Section A: Pure Mathematics

1 Find the three values of x for which the derivative of $x^2 e^{-x^2}$ is zero.

Given that a and b are distinct positive numbers, find a polynomial $\mathrm{P}(x)$ such that the derivative of $\mathrm{P}(x)e^{-x^2}$ is zero for $x = 0$, $x = \pm a$ and $x = \pm b$, but for no other values of x.

2 For any positive integer N, the function $\mathrm{f}(N)$ is defined by

$$\mathrm{f}(N) = N\Big(1 - \frac{1}{p_1}\Big)\Big(1 - \frac{1}{p_2}\Big)\cdots\Big(1 - \frac{1}{p_k}\Big)$$

where p_1, p_2, ... , p_k are the only prime numbers that are factors of N.
Thus $\mathrm{f}(80) = 80(1 - \frac{1}{2})(1 - \frac{1}{5})$.

(a) (i) Evaluate $\mathrm{f}(12)$ and $\mathrm{f}(180)$.

(ii) Show that $\mathrm{f}(N)$ is an integer for all N.

(b) Prove, or disprove by means of a counterexample, each of the following:

(i) $\mathrm{f}(m)\mathrm{f}(n) = \mathrm{f}(mn)$;

(ii) $\mathrm{f}(p)\mathrm{f}(q) = \mathrm{f}(pq)$ if p and q are distinct prime numbers;

(iii) $\mathrm{f}(p)\mathrm{f}(q) = \mathrm{f}(pq)$ only if p and q are distinct prime numbers.

(c) Find a positive integer m and a prime number p such that $\mathrm{f}(p^m) = 146410$.

3 Give a sketch, for $0 \leqslant x \leqslant \pi/2$, of the curve

$$y = (\sin x - x\cos x)\ ,$$

and show that $0 \leqslant y \leqslant 1$.

Show that:

(i) $\displaystyle\int_0^{\pi/2} y\ \mathrm{d}x = 2 - \frac{\pi}{2}$;

(ii) $\displaystyle\int_0^{\pi/2} y^2\ \mathrm{d}x = \frac{\pi^3}{48} - \frac{\pi}{8}$.

Deduce that $\pi^3 + 18\pi < 96$.

4 The positive numbers a, b and c satisfy $bc = a^2 + 1$. Prove that

$$\arctan\left(\frac{1}{a+b}\right) + \arctan\left(\frac{1}{a+c}\right) = \arctan\left(\frac{1}{a}\right).$$

The positive numbers p, q, r, s, t, u and v satisfy

$$st = (p+q)^2 + 1\,, \qquad uv = (p+r)^2 + 1\,, \qquad qr = p^2 + 1\,.$$

Prove that

$$\arctan\left(\frac{1}{p+q+s}\right) + \arctan\left(\frac{1}{p+q+t}\right) + \arctan\left(\frac{1}{p+r+u}\right) + \arctan\left(\frac{1}{p+r+v}\right) = \arctan\left(\frac{1}{p}\right).$$

Hence show that

$$\arctan\left(\frac{1}{13}\right) + \arctan\left(\frac{1}{21}\right) + \arctan\left(\frac{1}{82}\right) + \arctan\left(\frac{1}{187}\right) = \arctan\left(\frac{1}{7}\right).$$

[Note that $\arctan x$ is another notation for $\tan^{-1} x$.]

5 The angle A of triangle ABC is a right angle and the sides BC, CA and AB are of lengths a, b and c, respectively. Each side of the triangle is tangent to the circle S_1 which is of radius r. Show that $2r = b + c - a$.

Each vertex of the triangle lies on the circle S_2. The ratio of the area of the region between S_1 and the triangle to the area of S_2 is denoted by R. Show that

$$\pi R = -(\pi - 1)q^2 + 2\pi q - (\pi + 1)\,,$$

where $q = \dfrac{b+c}{a}$. Deduce that

$$R \leqslant \frac{1}{\pi(\pi - 1)}\,.$$

6 (i) Write down the general term in the expansion in powers of x of $(1-x)^{-1}$, $(1-x)^{-2}$ and $(1-x)^{-3}$, where $|x| < 1$.

