# STRUCTURE AND EVOLUTION OF SINGLE AND BINARY STARS

**1) Introduction** A set of ordinary differential equations can be determined that relate variables which describe the bulk physical properties of a single star or binary system. Static and dynamic solutions of these equations can yield insight into observable stellar evolution.
Note/ Part of these notes are taken from the Gravitational Astrophysics and Cosmology course. These sections will be preceded with a $\boxed{GAC}$ symbol.

**2) Nomenclature and observables** (See first sections of GAC course for Flux, Parallax)

* **Effective temperature** — assume stellar surface (spherical) radiates like a Black Body i.e. luminosity $L = 4\pi R^2 \sigma T_e^4$ (S1) $R$ = stellar radius $T_e$ = Effective temperature.
* **Abundance of elements.** In most of universe H ~ 70% He ~ 30% and trace 'metals' in order O C N Ne Fe S Si Mg). Metals are thought to originate from supernovae.
* **Spectral classes** categorise stars by temperature $T_e$ [$T_e$ effects ionization state of H ⇒ different spectral lines [between energy levels]].

HOT / COOL
O B A F G K M
40000K ⊙ 2800K

Sub classes
K 0 1 2 3 ..... 9 M
"Early type" "Late type"

* **luminosity classes**
I – Supergiants, less dense atmosphere
III – Giants
V – Dwarf, Main sequence.

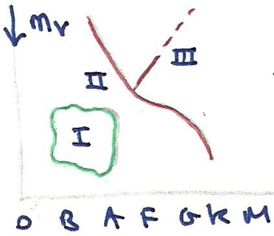
⇒ Sun is G 0 V i.e. "small yellow star".

* **Magnitudes** Define 'absolute' BOLOMETRIC magnitude $m_{bol} = -2.5\log_{10}F + C$ (S2)
F is flux in units of $ergs^{-1}\,cm^{-2}$ and C is a constant. Define bolometric ABSOLUTE magnitude $M_{bol} = -2.5\log_{10}F_{10pc} + C$ where $F_{10pc}$ is flux of star at d = 10pc.
(Note $F = \frac{L}{4\pi d^2}$). Using $F(L,d)$ ⇒ $m_{bol} - M_{bol} = 5\log_{10}\left(\frac{d}{10pc}\right)$ (S3) {Express d in pc or 10pc in cm... No probs with units!}
If observed magnitudes are m, M BOLOMETRIC correction measures deviation from $m_{bol}, M_{bol}$. $BC = M_{bol} - M = m_{bol} - m$. Find C by defining $M_{bol}$ to be zero at some L.

* **Colour indices** usually look at stellar luminosity in particular frequencies. U – UV, B – blue, V – visual.
U ~ 3600Å B ~ 4200Å V ~ 5200Å. Note $M_V - M_B$, $M_V - M_U$ etc. are INDEPENDANT OF DISTANCE d and ∴ (will approximate use of Planck T dependence of Black body spectrum) one can infer T from B–V, U–B measurements. [$B-V \equiv M_B - M_V$, $U-B \equiv M_U - M_B$].

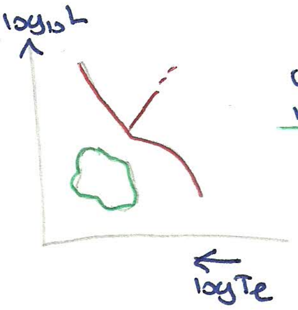
* **Hertzsprung Russell Diagrams.**

observable.

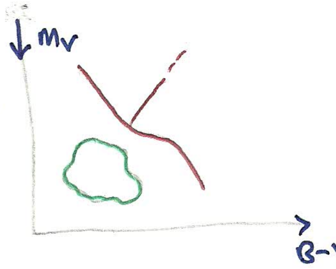



Spectral class

Theoretical.



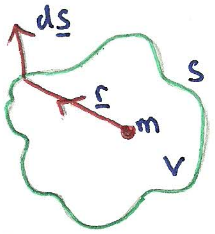

Colour magnitude.




I – white Dwarfs* II – main sequence
III – red Giants.

