STRUCTURE AND EVOLUTION OF SINGLE AND BINARY STARS

1) Introduction A set of ordinary differential equations can be determined that relate variables which describe the bulk physical properties of a single star or binary system. Static and dynamic solutions of these equations can yield insight into observable stellar evolution. Note/ Part of these notes are taken from the Graduate Astrophysics and Cosmology course. These sections will be preceded with a $\boxed{\text{GAC}}$ symbol.

2) Nomenclature and observables (See first sections of GAC course pr Flux, Parallax)

* Effective temperature – assume stellar surface (spherical) radiates like a Black Body i.e luminosity $L = 4\pi R^2 \sigma T_e^4$ (S1) R = Stellar radius T_e = Effective temperature.

* Abundance of elements. In most of universe H ~ 70% He ~ 30% and trace 'metals' (in order O C N Ne Fe S Si Mg). Metals are thought to originate from supernovae.

* Spectral classes categorise stars by temperature T_e [T_e effects ionization state of H ⇒ different spectral lines [between energy levels]].

HOT COOL

O B A F G K M

40000K ⊙ ~2800K

Sub classes

K 0 1 2 3..... 9 M

"Early type" "Late type"

* Luminosity classes

I – Supergiants, less dense atmosphere

III – Giants

V – Dwarf, Main sequence.

⇒ Sun is G0V i.e "small yellow star".

* Magnitudes Define 'absolute' BOLOMETRIC magnitude $m_{bol} = -2.5\log_{10}F + C$ (S2) F is flux in units of $erg s^{-1} cm^{-2}$ and C is a constant. Define bolometric ABSOLUTE magnitude $M_{bol} = -2.5\log_{10}F_{10pc} + C$ where F_{10pc} is flux of star at d = 10pc. (Note $F = \frac{L}{4\pi d^2}$). Using $F(L,d) \Rightarrow m_{bol} - M_{bol} = 5\log_{10}\left(\frac{d}{10pc}\right)$ (S3) {Express d in pc or 10pc in cm... No probs with units!}

If observed magnitudes are m, M BOLOMETRIC correction measures deviation from m_{bol}, M_{bol}. $BC = M_{bol} - M = m_{bol} - m$. Find C by defining M_{bol} to be zero at some L.

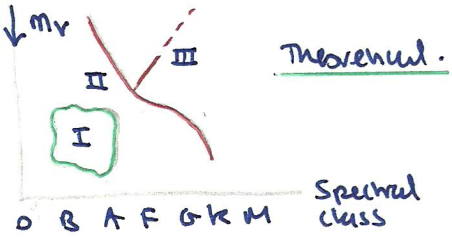
* Colour indices usually look at stellar luminosity in particular frequencies.

U – UV
B – blue
V – visual

U ~ 3600Å B ~ 4200Å V ~ 5200Å. Note $m_V - m_B$, $m_V - m_U$ etc. are INDEPENDANT OF DISTANCE d and ∴ (with appropriate use of Planck T dependence of Black body spectrum) one can infer T from B−V, U−B measurements. [$B-V \equiv m_B - m_V$, $U-B \equiv m_U - m_B$].

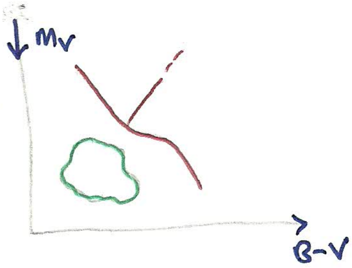
* Hertzsprung Russell Diagrams.

Observable.



Theoretical.

Colour magnitude.



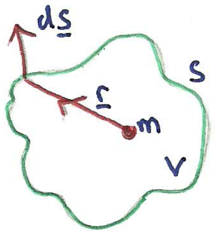
I – white dwarfs* II – main sequence

III – red Giants.

* Brown dwarfs, neutron stars and most extreme (L→0) Black holes.

3) Equations of (Static) Stellar Structure

[1] $\frac{dm}{dr} = 4\pi\rho r^2$ {As a consequence of proving $\nabla^2\phi = 4\pi\rho(r)G$}



Consider flux of gravity field $\underline{g}$ emanating from a point mass within surface S. Newton's law ⇒ $\underline{g} = -\frac{Gm}{r^2}\hat{\underline{r}}$. Hence flux $F = \int_S -\frac{Gm}{r^2}\hat{\underline{r}}\cdot d\underline{S}$ $\{F = \int_S \underline{g}\cdot d\underline{S}\}$

Applying Divergence theorem: $F = -Gm\int_V \nabla\cdot\left(\frac{\hat{r}}{r^2}\right)dV$. Now $\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta(\underline{r})$

⇒ $F = -4\pi Gm$. Now $\underline{g} = -\nabla\phi$. (Definition of scalar field ϕ). Considering sum of point masses within S:

$$-4\pi G\sum_i m_i = \sum_i\int_S -\nabla\phi_i\cdot d\underline{S} = -\sum_i\int_V \nabla\cdot\nabla\phi_i\,dV \text{ (Divergence theorem)} = -\int_V\sum_i\nabla^2\phi_i\,dV.$$

Let $\nabla^2\phi = \sum_i\nabla^2\phi_i$ (i.e $\phi = \sum_i\phi_i$) and take distributed limit $\int_V \rho dV \Leftarrow \sum_i m_i$

$\Rightarrow -4\pi G\int_V \rho dV = -\int_V \nabla^2\phi\,dV$. Hence by comparison of integrands

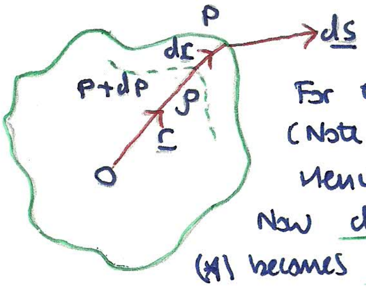
$$\nabla^2\phi = 4\pi G\rho(\underline{r}) \quad (S4)$$

