else has heard it because X has 'had a conversation' with everyone else!

∴ Probability that A rehears the rumour is $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{2}{4} + \frac{2}{8}$

$= \frac{1}{2} + \frac{1}{4}$

$= \boxed{\frac{3}{4}}$

P(C hears it twice)

$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

$+ \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

$+ \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

$= \frac{4}{8} = \boxed{\frac{1}{2}}$

P(B hears it twice) $= 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \boxed{\frac{1}{4}}$

13/ 4 Students (1 a Mathematician) take turns washing up over a long period.

\# plates broken each time by any student obeys a Poisson distribution
(n)

ie $n \sim Po(\lambda)$ ie $\boxed{P(n,\lambda) = \dfrac{e^{-\lambda}\lambda^n}{n!}}$

λ is a fixed constant

"Mean breakage rate"

We are given $n = 5$ and want
$P(\geq 3$ plates are broken by the Mathematician$) = p$

q — 5 plates are broken [by the students, any of them!]
- p — ≥ 3 broken by the Mathematician GIVEN 5 plates are broken
- $1-p$ — < 3 are broken by the Mathematician given 5 plates are broken

$1-q$ — 5 plates are not broken

Now $q = P(5, 4\lambda)$ Since all four students have an equal chance of being on washing up duty

$$\therefore\ q = \frac{e^{-4\lambda}(4\lambda)^5}{5!} = \frac{1024}{120}\lambda^5 e^{-4\lambda} = \frac{256}{30}\lambda^5 e^{-4\lambda}$$

$$= \boxed{\frac{128}{15}\lambda^5 e^{-4\lambda}}$$

Now pq is also P(Mathematician breaks 3 plates) × P(others break 2)
+ P(" " 4 plates) × P(" " 1)
+ P(" " 5 plates) × P(" " 0)

ie pq is the probability that the Mathematician breaks 3 or more plates. p is the probability that the Mathematician breaks 3 or more of the five that are broken. So $pq \neq p$

$$\therefore \; P_4 = \frac{e^{-\lambda}\lambda^5}{5!} \times \frac{e^{-3\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}}{4!}\lambda^4 \times \frac{e^{-3\lambda}(3\lambda)}{1!} + \frac{e^{-\lambda}}{3!}\lambda^3 + \frac{e^{-3\lambda}}{2!}(3\lambda)^2$$

Mathematician breaks five plates, the others break none

Mathematician breaks four plates, one is broken by the other three students

Mathematician breaks 3 plates, two are broken by the other students.

$$\therefore \; \frac{128}{15}\lambda^5 e^{-4\lambda} p = e^{-4\lambda}\lambda^5\left(\frac{1}{5!} + \frac{3}{4!} + \frac{9}{3!2!}\right)$$

$$\therefore \; p = \frac{15}{128}\left(\frac{1}{120} + \frac{3}{24} + \frac{9}{12}\right)$$

$$p = \frac{15 \times 53}{128 \times 60}$$

$$\boxed{p = \frac{53}{512}} \simeq 0.104$$

14/ N candidates are interviewed. Let us assume they can be ranked uniquely $1 \ldots N$ (1 being 'the best').

Candidates are interviewed in a random order.

(i) Probability that the best amongst the first n candidates is the best overall $= 1 - P($the overall best is not in the first n interviewed$)$

$$= 1 - \left(\frac{\cancel{N-1}}{N}\right)\left(\frac{\cancel{N-2}}{\cancel{N-1}}\right)\left(\frac{N-3}{\cancel{N-2}}\right) \times \cdots \times \left(\frac{\cancel{N-n+1}}{N-n+2}\right)\left(\frac{N-n}{\cancel{N-n+1}}\right)$$

Interview #1 ... Interview #n

$$= 1 - \frac{N-n}{N} = \boxed{\frac{n}{N}}$$

(ii) Now we want the probability that the best amongst the first n candidates is the best or second best overall.

By the same idea as in (i), this is

$1 - P($overall top 2 candidates are not in the first n interviews$)$

$$= 1 - \left(\frac{N-2}{N}\right)\left(\frac{N-3}{N-1}\right)\left(\frac{N-4}{N-2}\right) \times \cdots \left(\frac{N-n-1}{N-n+1}\right)$$

$$= 1 - \frac{1}{N(N-1)} \times (N-n)(N-n-1)$$

$$= \frac{N(N-1) - (N-n)(N-n-1)}{N(N-1)}$$

$$= \frac{N(N-1) - N(N-1) + n(N-1) + Nn - n^2}{N(N-1)}$$

$$= \frac{n(N-1) + n(N-n)}{N(N-1)}$$

$$= \boxed{\frac{n}{N}\left(1 + \frac{N-n}{N-1}\right)}$$

Consider $N = 4$ candidates. There are $4! = 24$ possible interview orders.

1234	**	3124	**
1243	**	3142	**
1342	**	3214	*
1324	**	3241	*
1423	**	3412	
1432	**	3421	
2134	**	4123	**
2143	**	4132	**
2314	*	4213	*
2341	*	4231	*
2431	*	4312	
2413	*	4321	

** means best (ie 1) is in first n

* means " or second best is in first n
↑
ie 2

There are 12 ** and 8 *

So $P(\text{best in first } 2) = \frac{12}{24} = \boxed{\frac{1}{2}}$

Prediction is $\frac{n}{N} = \frac{2}{4} = \frac{1}{2}$ ✓

So $P(\text{best or second best in first } 2) = \frac{12+8}{24} = \frac{20}{24} = \boxed{\frac{5}{6}}$

Prediction is $\frac{n}{N}\left(1 + \frac{N-n}{N-1}\right) = \frac{2}{4}\left(1 + \frac{4-2}{4-1}\right)$

$= \frac{1}{2}\left(1 + \frac{2}{3}\right) = \frac{1}{2} + \frac{1}{3}$

$= \frac{5}{6}$ ✓