Evaluate $\displaystyle\sum_{n=1}^{\infty} n2^{-n}$ and $\displaystyle\sum_{n=1}^{\infty} n^2 2^{-n}$.

(ii) Show that $(1-x)^{-\frac{1}{2}} = \displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}\frac{x^n}{2^{2n}}$, for $|x| < 1$.

Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 2^{2n} 3^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{n(2n)!}{(n!)^2 2^{2n} 3^n}$.

7 The position vectors, relative to an origin O, at time t of the particles P and Q are

$$\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 0\,\mathbf{k} \qquad \text{and} \qquad \cos(t + \tfrac{1}{4}\pi)\left[\tfrac{3}{2}\mathbf{i} + \tfrac{3\sqrt{3}}{2}\mathbf{k}\right] + 3\sin(t + \tfrac{1}{4}\pi)\,\mathbf{j}\,,$$

respectively, where $0 \leqslant t \leqslant 2\pi$.

(i) Give a geometrical description of the motion of P and Q.

(ii) Let θ be the angle POQ at time t that satisfies $0 \leqslant \theta \leqslant \pi$. Show that

$$\cos\theta = \tfrac{3\sqrt{2}}{8} - \tfrac{1}{4}\cos(2t + \tfrac{1}{4}\pi)\ .$$

(iii) Show that the total time for which $\theta \geqslant \frac{1}{4}\pi$ is $\frac{3}{2}\pi$.

8 For $x \geqslant 0$ the curve C is defined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3y^2}{(1+x^2)^{5/2}}$$

with $y = 1$ when $x = 0$. Show that

$$\frac{1}{y} = \frac{2+3x^2}{3(1+x^2)^{3/2}} + \frac{1}{3}$$

and hence that for large positive x

$$y \approx 3 - \frac{9}{x}\ .$$

Draw a sketch of C.

On a separate diagram draw a sketch of the two curves defined for $x \geqslant 0$ by

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{x^3z^3}{2(1+x^2)^{5/2}}$$

with $z = 1$ at $x = 0$ on one curve, and $z = -1$ at $x = 0$ on the other.

Section B: Mechanics

9 Two particles, A and B, of masses m and $2m$, respectively, are placed on a line of greatest slope, ℓ, of a rough inclined plane which makes an angle of $30°$ with the horizontal. The coefficient of friction between A and the plane is $\frac{1}{6}\sqrt{3}$ and the coefficient of friction between B and the plane is $\frac{1}{3}\sqrt{3}$. The particles are at rest with B higher up ℓ than A and are connected by a light inextensible string which is taut. A force P is applied to B.

(i) Show that the least magnitude of P for which the two particles move upwards along ℓ is $\frac{11}{8}\sqrt{3}\,mg$ and give, in this case, the direction in which P acts.

(ii) Find the least magnitude of P for which the particles do not slip downwards along ℓ.

10 The points A and B are 180 metres apart and lie on horizontal ground. A missile is launched from A at speed of $100\,\mathrm{m\,s^{-1}}$ and at an acute angle of elevation to the line AB of $\arcsin\frac{3}{5}$. A time T seconds later, an anti-missile missile is launched from B, at speed of $200\,\mathrm{m\,s^{-1}}$ and at an acute angle of elevation to the line BA of $\arcsin\frac{4}{5}$. The motion of both missiles takes place in the vertical plane containing A and B, and the missiles collide.

Taking $g = 10\,\mathrm{m\,s^{-2}}$ and ignoring air resistance, find T.

[Note that $\arcsin\frac{3}{5}$ is another notation for $\sin^{-1}\frac{3}{5}$.]

11 A plane is inclined at an angle $\arctan\frac{3}{4}$ to the horizontal and a small, smooth, light pulley P is fixed to the top of the plane. A string, APB, passes over the pulley. A particle of mass m_1 is attached to the string at A and rests on the inclined plane with AP parallel to a line of greatest slope in the plane. A particle of mass m_2, where $m_2 > m_1$, is attached to the string at B and hangs freely with BP vertical. The coefficient of friction between the particle at A and the plane is $\frac{1}{2}$.