Now in spherically symmetric cases $\phi = \phi(r)$ (as is ρ) so can write ∇^2 operator as $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)$

Hence (S4) $\Rightarrow$ $r^2\frac{d\phi}{dr} = G\int_0^r 4\pi\rho(r')r'^2\,dr'$. Considering the construction of a sphere by assembling ever increasing radii shells $\Rightarrow$ mass contained within radius r, $M(r)$ is $M(r) = \int_0^r 4\pi r'^2\rho(r')\,dr'$ $\Rightarrow$ $\frac{dM}{dr} = 4\pi\rho r^2$ [1]. Hence (S4) becomes $r^2\frac{d\phi}{dr} = GM(r)$

$\therefore \left[\phi(r)\right]_r^{\infty} = \int_r^{\infty}\frac{GM(r)}{r^2}dr$ $\Rightarrow$ $\phi(r) = -\int_r^{\infty}\frac{GM(r)}{r^2}dr$ (S5) since $\phi(\infty) = 0$ by definition of ϕ.

[2] $\frac{dP}{dr} = -GM(r)\rho/r^2$ Consider closed surface S within stellar interior. If surface is in 'hydrostatic' equilibrium: If P is pressure field:



$[(P+dP)-P]\,\underline{dS} + \underline{g}\,d\underline{S}\,dr\,\rho(\underline{r}) = 0$ where $\underline{g}$ is local gravity field. For this to be true $\underline{g} \parallel \underline{dS}$ and can $\therefore$ write $\underline{g}\,\underline{dS}$ as $-|\underline{g}|\,\underline{dS}$. Sign convention: "outwards from O is +ve".
(Note $\underline{g}$ points inwards – attractive always!)
Hence comparing coefficients of $\underline{dS}$: $dP - |\underline{g}|\,dr\,\rho = 0$ (*)
Now $dP = -\nabla P.\underline{dr}$ since $P = P(\underline{r})$ and $P \to P+dP$ as $\underline{r} \to \underline{r} - \underline{dr}$. Noting $\frac{\underline{dr}}{dr} = \hat{\underline{r}}$
(*) becomes $\nabla P.\hat{\underline{r}} + |\underline{g}|\rho = 0$. Now from above $\underline{g} = -\nabla\phi$, and $\nabla P.\hat{\underline{r}} = \frac{\partial P}{\partial r}$

$\Rightarrow \frac{\partial P}{\partial r} = -|\nabla\phi|\rho(\underline{r})$ (S6) Using $\phi(r)$ (S5) for spherical star $\left\{\Rightarrow \frac{\partial}{\partial r} \to \frac{d}{dr} \text{ and } \nabla \to \hat{\underline{r}}\frac{d}{dr}\right\}$

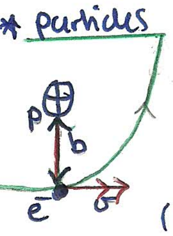
(S6) $\Rightarrow$ $\frac{dP}{dr} = -GM(r)\rho/r^2$ [2]

[3] $\frac{dT}{dr} = \frac{-3\kappa\rho L}{16\pi ac\,r^2T^3}$ (Radiative part of star)

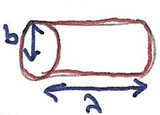
The interior of a star can either be CONVECTIVE or RADIATIVE. (Decided by the Schwarzchild stability criterion – see below).

In both cases we assume Local Thermodynamic Equilibrium. i.e length scales associated with mean free path of particles (well electrons) and photons $\ll \frac{T}{\left|\frac{dT}{dr}\right|}$ averaged over star. (i.e $\frac{T}{\left|\frac{dT}{dr}\right|} \sim R$, stellar radius). *photons take N^2 steps (random walk) to diffuse through R. $N\lambda = R$ where λ = step length. Since step time is $\frac{\lambda}{c}$
$\Rightarrow$ diffusion time $\sim N^2\frac{\lambda}{c} = \left(\frac{R}{\lambda}\right)^2\frac{\lambda}{c} = \frac{R^2}{\lambda c}$ This $\sim$ "leakout time" $\sim \frac{P_r}{P}t_{KH}$ $\left[t_{leak} = \frac{R^2}{\lambda c}\right]$
$\frac{P_r}{P}$ is ratio of radiation to gas pressure in a star. For giants: $\frac{P_r}{P} \sim 0.25$, M.S. $\frac{P_r}{P} \sim 0.02$.
t_{KH} (Kelvin-Helmholtz) is time to radiate away binding energy. i.e. $t_{KH} \sim \frac{GM^2}{RL}$.
So $\frac{R^2}{\lambda c} \sim \frac{GM^2}{RL}\frac{P_r}{P} \Rightarrow \lambda_{photon} \sim \frac{LR^3}{GM^2c}\frac{P}{P_r}$ (S7). For Sun
$L_\odot \sim 4\times10^{26}$ W
$R_\odot \sim 7\times10^{8}$ m
$M_\odot \sim 2\times10^{30}$ kg
$\Rightarrow$ $\lambda \sim 10^{-2}$ cm $\ll R_\odot$

*particles



Electrons accelerated by protons s.t. if closest approach is b $\frac{1}{2}m_ev^2 \sim \frac{e^2}{4\pi\varepsilon_0 b}$
If local thermodynamic equilibrium (LTE) $\frac{1}{2}m_ev^2 \sim \frac{3}{2}k_BT$
$\therefore$ $b \sim \frac{e^2}{6\pi\varepsilon_0 k_BT}$. Hence # collisions in cylinder of volume $\pi b^2\lambda$ $\sim 1 \Rightarrow$ on average 1 particle / cylinder.



