Section A: Pure Mathematics

1 (i) Express $\left(3+2\sqrt{5}\right)^3$ in the form $a+b\sqrt{5}$ where a and b are integers.

(ii) Find the positive integers c and d such that $\sqrt[3]{99-70\sqrt{2}} = c-d\sqrt{2}\,.$

(iii) Find the two real solutions of $x^6-198x^3+1=0\,.$

2 The square bracket notation $[x]$ means the greatest integer less than or equal to x. For example, $[\pi]=3\,,\ [\sqrt{24}\,]=4$ and $[5]=5\,.$

(i) Sketch the graph of $y=\sqrt{[x]}$ and show that

$$\int_0^a \sqrt{[x]}\;\mathrm{d}x = \sum_{r=0}^{a-1}\sqrt{r}$$

when a is a positive integer.

(ii) Show that $\displaystyle\int_0^a 2^{[x]}\;\mathrm{d}x = 2^a-1$ when a is a positive integer.

(iii) Determine an expression for $\displaystyle\int_0^a 2^{[x]}\;\mathrm{d}x$ when a is positive but not an integer.

3 (i) Show that $x-3$ is a factor of

$$x^3-5x^2+2x^2y+xy^2-8xy-3y^2+6x+6y\;. \qquad (*)$$

Express $(*)$ in the form $(x-3)(x+ay+b)(x+cy+d)$ where a, b, c and d are integers to be determined.

(ii) Factorise $6y^3-y^2-21y+2x^2+12x-4xy+x^2y-5xy^2+10$ into three linear factors.

4 Differentiate $\sec t$ with respect to t.

(i) Use the substitution $x = \sec t$ to show that $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,\mathrm{d}x = \frac{\sqrt{3}-2}{8} + \frac{\pi}{24}$.

(ii) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}}\,\mathrm{d}x$.

(iii) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+4x-5}}\,\mathrm{d}x$.

5 The positive integers can be split into five distinct arithmetic progressions, as shown:

$$\begin{aligned}
A&: \ 1,\ 6,\ 11,\ 16,\ \dots\\
B&: \ 2,\ 7,\ 12,\ 17,\ \dots\\
C&: \ 3,\ 8,\ 13,\ 18,\ \dots\\
D&: \ 4,\ 9,\ 14,\ 19,\ \dots\\
E&: \ 5, 10,\ 15,\ 20,\ \dots
\end{aligned}$$

Write down an expression for the value of the general term in each of the five progressions. Hence prove that the sum of any term in B and any term in C is a term in E.

Prove also that the square of every term in B is a term in D. State and prove a similar claim about the square of every term in C.

(i) Prove that there are no positive integers x and y such that

$$x^2 + 5y = 243\,723\,.$$

(ii) Prove also that there are no positive integers x and y such that

$$x^4 + 2y^4 = 26\,081\,974\,.$$

6 The three points A, B and C have coordinates $(p_1\,,\; q_1)$, $(p_2\,,\; q_2)$ and $(p_3\,,\; q_3)$, respectively. Find the point of intersection of the line joining A to the midpoint of BC, and the line joining B to the midpoint of AC. Verify that this point lies on the line joining C to the midpoint of AB.

The point H has coordinates $(p_1 + p_2 + p_3\,,\; q_1 + q_2 + q_3)$. Show that if the line AH intersects the line BC at right angles, then $p_2^2 + q_2^2 = p_3^2 + q_3^2$, and write down a similar result if the line BH intersects the line AC at right angles.

Deduce that if AH is perpendicular to BC and also BH is perpendicular to AC, then CH is perpendicular to AB.

7 (i) The function $\mathrm{f}(x)$ is defined for $|x| < \frac{1}{5}$ by

$$\mathrm{f}(x) = \sum_{n=0}^{\infty} a_n x^n\,,$$

where $a_0 = 2$, $a_1 = 7$ and $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geqslant 2$.

Simplify $\mathrm{f}(x) - 7x\mathrm{f}(x) + 10x^2\mathrm{f}(x)$, and hence show that $\mathrm{f}(x) = \dfrac{1}{1-2x} + \dfrac{1}{1-5x}$.

Hence show that $a_n = 2^n + 5^n$.

(ii) The function $\mathrm{g}(x)$ is defined for $|x| < \frac{1}{3}$ by

$$\mathrm{g}(x) = \sum_{n=0}^{\infty} b_n x^n\,,$$

where $b_0 = 5$, $b_1 = 10$, $b_2 = 40$, $b_3 = 100$ and $b_n = pb_{n-1} + qb_{n-2}$ for $n \geqslant 2$. Obtain an expression for $\mathrm{g}(x)$ as the sum of two algebraic fractions and determine b_n in terms of n.

