Section A: Pure Mathematics

1 *In this question, do not consider the special cases in which the denominators of any of your expressions are zero.*

Express $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ in terms of t_i, where $t_1 = \tan\theta_1$, etc.

Given that $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ and $\tan\theta_4$ are the four roots of the equation

$$at^4 + bt^3 + ct^2 + dt + e = 0$$

(where $a \neq 0$), find an expression in terms of a, b, c, d and e for $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$.

The four real numbers θ_1, θ_2, θ_3 and θ_4 lie in the range $0 \leqslant \theta_i < 2\pi$ and satisfy the equation

$$p\cos 2\theta + \cos(\theta - \alpha) + p = 0\,,$$

where p and α are independent of θ. Show that $\theta_1 + \theta_2 + \theta_3 + \theta_4 = n\pi$ for some integer n.

2 **(i)** Show that $1.3.5.7.\ \ldots\ .(2n-1) = \dfrac{(2n)!}{2^n n!}$ and that, for $|x| < \frac{1}{4}$,

$$\frac{1}{\sqrt{1-4x}} = 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}\, x^n\,.$$

(ii) By differentiating the above result, deduce that

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,(n-1)!} \left(\frac{6}{25}\right)^n = 60\,.$$

(iii) Show that

$$\sum_{n=1}^{\infty} \frac{2^{n+1}(2n)!}{3^{2n}(n+1)!\,n!} = 1\,.$$

3 A sequence of numbers, F_1, F_2, ..., is defined by $F_1 = 1$, $F_2 = 1$, and

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geqslant 3.$$

(i) Write down the values of F_3, F_4, ... , F_8.

(ii) Prove that $F_{2k+3}F_{2k+1} - F_{2k+2}^2 = -F_{2k+2}F_{2k} + F_{2k+1}^2$.

(iii) Prove by induction or otherwise that $F_{2n+1}F_{2n-1} - F_{2n}^2 = 1$ and deduce that $F_{2n}^2 + 1$ is divisible by F_{2n+1} .

(iv) Prove that $F_{2n-1}^2 + 1$ is divisible by F_{2n+1} .

4 A curve is given parametrically by

$$x = a\big(\cos t + \ln\tan\tfrac{1}{2}t\big)\,,$$
$$y = a\sin t\,,$$

where $0 < t < \frac{1}{2}\pi$ and a is a positive constant. Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \tan t$ and sketch the curve.

Let P be the point with parameter t and let Q be the point where the tangent to the curve at P meets the x-axis. Show that $PQ = a$.

The *radius of curvature*, ρ, at P is defined by

$$\rho = \frac{\big(\dot{x}^2 + \dot{y}^2\big)^{\frac{3}{2}}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}\,,$$

where the dots denote differentiation with respect to t. Show that $\rho = a\cot t$.

The point C lies on the normal to the curve at P, a distance ρ from P and above the curve. Show that CQ is parallel to the y-axis.

5 Let $y = \ln(x^2 - 1)$, where $x > 1$, and let r and θ be functions of x determined by $r = \sqrt{x^2 - 1}$ and $\coth\theta = x$. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cosh\theta}{r} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2\cosh 2\theta}{r^2}\,,$$

and find an expression in terms of r and θ for $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$.

Find, with proof, a similar formula for $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$ in terms of r and θ.

6 The distinct points P, Q, R and S in the Argand diagram lie on a circle of radius a centred at the origin and are represented by the complex numbers p, q, r and s, respectively. Show that

$$pq = -a^2 \frac{p-q}{p^* - q^*}\,.$$

Deduce that, if the chords PQ and RS are perpendicular, then $pq + rs = 0$.

The distinct points A_1, A_2, ..., A_n (where $n \geqslant 3$) lie on a circle. The points B_1, B_2, ..., B_n lie on the same circle and are chosen so that the chords B_1B_2, B_2B_3, ..., B_nB_1 are perpendicular, respectively, to the chords A_1A_2, A_2A_3, ..., A_nA_1. Show that, for $n = 3$, there are only two choices of B_1 for which this is possible. What is the corresponding result for $n = 4$? State the corresponding results for values of n greater than 4.