* Brown dwarfs, neutron stars and most extreme (L→0) Black holes.

**3) Equations of (static) stellar structure**




[1] $\frac{dm}{dr} = 4\pi\rho r^2$ {As a consequence of proving $\nabla^2\phi = 4\pi\rho(r)G$}

Consider flux of gravity field $\underline{g}$ emanating from a point mass within surface S.
Newton's law ⇒ $\underline{g} = -\frac{Gm}{r^2}\hat{\underline{r}}$. Hence flux $F = \int_S -\frac{Gm}{r^2}\hat{\underline{r}}\cdot d\underline{S}$ {$F = \int_S \underline{g}\cdot d\underline{S}$}
Applying Divergence theorem: $F = -Gm\int_V \nabla\cdot\left(\frac{\hat{r}}{r^2}\right)dV$. Now $\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta(\underline{r})$
⇒ $F = -4\pi Gm$. Now $\underline{g} = -\nabla\phi$. (Definition of scalar field $\phi$). Considering sum of point masses within S:

$$-4\pi G\sum_i m_i = \sum_i\int_S -\nabla\phi_i\cdot d\underline{S} = -\sum_i\int_V \nabla\cdot\nabla\phi_i\, dV \text{ (Divergence theorem)} = -\int_V\sum_i\nabla^2\phi_i\, dV.$$

Let $\nabla^2\phi = \sum_i\nabla^2\phi_i$ (i.e. $\phi = \sum_i\phi_i$) and take distributed limit $\int_V\rho\, dV \leftarrow \sum_i m_i$

⇒ $-4\pi G\int_V\rho\, dV = -\int_V\nabla^2\phi\, dV$. Hence by comparison of integrands

$$\nabla^2\phi = 4\pi G\rho(\underline{r}) \quad (S4)$$

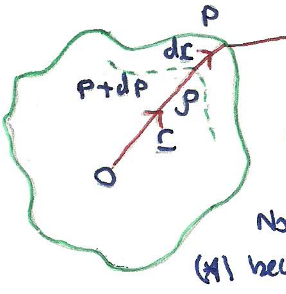
Now in spherically symmetric cases $\phi = \phi(r)$ (as is $\rho$) so can write $\nabla^2$ operator as $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)$

Hence (S4) ⇒ $r^2\frac{d\phi}{dr} = G\int_0^r 4\pi\rho(r')r'^2\,dr'$. Considering the construction of a sphere by assembling ever increasing radii shells ⇒ mass contained within radius $r$, $M(r)$ is $M(r) = \int_0^r 4\pi r'^2\rho(r')\,dr'$ ⇒ $\frac{dM}{dr} = 4\pi\rho r^2$ [1]. Hence (S4) becomes $r^2\frac{d\phi}{dr} = GM(r)$

∴ $[\phi(r)]_r^\infty = \int_r^\infty \frac{GM(r)}{r^2}\,dr$ ⇒ $\phi(r) = -\int_r^\infty \frac{GM(r)}{r^2}\,dr$ (S5) since $\phi(\infty) = 0$ by definition of $\phi$.

[2] $\frac{dP}{dr} = -GM(r)\rho/r^2$

Consider closed surface $S$ within stellar interior. If surface is in 'hydrostatic' equilibrium: if $P$ is pressure field:




$[(P+dP)-P]\,d\underline{S} + \underline{g}\,d\underline{S}\,dr\,\rho(\underline{r}) = 0$ where $\underline{g}$ is local gravity field

For this to be true $\underline{g} \parallel d\underline{S}$ and can ∴ write $\underline{g}\,d\underline{S}$ as $-|\underline{g}|\,d\underline{S}$. Sign convention: "outwards from O is +ve".

(Note $\underline{g}$ points inwards – attractive always!)