The system is released from rest with the string taut. Show that the acceleration of the particles is $\dfrac{m_2 - m_1}{m_2 + m_1}g$.

At a time T after release, the string breaks. Given that the particle at A does not reach the pulley at any point in its motion, find an expression in terms of T for the time after release at which the particle at A reaches its maximum height. It is found that, regardless of when the string broke, this time is equal to the time taken by the particle at A to descend from its point of maximum height to the point at which it was released. Find the ratio $m_1 : m_2$.

[Note that $\arctan\frac{3}{4}$ is another notation for $\tan^{-1}\frac{3}{4}$.]

Section C: Probability and Statistics

12 The twins Anna and Bella share a computer and never sign their e-mails. When I e-mail them, only the twin currently online responds. The probability that it is Anna who is online is p and she answers each question I ask her truthfully with probability a, independently of all her other answers, even if a question is repeated. The probability that it is Bella who is online is q, where $q = 1 - p$, and she answers each question truthfully with probability b, independently of all her other answers, even if a question is repeated.

(i) I send the twins the e-mail: 'Toss a fair coin and answer the following question. Did the coin come down heads?'. I receive the answer 'yes'. Show that the probability that the coin did come down heads is $\frac{1}{2}$ if and only if $2(ap + bq) = 1$.

(ii) I send the twins the e-mail: 'Toss a fair coin and answer the following question. Did the coin come down heads?'. I receive the answer 'yes'. I then send the e-mail: 'Did the coin come down heads?' and I receive the answer 'no'. Show that the probability (taking into account these answers) that the coin did come down heads is $\frac{1}{2}$.

(iii) I send the twins the e-mail: 'Toss a fair coin and answer the following question. Did the coin come down heads?'. I receive the answer 'yes'. I then send the e-mail: 'Did the coin come down heads?' and I receive the answer 'yes'. Show that, if $2(ap + bq) = 1$, the probability (taking into account these answers) that the coin did come down heads is $\frac{1}{2}$.

13 The number of printing errors on any page of a large book of N pages is modelled by a Poisson variate with parameter λ and is statistically independent of the number of printing errors on any other page. The number of pages in a random sample of n pages (where n is much smaller than N and $n \geqslant 2$) which contain fewer than two errors is denoted by Y. Show that $\mathrm{P}(Y = k) = \binom{n}{k} p^k q^{n-k}$ where $p = (1 + \lambda)e^{-\lambda}$ and $q = 1 - p$.

Show also that, if λ is sufficiently small,

(i) $q \approx \frac{1}{2}\lambda^2$;

(ii) the largest value of n for which $\mathrm{P}(Y = n) \geqslant 1 - \lambda$ is approximately $2/\lambda$;

(iii) $\mathrm{P}(Y > 1 \mid Y > 0) \approx 1 - n(\lambda^2/2)^{n-1}$.

14 The probability density function $\mathrm{f}\,(x)$ of the random variable X is given by

$$\mathrm{f}\,(x) = k\,[\phi(x) + \lambda \mathrm{g}\,(x)]\,,$$

where $\phi(x)$ is the probability density function of a normal variate with mean 0 and variance 1, λ is a positive constant, and $\mathrm{g}\,(x)$ is a probability density function defined by

$$\mathrm{g}\,(x) = \begin{cases} 1/\lambda & \text{for } 0 \leqslant x \leqslant \lambda\,; \\ 0 & \text{otherwise.} \end{cases}$$

Find μ, the mean of X, in terms of λ, and prove that σ, the standard deviation of X, satisfies.

$$\sigma^2 = \frac{\lambda^4 + 4\lambda^3 + 12\lambda + 12}{12(1+\lambda)^2}\,.$$

In the case $\lambda = 2$:

(i) draw a sketch of the curve $y = \mathrm{f}\,(x)$;

(ii) express the cumulative distribution function of X in terms of $\Phi(x)$, the cumulative distribution function corresponding to $\phi(x)$;

(iii) evaluate $\mathrm{P}(0 < X < \mu + 2\sigma)$, given that $\Phi(\frac{2}{3} + \frac{2}{3}\sqrt{7}) = 0.9921$.