7 The functions $\mathrm{s}(x)$ $(0 \leqslant x < 1)$ and $\mathrm{t}(x)$ $(x \geqslant 0)$, and the real number p, are defined by

$$\mathrm{s}(x) = \int_0^x \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u\,, \qquad \mathrm{t}(x) = \int_0^x \frac{1}{1+u^2}\,\mathrm{d}u\,, \qquad p = 2\int_0^\infty \frac{1}{1+u^2}\,\mathrm{d}u\,.$$

For this question, do not evaluate any of the above integrals explicitly in terms of inverse trigonometric functions or the number π.

(i) Use the substitution $u = v^{-1}$ to show that $\mathrm{t}(x) = \displaystyle\int_{1/x}^\infty \frac{1}{1+v^2}\,\mathrm{d}v$. Hence evaluate $\mathrm{t}(1/x) + \mathrm{t}(x)$ in terms of p and deduce that $2\mathrm{t}(1) = \frac{1}{2}p$.

(ii) Let $y = \dfrac{u}{\sqrt{1+u^2}}$. Express u in terms of y, and show that $\dfrac{\mathrm{d}u}{\mathrm{d}y} = \dfrac{1}{\sqrt{(1-y^2)^3}}$.

By making a substitution in the integral for $\mathrm{t}(x)$, show that

$$\mathrm{t}(x) = \mathrm{s}\left(\frac{x}{\sqrt{1+x^2}}\right).$$

Deduce that $\mathrm{s}\big(\frac{1}{\sqrt{2}}\big) = \frac{1}{4}p$.

(iii) Let $z = \dfrac{u + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}u}$. Show that $\mathrm{t}\big(\frac{1}{\sqrt{3}}\big) = \displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+z^2}\,\mathrm{d}z$, and hence that $3\mathrm{t}\big(\frac{1}{\sqrt{3}}\big) = \frac{1}{2}p$.

8 **(i)** Find functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$ such that $u = x$ and $u = \mathrm{e}^{-x}$ both satisfy the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \mathrm{a}(x)\frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{b}(x)u = 0\,.$$

For these functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$, write down the general solution of the equation.

Show that the substitution $y = \dfrac{1}{3u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$ transforms the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + \frac{x}{1+x}y = \frac{1}{3(1+x)} \qquad (*)$$

into

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1+x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{1+x}u = 0$$

and hence show that the solution of equation $(*)$ that satisfies $y = 0$ at $x = 0$ is given by $y = \dfrac{1-\mathrm{e}^{-x}}{3(x+\mathrm{e}^{-x})}$.

(ii) Find the solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{x}{1-x}y = \frac{1}{1-x}$$

that satisfies $y = 2$ at $x = 0$.

Section B: Mechanics

9 Two small beads, A and B, each of mass m, are threaded on a smooth horizontal circular hoop of radius a and centre O. The angle θ is the acute angle determined by $2\theta = \angle AOB$.

The beads are connected by a light straight spring. The energy stored in the spring is

$$mk^2a^2(\theta - \alpha)^2,$$

where k and α are constants satisfying $k > 0$ and $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$.

The spring is held in compression with $\theta = \beta$ and then released. Find the period of oscillations in the two cases that arise according to the value of β and state the value of β for which oscillations do not occur.

10 A particle is projected from a point on a plane that is inclined at an angle ϕ to the horizontal. The position of the particle at time t after it is projected is (x, y), where $(0, 0)$ is the point of projection, x measures distance up the line of greatest slope and y measures perpendicular distance from the plane. Initially, the velocity of the particle is given by $(\dot{x}, \dot{y}) = (V\cos\theta, V\sin\theta)$, where $V > 0$ and $\phi + \theta < \pi/2$. Write down expressions for x and y.

The particle bounces on the plane and returns along the same path to the point of projection. Show that

$$2\tan\phi\tan\theta = 1$$

and that

$$R = \frac{V^2\cos^2\theta}{2g\sin\phi},$$

where R is the range along the plane.