Hence comparing coefficients of $d\underline{S}$: $dP - |\underline{g}|\,dr\,\rho = 0$ (*)

Now $dP = -\nabla P . d\underline{r}$ since $P = P(\underline{r})$ and $P \to P + dP$ as $\underline{r} \to \underline{r} - d\underline{r}$. Noting $\frac{d\underline{r}}{dr} = \hat{\underline{r}}$

(*) becomes $\nabla P . \hat{\underline{r}} + |\underline{g}|\rho = 0$. Now from above $\underline{g} = -\nabla\phi$, and $\nabla P . \hat{\underline{r}} = \frac{\partial P}{\partial r}$

⇒ $\frac{\partial P}{\partial r} = -|\nabla\phi|\rho(r)$ (S6) Using $\phi(r)$ (S5) for spherical star $\left\{\Rightarrow \frac{\partial}{\partial r} \to \frac{d}{dr} \text{ and } \nabla \to \hat{\underline{r}}\frac{d}{dr}\right\}$

(S6) ⇒ $\frac{dP}{dr} = -GM(r)\rho/r^2$ [2]

[3] $\frac{dT}{dr} = \frac{-3\kappa\rho L}{16\pi a c r^2 T^3}$ (Radiative part of star)

The interior of a star can either be CONVECTIVE or RADIATIVE. (Decided by the Schwarzschild stability criterion – see below).

In both cases we assume Local Thermodynamic Equilibrium. i.e. length scales associated with mean free path of particles (well electrons) and photons $\ll \frac{T}{\left|\frac{dT}{dr}\right|}$ averaged over star. (i.e. $\frac{T}{\left|\frac{dT}{dr}\right|} \sim R$, stellar radius). * photons take $N^2$ steps (random walk) to diffuse through $R$. $N\lambda = R$ where $\lambda$ = step length. Since step time is $\frac{\lambda}{c}$

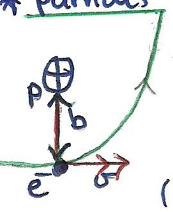
⇒ diffusion time $\sim N^2\frac{\lambda}{c} = \left(\frac{R}{\lambda}\right)^2\frac{\lambda}{c} = \frac{R^2}{\lambda c}$. This ~ "leakout time" $\sim \frac{P_r}{P}t_{KH}$ $\left[t_{leak} = \frac{R^2}{\lambda c}\right]$

$\frac{P_r}{P}$ is ratio of radiation to gas pressure in a star. For giants: $\frac{P_r}{P} \sim 0.25$, M.S. $\frac{P_r}{P} \sim 0.02$.

$t_{KH}$ (Kelvin-Helmholtz) is time to radiate away binding energy. i.e. $t_{KH} \sim \frac{GM^2}{RL}$.

So $\frac{R^2}{\lambda c} \sim \frac{GM^2}{RL}\frac{P_r}{P}$ ⇒ $\lambda_{photon} \sim \frac{LR^3}{GM^2c}\frac{P}{P_r}$ (S7). For Sun $L_\odot \sim 4\times10^{26}$ W, $R_\odot \sim 7\times10^8$ m, $M_\odot \sim 2\times10^{30}$ kg ⇒ $\lambda \sim 10^{-2}$ cm $\ll R_\odot$

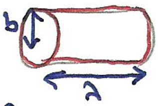
* particles




Electrons accelerated by protons s.t. if closest approach is $b$: $\frac{1}{2}m_e v^2 \sim \frac{e^2}{4\pi\varepsilon_0 b}$

if local thermodynamic equilibrium (LTE) $\frac{1}{2}m_e v^2 \sim \frac{3}{2}k_B T$

∴ $b \sim \frac{e^2}{6\pi\varepsilon_0 k_B T}$. Hence # collisions in cylinder of volume $\pi b^2\lambda$ ~1 ⇒ on average 1 particle / cylinder.




(collisions ⇒ $e^-$, $p$ interact and ~ even # $e^-$ and $p$). ∴ If electron number density is $n$ ⇒ $1 \sim \pi b^2\lambda n$ ⇒ $\lambda \sim \frac{1}{\pi b^2 n}$. $n = \bar{\rho}/m_p$ (since $m_p \gg m_e$ and even # $p, e^-$)

⇒ $\lambda_{particles} \sim \frac{m_p}{\pi b^2\bar{\rho}} \sim \frac{36\pi m_p \varepsilon_0^2 k_B^2 T^2}{\bar{\rho} e^4}$ (S8). For Sun $T_\odot \sim 10^7$ K (CENTRAL temp. $T_{e\odot} \sim 5770$ K) ⇒ $\lambda_{particles} \sim 3\times10^{-5}$ cm $\ll R_\odot$