(collisions $\Rightarrow$ e^-, p interact and $\sim$ even # e^- and p). $\therefore$ If electron number density is n $\Rightarrow$ $1 \sim \pi b^2\lambda n$ $\Rightarrow$ $\lambda \sim \frac{1}{\pi b^2 n}$. $n = \bar{\rho}/m_p$ (Since $m_p \gg m_e$ and even # p, e^-)
$\Rightarrow$ $\lambda_{particles} \sim \frac{m_p}{\pi b^2\bar{\rho}} \sim \frac{36\pi m_p\varepsilon_0^2k_B^2T^2}{\bar{\rho}e^4}$ (S8). For Sun $T_\odot \sim 10^7$ K (CENTRAL temp. $T_{e\odot} \sim 5770$ K)
$\Rightarrow$ $\lambda_{particles} \sim 3\times10^{-5}$ cm $\ll R_\odot$

$\Rightarrow$ LTE is justified! *what about conduction? Compare particle and radiative flux. Energy flux F = energy density · diffusion velocity. Since P $\propto$ energy density
$F_c/F_r = \frac{P}{P_r}\cdot\frac{v_c}{v_r}$. Now $v_r \sim \frac{R}{t_{leak}} = \frac{R}{R^2}\lambda c = \frac{\lambda c}{R}$ ($\lambda_r = \lambda_{photon}$). Similarly $v_c \sim \lambda_{particles}\,v/R$
LTE $\Rightarrow$ $v \sim \sqrt{3k_BT/m_e}$ $\therefore$ $F_c/F_r = \frac{P}{P_r}\frac{\lambda_{particle}\sqrt{3k_BT/m_e}}{\lambda_{photon}\,c}$. For Sun (using results above) $F_c/F_r \sim 1\%$
i.e. can ignore conduction compared to radiation. [Note: near surface analysis breaks down (N small) and for ions F_c/F_r becomes significant since $m_p/m_e \sim 1000$ [multiply by factor 1000. $\frac{1}{\sqrt{1000}} \sim 32$]. However diffusion time for ions is 1000× less ~~than~~ for electrons $\left\{t_{leakout} \propto \frac{1}{\lambda}\right\}$ which exceeds stars lifetime – so can ignore].

Since we have established LTE is a valid approximation we can now assume radiation field is that of a Black Body and particle velocities follow Maxwell speed distribution.

* flux emitted radiatively per unit area* by a surface into a unit solid angle normal to the surface is given by the Planck law because of LTE.

$$B(\nu,T) = \frac{2h\nu^3/c^2}{e^{h\nu/k_BT}-1} \quad (S9)$$

ν = frequency.

* needless since flux is energy/unit area/unit time!

Define $B^*(\nu,T)$ as the flux emitted by a surface into all space.

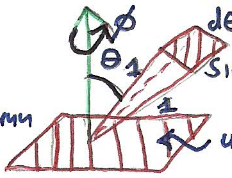
By Stephen Boltzmann law $\int_0^\infty B^*(\nu,T)\,d\nu = \sigma T^4$. (S10)

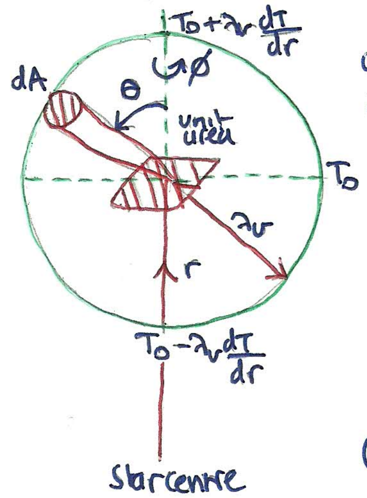
$$B^*(\nu,T) = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} B(\nu,T)\cos\theta\sin\theta\,d\theta\,d\phi = \pi B(\nu,T). \quad (S11)$$

upper hemisphere

projection of unit surface

(∴ Prove Stephen Boltzmann law by integrating (S9) and $\times\pi$).





Consider unit area at radius r in stellar interior. If λ_ν is mean free path of photon of frequency ν can assume all photons emit from λ_ν away into unit area. (In general need to integrate over Poisson distribution but yields same result).

Consider element dA of spherical surface of radius λ_ν where photons are emitted from.

$dA = \lambda_\nu^2\sin\theta\,d\theta\,d\phi$. $T_{dA} = T_0 + \frac{dT}{dr}\lambda_\nu\cos\theta$. Power emitted by dA passing through unit area is $B(\nu,T_0+\Delta T)$. Solid angle of unit area. dA

$$= B^*(\nu,T_0+\Delta T)\frac{1}{\pi}\cdot\frac{\cos\theta}{\lambda_\nu^2}\cdot\lambda_\nu^2\sin\theta\,d\theta\,d\phi$$

(S11) ↗

where $\Delta T = \frac{dT}{dr}\lambda_\nu\cos\theta$. is 44

For each area on upper hemisphere at θ there is equivalent at $\pi-\theta$ at temperature $T_0-\Delta T$. Hence net power into unit area from these two is

$${}^{\theta\phi}F_\nu = \{B^*(\nu,T_0+\Delta T)\cos\theta\sin\theta + B^*(\nu,T_0-\Delta T)\cos(\pi-\theta)\sin(\pi-\theta)\}\frac{1}{\pi}d\theta\,d\phi = \frac{\{B^*(\nu,T_0+\Delta T)-B^*(\nu,T_0-\Delta T)\}}{\Delta T}\frac{\Delta T}{\pi}\cos\theta\sin\theta\,d\theta\,d\phi$$

Now flux emitted by unit area* is $\overset{\theta\phi}{-}F_\nu \equiv {}^{\theta,\phi}F'_\nu$ (for these two)

* well power radiated / unit area = flux! ⇐ already differential!