Show further that

$$\frac{2V^2}{gR} = 3\sin\phi + \operatorname{cosec}\phi$$

and deduce that the largest possible value of R is $V^2/(\sqrt{3}\,g)$.

11 **(i)** A wheel consists of a thin light circular rim attached by light spokes of length a to a small hub of mass m. The wheel rolls without slipping on a rough horizontal table directly towards a straight edge of the table. The plane of the wheel is vertical throughout the motion. The speed of the wheel is u, where $u^2 < ag$.

Show that, after the wheel reaches the edge of the table and while it is still in contact with the table, the frictional force on the wheel is zero. Show also that the hub will fall a vertical distance $(ag - u^2)/(3g)$ before the rim loses contact with the table.

(ii) Two particles, each of mass $m/2$, are attached to a light circular hoop of radius a, at the ends of a diameter. The hoop rolls without slipping on a rough horizontal table directly towards a straight edge of the table. The plane of the hoop is vertical throughout the motion. When the centre of the hoop is vertically above the edge of the table it has speed u, where $u^2 < ag$, and one particle is vertically above the other.

Show that, after the hoop reaches the edge of the table and while it is still in contact with the table, the frictional force on the hoop is non-zero and deduce that the hoop will slip before it loses contact with the table.

Section C: Probability and Statistics

12 I choose a number from the integers 1, 2, ..., $(2n-1)$ and the outcome is the random variable N. Calculate $\mathrm{E}(N)$ and $\mathrm{E}(N^2)$.

I then repeat a certain experiment N times, the outcome of the ith experiment being the random variable X_i $(1 \leqslant i \leqslant N)$. For each i, the random variable X_i has mean μ and variance σ^2, and X_i is independent of X_j for $i \neq j$ and also independent of N. The random variable Y is defined by $Y = \sum\limits_{i=1}^{N} X_i$. Show that $\mathrm{E}(Y) = n\mu$ and that $\mathrm{Cov}\,(Y, N) = \frac{1}{3}n(n-1)\mu$. Find $\mathrm{Var}\,(Y)$ in terms of n, σ^2 and μ.

13 A frog jumps towards a large pond. Each jump takes the frog either 1 m or 2 m nearer to the pond. The probability of a 1 m jump is p and the probability of a 2 m jump is q, where $p + q = 1$, the occurence of long and short jumps being independent.

(i) Let $p_n(j)$ be the probability that the frog, starting at a point $(n - \frac{1}{2})$ m away from the edge of the pond, lands in the pond for the first time on its jth jump. Show that $p_2(2) = p$.

(ii) Let u_n be the expected number of jumps, starting at a point $(n - \frac{1}{2})$ m away from the edge of the pond, required to land in the pond for the first time. Write down the value of u_1. By finding first the relevant values of $p_n(m)$, calculate u_2 and show that $u_3 = 3 - 2q + q^2$.

(iii) Given that u_n can be expressed in the form $u_n = A(-q)^{n-1} + B + Cn$, where A, B and C are constants (independent of n), show that $C = (1+q)^{-1}$ and find A and B in terms of q. Hence show that, for large n, $u_n \approx \dfrac{n}{p+2q}$ and explain carefully why this result is to be expected.

14 **(i)** My favourite dartboard is a disc of unit radius and centre O. I never miss the board, and the probability of my hitting any given area of the dartboard is proportional to the area. Each throw is independent of any other throw. I throw a dart n times (where $n > 1$). Find the expected area of the smallest circle, with centre O, that encloses all the n holes made by my dart.

Find also the expected area of the smallest circle, with centre O, that encloses all the $(n-1)$ holes nearest to O.

(ii) My other dartboard is a square of side 2 units, with centre Q. I never miss the board, and the probability of my hitting any given area of the dartboard is proportional to the area. Each throw is independent of any other throw. I throw a dart n times (where $n > 1$). Find the expected area of the smallest square, with centre Q, that encloses all the n holes made by my dart.

(iii) Determine, without detailed calculations, whether the expected area of the smallest circle, with centre Q, on my square dartboard that encloses all the n holes made by my darts is larger or smaller than that for my circular dartboard.