⇒ LTE is justified! * what about conduction? Compare particle and radiative flux. Energy flux $F$ = energy density · diffusion velocity. Since $P \propto$ energy density

$\frac{F_c}{F_r} = \frac{P}{P_r}\cdot\frac{v_c}{v_r}$. Now $v_r \sim \frac{R}{t_{leak}} = \frac{R}{R^2}\lambda c = \frac{\lambda c}{R}$ ($\lambda_r = \lambda_{photon}$). Similarly $v_c \sim \lambda_{particles}\frac{v}{R}$

LTE ⇒ $v \sim \sqrt{3k_BT/m_e}$ ∴ $\frac{F_c}{F_r} = \frac{P}{P_r}\frac{\lambda_{particle}\sqrt{3k_BT/m_e}}{\lambda_{photon}\,c}$. For Sun (using results above) $\frac{F_c}{F_r} \sim 1\%$

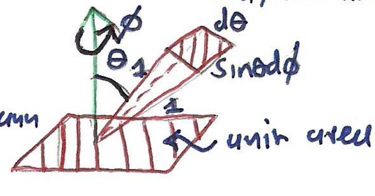
i.e. can ignore conduction compared to radiation. [Note: near surface analysis breaks down ($N$ small) and for WDs $\frac{F_c}{F_r}$ becomes significant since $m_p/m_e \sim 1000$ [multiply by factor 1000. $\frac{1}{\sqrt{1000}} \sim 32$]. However diffusion time for WDs is 1000× less ~~than~~ for electrons $\{t_{leakout} \propto \frac{1}{\lambda}\}$ which exceeds stars lifetime – so can ignore].

Since we have established LTE is a valid approximation we can now assume radiation field is that of a Black Body and particle velocities follow Maxwell speed distribution.
* Flux emitted radiatively per unit area* by a surface into a unit solid angle normal to the surface is given by the Planck law because of LTE.

$$B(\nu,T) = \frac{2h\nu^3/c^2}{e^{h\nu/k_BT}-1} \quad (S9) \qquad \nu = \text{frequency.}$$

* needless since flux is energy/unit area/unit time!

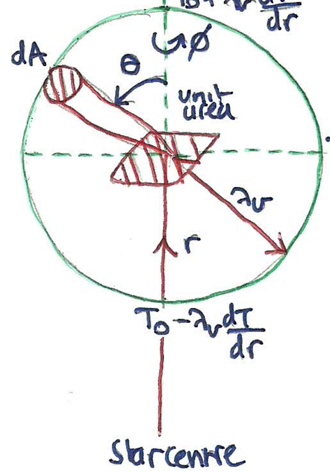
Define $B^*(\nu,T)$ as the flux emitted by a surface into all space.
By Stephen Boltzmann law $\int_0^\infty B^*(\nu,T)\,d\nu = \sigma T^4$. (S10)




$$B^*(\nu,T) = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} B(\nu,T)\cos\theta\sin\theta\,d\theta\,d\phi = \pi B(\nu,T). \quad (S11)$$

upper hemisphere; projection of unit surface

(∴ Prove Stephen Boltzmann law by integrating (S9) and $\times\pi$).




Consider unit area at radius r in steller interior. If $\lambda_\nu$ is mean free path of photon of frequency $\nu$ can assume all photons emit from $\lambda_\nu$ away into unit area. (In general need to integrate over Poisson distribution but yields same result).

Consider element dA of spherical surface of radius $\lambda_\nu$ where photons are emitted from.

$dA = \lambda_\nu^2\sin\theta\,d\theta\,d\phi$. $T_{dA} = T_0 + \frac{dT}{dr}\lambda_\nu\cos\theta$. Power emitted by dA passing through unit area is $B(\nu,T_0+\Delta T)$. Solid angle of unit area. dA

$= B^*(\nu,T_0+\Delta T)\frac{1}{\pi}\cdot\frac{\cos\theta}{\lambda_\nu^2}\cdot\lambda_\nu^2\sin\theta\,d\theta\,d\phi$ where $\Delta T = \frac{dT}{dr}\lambda_\nu\cos\theta$.