⇒ Total power radiated by unit area = flux $= F_\nu$ is: (Taking limit $\Delta T\to 0$)

Taylors first approximation

$$F_\nu \cong \left\{B^*(\nu,T_0) - \Delta T\frac{\partial B^*}{\partial T}(\nu,T_0) - B^*(\nu,T_0) - \Delta T\frac{\partial B^*}{\partial T}(\nu,T_0)\right\}(\Delta T)^{-1}\frac{dT}{dr}\frac{\lambda_\nu}{\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta\,d\theta\,d\phi$$

$$\Rightarrow F_\nu \approx -\frac{4}{3}\frac{\partial B^*(\nu,T)}{\partial T}\frac{dT}{dr}\frac{1}{\kappa_\nu\rho}$$

renaming (general) T_0 as T and defining OPACITY $\kappa_\nu = \frac{1}{\rho\lambda_\nu}$ (S12)

Define a "Rosseland" mean opacity $\frac{1}{\kappa} = \int_0^\infty\frac{1}{\kappa_\nu}\frac{\partial B^*(\nu,T)}{\partial T}d\nu \Big/ \int_0^\infty\frac{\partial B^*}{\partial T}d\nu$ (S13)

$$\Rightarrow \int_0^\infty F_\nu = F = -\frac{4}{3}\frac{dT}{dr}\frac{1}{\rho\kappa}\int_0^\infty\frac{\partial B^*}{\partial T}d\nu.$$

(S10) ⇒ $\int_0^\infty\frac{\partial B^*}{\partial T}d\nu = \frac{\partial}{\partial T}\int_0^\infty B^*d\nu = \frac{d}{dT}(\sigma T^4) = 4\sigma T^3$

$$\therefore F = \frac{-16\sigma T^3}{3\rho\kappa}\frac{dT}{dr}$$

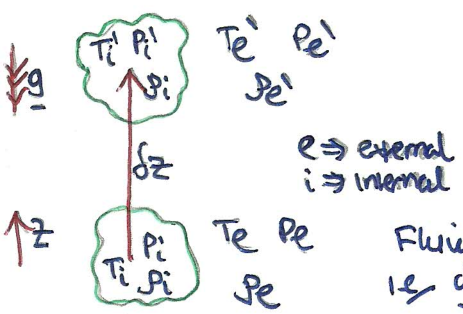
Now $L = F.4\pi r^2$ (radiation of surface within star if radiative) and $\sigma = ac/4$

$$\Rightarrow \frac{dT}{dr} = -\frac{3\rho\kappa L}{16\pi ac r^2T^3} \quad [3]$$

Valid deep inside radiative star – not at surface (cannot construct λ_ν sphere)

$$\frac{dT}{dr} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \Delta\nabla T \quad [4]$$

* Schwarzschild criterion for convective stability [i.e. convectively unstable ⇒ star NOT radiative]



Consider rise of small isolated fluid element (in a main body of fluid) in a locally linear gravity field. Referring to diagram: $T_i = T_e$ $T_i' = T_e'$ $P_i = P_e$ $P_i' = P_e'$ if in LTE. Also $\rho_i = \rho_e$ but $\rho_i' \neq \rho_e'$ in general.

$\rho_i' = \rho_i + \frac{d\rho_i}{dz}\delta z$ $\rho_e' = \rho_e + \frac{d\rho_e}{dz}\delta z$ (only z variation)

Fluid element will continue to rise if (Archimedes) $\rho_e' > \rho_i'$ i.e. $\frac{d\rho_e}{dz} > \frac{d\rho_i}{dz}$. Now assume external fluid always undergoes adiabatic changes.......

Instability criterion

... OK if external fluid >> fluid element. (Acts as a reservoir). For adiabatic change of a mixture of radiation and ideal gas: $\frac{dP}{P}+\gamma_1\frac{dV}{V}=0 \Rightarrow \frac{dP}{P}=\gamma_1\frac{d\rho}{\rho}$ (S14) $(\rho \propto \frac{1}{V})$
where $\gamma_1 = \beta + \frac{(4-3\beta)^2(\gamma-1)}{\beta+12(\gamma-1)(1-\beta)}$ and $\beta = \frac{P_{gas}}{P}$, γ = ratio of specific heats of ideal gas.

Hence $dP_e = \gamma_1 d\rho_e \frac{P_e}{\rho_e}$. Now a polytrope has equation of state $P = k\rho^{\Gamma}$ (S15) ; k, Γ constants
fluid element can be thought to be "locally polytropic"
in which case since $\Gamma = \frac{d\ln P}{d\ln\rho} = \frac{dP}{P}\cdot\frac{\rho}{d\rho} \Rightarrow dP_i = \Gamma \frac{P_i}{\rho_i} d\rho_i$

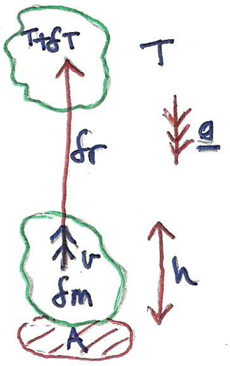
Hence instability criterion becomes $\frac{\rho_e}{P_e}\frac{1}{\gamma_1}dP_e > \frac{\rho_i}{P_i}\frac{1}{\Gamma}dP_i$ since $P_e = P_i$ and $\rho_e = \rho_i$
$\Rightarrow \Gamma > \gamma_1$

∴ Schwarzschild stability criterion for convection is $\gamma_1 > \Gamma$. [Note if $\beta=1$, $\gamma_1=\gamma$.] (S16)

Now define SUPERADIABATIC gradient $\Delta\nabla T$ as the difference between an adiabatic temperature gradient and one in a general situation. i.e $\Delta\nabla T = \left(\frac{dT}{dr}\right)_{Adiabatic} - \frac{dT}{dr}$ (S17)
Now $\frac{dT}{dr} = \frac{\partial\ln T}{\partial\ln P}\frac{T}{P}\frac{dP}{dr}$ {Proof: $\frac{\partial\ln T}{\partial\ln P} = \frac{dT}{T}\cdot\frac{P}{dP}$ ..... rest follows}.