8 A sequence t_0, t_1, t_2, ... is said to be *strictly increasing* if $t_{n+1} > t_n$ for all $n \geqslant 0$.

(i) The terms of the sequence $x_0\,,\, x_1\,,\, x_2\,,\, \ldots$ satisfy

$$x_{n+1} = \frac{x_n^2 + 6}{5}$$

for $n \geqslant 0$. Prove that if $x_0 > 3$ then the sequence is strictly increasing.

(ii) The terms of the sequence $y_0\,,\, y_1\,,\, y_2\,,\, \ldots$ satisfy

$$y_{n+1} = 5 - \frac{6}{y_n}$$

for $n \geqslant 0$. Prove that if $2 < y_0 < 3$ then the sequence is strictly increasing but that $y_n < 3$ for all n.

Section B: Mechanics

9 A particle is projected over level ground with a speed u at an angle θ above the horizontal. Derive an expression for the greatest height of the particle in terms of u, θ and g.

A particle is projected from the floor of a horizontal tunnel of height $\frac{9}{10}d$. Point P is $\frac{1}{2}d$ metres vertically and d metres horizontally along the tunnel from the point of projection. The particle passes through point P and lands inside the tunnel without hitting the roof. Show that

$$\arctan \tfrac{3}{5} < \theta < \arctan 3 .$$

10 A particle is travelling in a straight line. It accelerates from its initial velocity u to velocity v, where $v > |u| > 0$, travelling a distance d_1 with uniform acceleration of magnitude $3a$. It then comes to rest after travelling a further distance d_2 with uniform deceleration of magnitude a. Show that

(i) if $u > 0$ then $3d_1 < d_2$;

(ii) if $u < 0$ then $d_2 < 3d_1 < 2d_2$.

Show also that the average speed of the particle (that is, the total distance travelled divided by the total time) is greater in the case $u > 0$ than in the case $u < 0$.

Note: In this question d_1 and d_2 are distances travelled by the particle which are not the same, in the second case, as displacements from the starting point.

11 Two uniform ladders AB and BC of equal length are hinged smoothly at B. The weight of AB is W and the weight of BC is $4W$. The ladders stand on rough horizontal ground with $\widehat{ABC} = 60^\circ$. The coefficient of friction between each ladder and the ground is μ.

A decorator of weight $7W$ begins to climb the ladder AB slowly. When she has climbed up $\frac{1}{3}$ of the ladder, one of the ladders slips. Which ladder slips, and what is the value of μ?

Section C: Probability and Statistics

12 In a certain factory, microchips are made by two machines. Machine A makes a proportion λ of the chips, where $0 < \lambda < 1$, and machine B makes the rest. A proportion p of the chips made by machine A are perfect, and a proportion q of those made by machine B are perfect, where $0 < p < 1$ and $0 < q < 1$. The chips are sorted into two groups: group 1 contains those that are perfect and group 2 contains those that are imperfect.

In a large random sample taken from group 1, it is found that $\frac{2}{5}$ were made by machine A. Show that λ can estimated as

$$\frac{2q}{3p+2q}\,.$$

Subsequently, it is discovered that the sorting process is faulty: there is a probability of $\frac{1}{4}$ that a perfect chip is assigned to group 2 and a probability of $\frac{1}{4}$ that an imperfect chip is assigned to group 1. Taking into account this additional information, obtain a new estimate of λ.

13 (i) Three real numbers are drawn independently from the continuous rectangular distribution on $[0,1]$. The random variable X is the maximum of the three numbers. Show that the probability that $X \leqslant 0.8$ is 0.512, and calculate the expectation of X.

(ii) N real numbers are drawn independently from a continuous rectangular distribution on $[0,a]$. The random variable X is the maximum of the N numbers. A hypothesis test with a significance level of 5% is carried out using the value, x, of X. The null hypothesis is that $a = 1$ and the alternative hypothesis is that $a < 1$. The form of the test is such that H_0 is rejected if $x < c$, for some chosen number c.

Using the approximation $2^{10} \approx 10^3$, determine the smallest integer value of N such that if $x \leqslant 0.8$ the null hypothesis will be rejected.

With this value of N, write down the probability that the null hypothesis is rejected if $a = 0.8$, and find the probability that the null hypothesis is rejected if $a = 0.9$.

14 Three pirates are sharing out the contents of a treasure chest containing n gold coins and 2 lead coins. The first pirate takes out coins one at a time until he takes out one of the lead coins. The second pirate then takes out coins one at a time until she draws the second lead coin. The third pirate takes out all the gold coins remaining in the chest.

Find:

(i) the probability that the first pirate will have some gold coins;

(ii) the probability that the second pirate will have some gold coins;

(iii) the probability that all three pirates will have some gold coins.