(S11) For each area on upper hemisphere at $\theta$ there is an equivalent at $\pi-\theta$ at temperature $T_0-\Delta T$. Hence net power into unit area from these two is

$$F_\nu = \{B^*(\nu,T_0+\Delta T) + B^*(\nu,T_0-\Delta T)\cos(\pi-\theta)\sin(\pi-\theta)\}\frac{1}{\pi}d\theta\,d\phi = \frac{\{B^*(\nu,T_0+\Delta T) - B^*(\nu,T_0-\Delta T)\}}{\Delta T}\frac{\Delta T\cos\theta\sin\theta\,d\theta\,d\phi}{\pi}$$

$\{.\cos\theta\sin\theta$

Now flux emitted by unit area* is $-\overset{\theta,\phi}{F_\nu} \equiv \overset{\theta,\phi}{F'_\nu}$ (for these two) * well power radiated / unit area = flux! ← already differential!

⇒ Total power radiated by unit area = flux = $F_\nu$ is: (Taking limit $\Delta T\to 0$)

★ Taylors first approximation

$$F_\nu \cong \left\{B^*(\nu,T_0) - \Delta T\frac{\partial B^*}{\partial T}(\nu,T_0) - B^*(\nu,T_0) - \Delta T\frac{\partial B^*}{\partial T}(\nu,T_0)\right\}(\Delta T)^{-1}\frac{dT}{dr}\frac{\lambda_\nu}{\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta\,d\theta\,d\phi$$

$$\Rightarrow F_\nu \approx -\frac{4}{3}\frac{\partial B^*(\nu,T)}{\partial T}\frac{dT}{dr}\frac{1}{\kappa_\nu\rho}$$

renaming (general) $T_0$ as $T$ and defining OPACITY $\kappa_\nu = \frac{1}{\rho\lambda_\nu}$ (S12)

Define a "Rosseland" mean opacity $\frac{1}{\kappa} = \int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B^*(\nu,T)}{\partial T}d\nu \Big/ \int_0^\infty\frac{\partial B^*}{\partial T}d\nu$ (S13)

$$\Rightarrow \int_0^\infty F_\nu = F = -\frac{4}{3}\frac{dT}{dr}\frac{1}{\rho\kappa}\int_0^\infty\frac{\partial B^*}{\partial T}d\nu. \quad (S10)\Rightarrow \int_0^\infty\frac{\partial B^*}{\partial T}d\nu = \frac{\partial}{\partial T}\int_0^\infty B^*d\nu = \frac{d}{dT}(\sigma T^4) = 4\sigma T^3$$

$$\therefore\ F = \frac{-16\sigma T^3}{3\rho\kappa}\frac{dT}{dr}$$

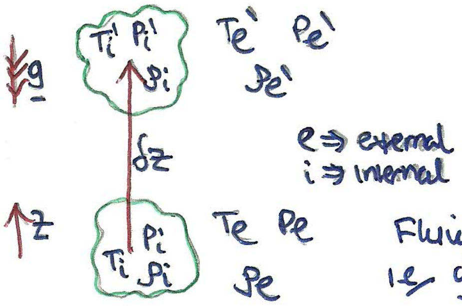
Now $L = F.4\pi r^2$ (radiation of surface within star if radiative) and $\sigma = ac/4$

$$\Rightarrow \frac{dT}{dr} = -\frac{3\rho\kappa L}{16\pi ac r^2 T^3} \quad [3]$$

Valid deep inside radiative star – not at surface (cannot construct $\lambda_\nu$ sphere)

★ Schwarzschild criterion for convective stability
[i.e. convectively unstable ⇒ star NOT radiative]

$$\frac{dT}{dr} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \Delta\nabla T \quad [4]$$




Consider rise of small isolated fluid element (in a main body of fluid) in a bulky linear gravity field. Referring to diagram: $T_i = T_e$ $T_i' = T_e'$ $P_i = P_e$ $P_i' = P_e'$ if in LTE. Also $\rho_i = \rho_e$ but $\rho_i' \neq \rho_e'$ in general.