∴ can write $\Delta\nabla T = \left(\frac{\partial\ln T}{\partial\ln P}\right)_{adiabatic}\frac{T}{P}\frac{dP}{dr} - \frac{dT}{dr}$. Now for an adiabatic change of a mixture of radiation and gas
↑Full derivatives better.
(S14) $\Rightarrow \frac{dP}{P} = \gamma_1\frac{d\rho}{\rho}$. Can show that
$\frac{dP}{P} = \frac{\gamma_1(4-3\beta)}{-\beta+\gamma_1}\frac{dT}{T}$ if ρ (↖well V) is eliminated instead of T in the derivation. (Q2 of ES II).
Hence $\left(\frac{d\ln T}{d\ln P}\right)_{adiabatic} = \frac{-\beta+\gamma_1}{\gamma_1(4-3\beta)}$. For pure ideal gas $\beta=1$, $\gamma_1=\gamma \Rightarrow \left(\frac{d\ln T}{d\ln P}\right)_{adiabatic} = 1-\frac{1}{\gamma}$

∴ $\frac{dT}{dr} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \Delta\nabla T$ [4]. Now consider small mass δm convecting in an external fluid in a locally linear gravity field. Excess heat transported by the mass is $\Delta\nabla T\,\delta r\,\delta m\,C_p = \delta H$ {heat transport analogous to mass transport analysed in derivation of Schwarzschild stability criterion}. If mass velocity is v and height h, cross section A, 1 one second mass moves up by $\frac{v}{h}$ of total height ∴ heat $\frac{\delta H v}{h}\frac{1}{A}$ is moved / s / unit area = FLUX. (F).



∴ $F_{convection} = \Delta\nabla T\,\delta r\,\delta m\,C_p\frac{v}{hA} = \Delta\nabla T\,\delta r\,C_p\,\rho\,v$. Density deficit $\delta\rho$ is
$\delta\rho = \frac{d\rho}{dr}\delta r - \left(\frac{d\rho}{dr}\right)_{adiabatic}$. Using (S14) $d\rho = \frac{\rho}{P}\frac{1}{\gamma}dP$ (ideal gas, pure).
$\Rightarrow \delta\rho = \frac{d\rho}{dr}\delta r - \frac{\rho}{P}\frac{1}{\gamma}\frac{dP}{dr}\delta r$. Assume average buoyancy force is $\frac{1}{2}\delta\rho g \Rightarrow$ work done $= \delta r\frac{1}{2}\delta\rho g = \frac{1}{2}\rho v^2$. ∴ $v = \left(\frac{\delta\rho}{\rho}g\,\delta r\right)^{\frac{1}{2}}$.

Now from [4] $\frac{\Delta\nabla T}{T}\rho = \left(1-\frac{1}{\gamma}\right)\frac{\rho}{P}\frac{dP}{dr} - \frac{dT}{dr}\frac{\rho}{T}$. For ideal gas $P = \frac{\rho RT}{\mu} \Rightarrow \frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$
∴ $\frac{dT}{T} = \frac{dP}{P} - \frac{d\rho}{\rho} \Rightarrow \frac{\Delta\nabla T}{T}\rho = \left(1-\frac{1}{\gamma}\right)\frac{\rho}{P}\frac{dP}{dr} - \frac{\rho}{P}\frac{dP}{dr} + \frac{d\rho}{dr} = \frac{d\rho}{dr} - \frac{1}{\gamma}\frac{\rho}{P}\frac{dP}{dr}$. Hence $\delta\rho = \frac{\Delta\nabla T\rho\,\delta r}{T}$
∴ $v = \left(\frac{\Delta\nabla T\rho\,\delta r^2}{\rho}g\right)^{\frac{1}{2}} \Rightarrow F_{convection} = C_p\rho\left(\frac{g}{T}\right)^{\frac{1}{2}}(\Delta\nabla T)^{\frac{3}{2}}\delta r^2$. Let $F_{convection} = \frac{L}{4\pi R^2}$
∴ $\Delta\nabla T = \left[\frac{L}{4\pi R^2 C_p\rho\left(\frac{g}{T}\right)^{\frac{1}{2}}\delta r^2}\right]^{\frac{2}{3}}$ (S18). For Sun $\Delta\nabla T \sim 10^{-10}\,\mathrm{K\,cm^{-1}}$ * Now $\left|\frac{dT}{dr}\right|_{Sun} \sim \frac{T_c}{R}$ $\sim \frac{10^7\,\mathrm{K}}{10^{11}\,\mathrm{cm}} \sim 10^{-4}\,\mathrm{K\,cm^{-1}}$. Since (typically)
$\left|\frac{dT}{dr}\right| \gg \Delta\nabla T$ we can assume $\Delta\nabla T = 0$ in context of [4].
i.e $\frac{dT}{dr} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$ (S19)

* We have assumed "mixing length" (unknown) ~ 0·1R☉. → Guess, fit data.....

Note $\Delta\nabla T > 0 \Rightarrow$ convective instability.
(use definitions of [illegible]). * convection very efficient inside star - not so much at surface

$[5]\ \frac{dL}{dr} = 4\pi r^2 \rho \varepsilon$

Define L to be the heat flow (Power) through shell of star at radius r. If ε is the rate of energy generation per unit mass

$\Rightarrow L = \int_0^r \rho \varepsilon 4\pi r^2 dr \Rightarrow \frac{dL}{dr} = 4\pi r^2 \rho \varepsilon$ [5]

ontributions to ε are from * local nuclear reactions (don't include neutrinos which escape freely from star) * release of gravitational energy due to contraction.

If gravitational contraction is important nuclear reactions will have exhausted themselves so can neglect compositional changes due to them. If u is total energy / unit mass in star 1st law $\Rightarrow du = đQ - p\,dV/m \Rightarrow du = đQ - p\,d(\frac{1}{\rho})$. Gravitational energy generated per unit mass per unit time is $\varepsilon_{grav} = -\frac{đQ}{dt}$. $\therefore -\varepsilon_{grav} = \frac{du}{dt} + p\,d(\frac{1}{\rho})/dt$

Now $u = u(p,\rho) \Rightarrow du = \frac{\partial u}{\partial p}\Big|_\rho dp + \frac{\partial u}{\partial \rho}\Big|_p d\rho$. $d(\frac{1}{\rho}) = -\frac{1}{\rho^2} d\rho \Rightarrow -\varepsilon_{grav} = \frac{d\rho}{dt}\left(\frac{\partial u}{\partial \rho}\Big|_p - \frac{p}{\rho^2}\right) + \frac{\partial u}{\partial p}\Big|_\rho \frac{dp}{dt}$

$$\Rightarrow -\varepsilon_{grav} = p\frac{\partial u}{\partial p}\Big|_\rho \left\{ \frac{d\ln p}{dt} + \frac{d\ln\rho}{dt} \frac{\left(\rho\left(\frac{\partial u}{\partial \rho}\Big|_p - \frac{p}{\rho}\right)\right)}{p\frac{\partial u}{\partial p}\Big|_\rho} \right\}$$

Now for adiabatic change (S14) $\Rightarrow \frac{dp}{p} = \gamma_1 \frac{d\rho}{\rho}$

diabatic change $\Rightarrow đQ = 0$

$\Rightarrow du = -p(\frac{1}{\rho})$. $\therefore \frac{\partial u}{\partial p}\Big|_\rho dp + \frac{\partial u}{\partial \rho}\Big|_p d\rho = \frac{p}{\rho^2} d\rho \Rightarrow \frac{dp}{p} = \frac{d\rho}{\rho}\frac{\left(\frac{p}{\rho} - \rho\frac{\partial u}{\partial \rho}\Big|_p\right)}{p\frac{\partial u}{\partial p}\Big|_\rho}$

$\Rightarrow \gamma_1 = \frac{\frac{p}{\rho} - \rho\frac{\partial u}{\partial \rho}\Big|_p}{p\frac{\partial u}{\partial p}\Big|_\rho}$ (S20) Hence $-\varepsilon_{grav} = p\frac{\partial u}{\partial p}\Big|_\rho \left\{\frac{d\ln p}{dt} - \gamma_1 \frac{d\ln\rho}{dt}\right\}$

Now $p\frac{\partial u}{\partial p}\Big|_\rho = p\frac{\partial u}{\partial T}\Big|_\rho \frac{\partial T}{\partial p}\Big|_\rho = pC_V\frac{\partial T}{\partial p}\Big|_\rho$. Define $\chi_T = \frac{\partial \ln p}{\partial \ln T}\Big|_\rho = \frac{T}{p}\frac{\partial p}{\partial T}\Big|_\rho = \frac{T}{p}\frac{1}{(\frac{\partial T}{\partial p})_\rho}$ $\therefore \frac{\partial T}{\partial p}\Big|_\rho = \frac{T}{p\chi_T} \Rightarrow p\frac{\partial u}{\partial p}\Big|_\rho = \frac{C_V T}{\chi_T}$. Now $\gamma_1 \frac{d\ln p}{dt} = \frac{d}{dt}(\ln\rho^{\gamma_1}) - \frac{d\gamma_1}{dt}\ln\rho$.

But $\frac{d\gamma_1}{dt} = 0$ ($\gamma_1 = \gamma_1(\beta,\gamma)$ - assume no composition change between gas, radiation) - or d.o.f of ideal gas!

$$\Rightarrow \varepsilon_{grav} = -\frac{C_V T}{\chi_T}\frac{d}{dt}\ln\left(\frac{p}{\rho^{\gamma_1}}\right) \quad (S21)$$

For ideal gas $p = \frac{\rho R T}{\mu} \Rightarrow d\ln p = d\ln\rho + d\ln T$
$\Rightarrow \chi_T = 1$.

Since C_V constant for ideal gas $\Rightarrow C_V T = u$
Also $\gamma_1 = \gamma$. ($\frac{5}{3}$ for 3 d.o.f).

. (ideal gas) $\varepsilon_{grav} = -u\frac{d}{dt}\ln\left(\frac{p}{\rho^{\gamma}}\right)$ (S22).

There are four important nuclear reactions occurring inside stars.

I pp chain (low temperatures) $p + p \to D + e^+ + \nu_e$ - requires QM tunnelling
$p + D \to He^3 + \gamma$ ~ 26.2 MeV / He^4 formed
$\varepsilon = \varepsilon_{pp}\rho T^n$ $3 < n < 5$ $He^3 + He^3 \to He^4 + 2p$
ε_{pp} constant

II CN cycle (more massive stars) Net reaction $6H + C^{12} \to He4 + N^{14}$
$\varepsilon = \varepsilon_{CN}\rho T^n$ $n \cong 16$ (Transition between pp and CN at ~10^7K)

III Triple α (high, >10^8k temperatures) $3He^4 \to C^{12}$ (3 body process hence high T)
$\varepsilon = \varepsilon_{He} Y^3 \rho^2 T^n$ $30 < n < 40$ Y is He abundance (fractional)

IV Carbon burning $2C^{12} \longrightarrow Ne^{20} + He^4$ OR $Na^{23} + H$ OR $Mg^{23} + n$

Note in general $\varepsilon = \sum_{chain} \frac{energy}{chain} \cdot \underbrace{\#\text{reactions/unit volume/unit time/chain}}_{\eta} \cdot \frac{1}{\rho} \cdot \psi . f$

η = # collisions/unit volume/unit time . Probability of coulomb barrier penetration . Reaction probability given penetration.

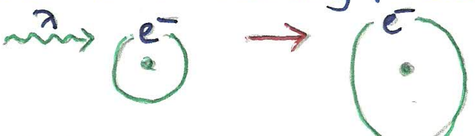
ψ = fraction of energy released NOT as neutrinos f = electron screening factor.

i.e. $\varepsilon_{pp} = 2.36\times10^6 X_H^2 \rho\left(\frac{T}{10^6k}\right)^{-2/3}\exp\left[\frac{-33.8}{(T/10^6k)^{1/3}}\right]\psi f_{pp}$ X_H = fractional H abundance
ε_{pp} in erg $s^{-1} g^{-1}$

4) More about opacity (S12) defines opacity $\kappa_\nu = \frac{1}{\rho\lambda_\nu}$ and in most cases we use a Rosseland mean (S13) to define an average opacity over all frequencies.