$\rho_i' = \rho_i + \frac{d\rho_i}{dz}\delta z$ $\rho_e' = \rho_e + \frac{d\rho_e}{dz}\delta z$ (only z variation)

Fluid element will continue to rise if (Archimedes) $\rho_e' > \rho_i'$ i.e. $\frac{d\rho_e}{dz} > \frac{d\rho_i}{dz}$. Instability criterion

Now assume external fluid always undergoes adiabatic changes.......

... OK if external fluid >> fluid element. (Acts as a reservoir). For adiabatic change of a mixture of radiation and ideal gas: $\frac{dP}{P} + \gamma_1 \frac{dV}{V} = 0 \Rightarrow \frac{dP}{P} = \gamma_1 \frac{d\rho}{\rho}$ (S14) $\left(\rho \propto \frac{1}{V}\right)$

where $\gamma_1 = \beta + \frac{(4-3\beta)^2(\gamma-1)}{\beta + 12(\gamma-1)(1-\beta)}$ and $\beta = \frac{P_{gas}}{P}$, $\gamma$ = ratio of specific heats of ideal gas.

Hence $dP_e = \gamma_1 d\rho_e \frac{P_e}{\rho_e}$. Now a polytrope has equation of state $P = k\rho^\Gamma$ (S15); $k, \Gamma$ constants. Fluid element can be thought to be "locally polytropic" in which case since $\Gamma = \frac{d\ln P}{d\ln \rho} = \frac{dP}{P}\cdot\frac{\rho}{d\rho} \Rightarrow dP_i = \Gamma \frac{P_i}{\rho_i} d\rho_i$

Hence instability criterion becomes $\frac{\rho_e}{P_e}\frac{1}{\gamma_1} dP_e > \frac{\rho_i}{P_i}\frac{1}{\Gamma} dP_i$ since $P_e = P_i$ and $\rho_e = \rho_i$ $\Rightarrow \Gamma > \gamma_1$

∴ Schwarzschild stability criterion for convection is $\gamma_1 > \Gamma$. [Note if $\beta = 1$, $\gamma_1 = \gamma$.] (S16)

Now define SUPERADIABATIC gradient $\Delta\nabla T$ as the difference between an adiabatic temperature gradient and one in a general situation. i.e. $\Delta\nabla T = \left(\frac{dT}{dr}\right)_{adiabatic} - \frac{dT}{dr}$ (S17)

Now $\frac{dT}{dr} = \frac{\partial \ln T}{\partial \ln P}\frac{T}{P}\frac{dP}{dr}$ {Proof: $\frac{\partial \ln T}{\partial \ln P} = \frac{dT}{T}\cdot\frac{P}{dP}$ ..... rest follows}.

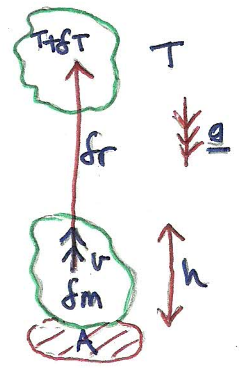
∴ can write $\Delta\nabla T = \left(\frac{\partial \ln T}{\partial \ln P}\right)_{adiabatic}\frac{T}{P}\frac{dP}{dr} - \frac{dT}{dr}$. Now for an adiabatic change of a mixture of radiation and gas (S14) $\Rightarrow \frac{dP}{P} = \gamma_1 \frac{d\rho}{\rho}$. Can show that

↑ Full derivatives better.

$\frac{dP}{P} = \frac{\gamma_1(4-3\beta)}{-\beta+\gamma_1}\frac{dT}{T}$ if $\rho$ (↑ well $V$) is eliminated instead of $T$ in the derivation. (Q2 of ES II).