Note mean free path $\lambda_\nu = \frac{1}{n\sigma_\nu}$ where σ_ν is the cross section for a particular scattering process (at photon frequency ν). calculate σ from Nuclear / Particle physics. i.e. QED etc). FOUR distinct scattering processes.

I BOUND-BOUND



- Photon absorbed of a particular frequency
- Not that important in stellar interior.

II BOUND-FREE - need photon energy > ionization potential.

III FREE-FREE

(Note nucleus involved to conserve energy and momentum)

Need Boltzmann distribution / Statistical mechanics to work out # densities of ionization states etc.

IV Elastic electron scattering (Thompson cross section)

Photon changes direction.

Useful result is SAHA equation which describes the equilibrium $S^{i+} \rightleftharpoons S^{(i+1)+} + e^-$

$$\frac{N_{i+1}}{N_i} = \frac{(2\pi m_e k_B T)^{3/2}}{N_e h^3} \frac{2g_{i+1}}{g_i} e^{-\chi_i/k_B T} \quad (S22)$$

N_i = # density of electrons in ionization state i
g_i = degeneracy of ionization state i.
χ_i = ionization energy i+ → (i+1)+.

In practice parametrise Rosseland mean opacity in form $\kappa = \kappa_0 P^{\alpha-1}/T^{\beta-4}$ (S23) → [See ES III Q1] Using structure equations [2] and [3] and assuming m(r) varies little with P (and B.C's P=T=0 at surface] near surface to yield $T^\beta = \lambda P^\alpha$ with we can integrate inwards into a star

$\lambda = \frac{\beta}{\alpha} \cdot \frac{3L\kappa_0}{16\pi acGM}$. Applying Schwarzschild criterion for convective stability $\gamma > \Gamma$ and asserting T↑ and P↑ ⇒ for a convective star $\alpha > \frac{2}{5}\beta > 0$ (S24)

Special cases: (I) κ = constant ⇒ $\alpha=1, \beta=4$ (S24) not true ⇒ RADIATIVE star.

* electron scattering opacity * occurs for high T, low ρ * occurs in cores of most stars (and atmospheres of hot stars).

(II) $\kappa = \kappa_0 \rho T^{-3.5}$ ⇒ $\alpha=2, \beta=8.5$ [assume ideal gas ⇒ $\rho \propto P/T$] KRAMERS OPACITY

(S24) Not true ⇒ RADIATIVE STAR (or envelope).

* ~ bound-free and free-free opacity.
* low density, low temperature stars.
* radiative atmospheres.

STARS ⑥

(III) $\kappa = \kappa_0 \rho^{\frac{1}{2}} T^{10}$ [ideal gas $\rho \propto \frac{P}{T}$] ⇒ $\alpha = 3/2, \beta = -\frac{11}{2}$. (S24) not true since $\beta < 0$ ⇒ RADIATIVE. * H^- absorption opacity * occurs in upper atmospheres of low temp stars.

5) Pressures, Energies and Equation of State. Usually consider star to be a mixture of ideal gas and radiation. Hence $P = P_{rad} + P_{gas} = \frac{R\rho T}{\mu} + \frac{1}{3}aT^4$ (S25) Note $P = (\gamma - 1)u$ (S26) where u = energy density. For radiation $\gamma = 4/3$ ($u = aT^4$ from Stephan Boltzmann) and ideal gas $\gamma = 5/3$ ($u = \frac{3}{2} N k_B T/V$ where N = particle #, V = volume ⇒ $u = \frac{3}{2} R\rho T/\mu$).

μ is mean molar mass. If $P = N'RT/V$ where N' = # moles of ideal gas ⇒ $P = \rho RT/\mu$ where μ is clearly mass of 1 mole of gas. If X, Y, Z are mass fractions of H, He and "metals" (the rest) [s.t. X+Y+Z=1] for a fully ionized plasma (Typical in star)

Element	H	He	metal $^A_z Z$
# density of nuclei	$X\rho/m_p$	$Y\rho/4m_p$	$Z\rho/Am_p$
# density of electrons	$X\rho/m_p$	$2Y\rho/4m_p$	$\frac{1}{2}A \cdot \frac{Z\rho}{Am_H}$

(assume ≈ ½ atomic mass # electrons) ∴ Total # density N'' is $N'' = \frac{\rho}{m_p}\left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right]$ (assume A>>1 so $\frac{Z}{A}$ term is negligible.

Now $\frac{N'}{V} = \frac{N''}{N_A}$ and $\frac{N'}{V} = \frac{\rho}{\mu}$

∴ $\frac{1}{\mu} = \frac{1}{N_A m_p}\left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right]$ (S27). Note in units of g $N_A m_p \sim 1$ ⇒ $\mu = \left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right]^{-1}$

limits: H: X=1 ⇒ μ=½ Fe: Z=1 ⇒ μ=2 Hence $\frac{1}{2} \leq \mu \leq 2$. Usually Z≈0 ⇒ $\mu \simeq \frac{4}{3+5X}$

As described in (S15) a polytrope has equation of state $P = K\rho^{\Gamma}$. Can express Γ as $\Gamma = 1+\frac{1}{n}$ where n is 'Polytropic index'. Can show (ESII Q3) that a star of mass M, radius R and polytropic index n has gravitational binding energy $\Omega = -\frac{GM^2}{R}\left(\frac{3}{5-n}\right)$ (S28)

6) Degeneracy Pressure

At high density / low temperature velocity distribution of particles becomes very non Maxwellian. QM effects become important. Consider small volume $d^3\underline{x}$ containing electrons packed so close that spatial parts of their wavefunctions are identical. Pauli Exclusion principle ⇒ no two fermions can have the same wavefunction ⇒ 2 electrons in $d^3\underline{x}$ of opposite spin.