Hence $\left(\frac{d\ln T}{d\ln P}\right)_{adiabatic} = \frac{-\beta+\gamma_1}{\gamma_1(4-3\beta)}$. For pure ideal gas $\beta = 1$, $\gamma_1 = \gamma \Rightarrow \left(\frac{d\ln T}{d\ln P}\right)_{adiabatic} = 1 - \frac{1}{\gamma}$

∴ $\frac{dT}{dr} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \Delta\nabla T$ [4]. Now consider small mass $\delta m$ convecting in an external fluid in a locally linear gravity field. Excess heat transported by the mass is $\Delta\nabla T \delta r \delta m C_p = \delta H$ {heat transport analogous to mass transport analysed in derivation of Schwarzschild stability criterion}. If mass velocity is $v$ and height $h$, cross section $A$, 1 one second mass moves up by $\frac{v}{h}$ of total height ∴ heat $\delta H \frac{v}{h}\frac{1}{A}$ is moved /s / unit area = FLUX ($F$).




∴ $F_{convection} = \Delta\nabla T \delta r \delta m C_p \frac{v}{hA} = \Delta\nabla T \delta r C_p \rho v$. Density deficit $\delta\rho$ is

$\delta\rho = \frac{d\rho}{dr}\delta r - \left(\frac{d\rho}{dr}\right)_{adiabatic}$. Using (S14) $d\rho = \frac{\rho}{P}\frac{1}{\gamma}dP$ (ideal gas, pure).

$\Rightarrow \delta\rho = \frac{d\rho}{dr}\delta r - \frac{\rho}{P}\frac{1}{\gamma}\frac{dP}{dr}\delta r$. Assume average buoyancy force is $\frac{1}{2}\delta\rho g \Rightarrow$ work done $= \delta r \frac{1}{2}\delta\rho g = \frac{1}{2}\rho v^2$. ∴ $v = \left(\frac{\delta\rho}{\rho} g \delta r\right)^{\frac{1}{2}}$.

Now from [4] $\frac{\Delta\nabla T}{T}\rho = \left(1-\frac{1}{\gamma}\right)\frac{\rho}{P}\frac{dP}{dr} - \frac{dT}{dr}\frac{\rho}{T}$. For ideal gas $P = \frac{\rho R T}{\mu} \Rightarrow \frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$

∴ $\frac{dT}{T} = \frac{dP}{P} - \frac{d\rho}{\rho} \Rightarrow \frac{\Delta\nabla T}{T}\rho = \left(1-\frac{1}{\gamma}\right)\frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{P}\frac{dP}{dr} + \frac{d\rho}{dr} = \frac{d\rho}{dr} - \frac{1}{\gamma}\frac{\rho}{P}\frac{dP}{dr}$. Hence $\delta\rho = \frac{\Delta\nabla T \rho \delta r}{T}$

∴ $v = \left(\frac{\Delta\nabla T \rho \delta r^2}{\rho} g\right)^{\frac{1}{2}} \Rightarrow F_{convection} = C_p \rho \left(\frac{g}{T}\right)^{\frac{1}{2}}(\Delta\nabla T)^{\frac{3}{2}}\delta r^2$. Let $F_{convection} = \frac{L}{4\pi R^2}$

∴ $\Delta\nabla T = \left[\frac{L}{4\pi R^2 C_p \rho \left(\frac{g}{T}\right)^{\frac{1}{2}}\delta r^2}\right]^{\frac{2}{3}}$ (S18). For Sun $\Delta\nabla T \sim 10^{-10}\,\mathrm{K\,cm^{-1}}$ *. Now $\left|\frac{dT}{dr}\right|_{sun} \sim \frac{T_c}{R} \sim \frac{10^7\,\mathrm{K}}{10^{11}\,\mathrm{cm}} \sim 10^{-4}\,\mathrm{K\,cm^{-1}}$. Since (typically)

$\left|\frac{dT}{dr}\right| \gg \Delta\nabla T$ we can assume $\Delta\nabla T = 0$ in context of [4].

i.e. $\frac{dT}{dr} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$ (S19)

* We have assumed "mixing length" (unknown) $\sim 0.1 R_\odot$ → Guess, fit data.....

Note $\Delta\nabla T > 0 \Rightarrow$ convective instability.

(use definitions of $\rho$, $\gamma$). * Convection very efficient inside star – not so much at surface.