Now uncertainty principle ⇒ $d^3\underline{x}\, d^3\underline{p} \sim h^3$ where $\underline{p}$ is electron momentum. If electron spatial w.f ⇒ momentum state in range $p \to p+dp$, # of electrons in momentum state $p \to p+dp$ unit volume, $N(p)dp = \frac{2}{d^3\underline{x}}$. Hence $N(p)dp = \frac{2d^3\underline{p}}{h^3} = \frac{8\pi p^2 dp}{h^3}$ ✓ if $d^3\underline{p} = 4\pi p^2 dp$ (spherical shell of momentum space).

Assume electron system has momenta in range $0 < p < p_F$. ⇒ Total electron # density $N_e = \int_0^{p_F} N(p)dp = \frac{8}{3}\pi p_F^3 / h^3$ (S29)

Now electron pressure $P = \int_0^{p_F} \underbrace{p\, v}_{\text{momentum flux}} \overset{\text{average over } x,y,z}{\frac{1}{3}} N(p)\, dp$. In non relativistic limit $p = m_e v$

(v = electron speed) [average |velocity|].

$$\Rightarrow P = \int_0^{p_F} \frac{p^2}{m_e}\frac{1}{3}\frac{8\pi p^2}{h^3}dp = \frac{1}{15}\frac{8\pi}{m_e h^3}p_F^5 \checkmark \quad \text{(S30)}$$

From (S29) $p_F = \left(\frac{3N_e h^3}{8\pi}\right)^{\frac{1}{3}}$ ⇒ $P = \frac{1}{15}\frac{8\pi}{m_e h^3}\left(\frac{3N_e h^3}{8\pi}\right)^{\frac{5}{3}} = \left(\frac{8\pi}{3}\right)\frac{1}{5m_e}\left(\frac{3}{8\pi}\right)^{5/3} h^2 N_e^{5/3}$

$$\Rightarrow P_{degen} = \left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{5m_e}N_e^{5/3} \quad \text{(S31)}$$

In relativistic limit
$v \to c$
$\Rightarrow P = \int_0^{p_F} p\frac{c}{3}N(p)dp = \frac{8\pi c p_F^4}{12h^3}$
$\Rightarrow P_{deg} = \left(\frac{3}{8\pi}\right)^{\frac{1}{3}}\frac{hc}{4}N_e^{4/3}$ (S31+)

Note ions have momentum $m_i v_i = \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} m_e v_e$ [equipartition $\frac{1}{2}m_i v_i^2 = \frac{1}{2}m_e v_e^2$] ⇒ occupy ≫ momentum space. So lots of possible w.f states for ions within above $d^3\underline{x}$. Need SMALLER volume for ions → yields same result with $m_e \to m_i$.

7) Static Solutions of Stellar Structure

*mass-radius relation for polytropes.

Require [1] $\frac{dm}{dr} = 4\pi r^2\rho$, [2] $\frac{dP}{dr} = -\frac{\rho G m(r)}{r^2}$ and $P = K\rho^{1+\frac{1}{n}}$ (S15)

Firstly eliminate m between [1], [2]. [2] ⇒ $m = -\frac{r^2}{\rho G}\frac{dP}{dr}$ ✓

∴ using [1] $\frac{d}{dr}\left(-\frac{r^2}{\rho G}\frac{dP}{dr}\right) = 4\pi r^2 \rho$ (*) Now (S15) ⇒ $P = P(\rho)$ so can solve. Define $\rho(r) = \lambda\theta^n$ with λ, n constants. (θ dimensionless) (n = polytropic index). $\theta = \theta(r)$. ∴ using (S15) $P(r) = K\lambda^{1+\frac{1}{n}}\theta^{n+1}$

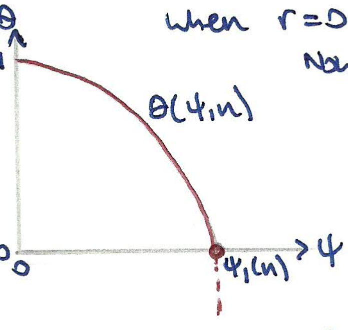
Substitution into (*) yields $\left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\theta}{dr}\right] = -\theta^n$

Since θ is dimensionless can define another dimensionless variable ψ s.t $r = \alpha\psi$. If α is (clearly) $= \left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{\frac{1}{2}}$ ⇒ (*) becomes

$$\frac{1}{\psi^2}\frac{d}{d\psi}\left(\psi^2\frac{d\theta}{d\psi}\right) = -\theta^n \quad \text{(S32)}$$

LANE EMDEN EQUATION OF INDEX n

Solutions look like this: (to the first zero of $\theta(\psi)$, ψ_1).



When $r=0 \Rightarrow \psi = 0$. Hence at $r=0$, $\theta = 1$. ∴ If $\rho = \lambda\theta^n \Rightarrow \lambda = \rho(0) \equiv \rho_c$

Now when $\theta = 0 \Rightarrow P = 0$. This is a surface B.C ⇒ $\theta=0 \Rightarrow r=R$.

$$\therefore R = \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}}\rho_c^{\frac{1-n}{2n}}\psi_1(n) \quad \text{(S33)}$$

Now star mass $M = \int_0^{\alpha\psi_1} 4\pi r^2\rho\, dr = 4\pi\alpha^3\lambda\int_0^{\psi_1}\psi^2\theta^n d\psi$

Now (S32) ⇒ $\psi^2\theta^n = -\frac{d}{d\psi}\left(\psi^2\frac{d\theta}{d\psi}\right)$ ⇒ $M = -4\pi\alpha^3\lambda\int_0^{\psi_1}\frac{d}{d\psi}\left(\psi^2\frac{d\theta}{d\psi}\right)d\psi$

Now at $\psi = 0$, $\frac{d\theta}{d\psi} = \left.\frac{d\left(\left(\frac{\rho}{\lambda}\right)^{\frac{1}{n}}\right)}{d\left(\frac{r}{\alpha}\right)}\right|_{r=0} = \left.\frac{1}{n}\alpha\lambda^{-\frac{1}{n}}\rho^{\frac{1}{n}-1}\frac{d\rho}{dr}\right|_{r=0} = 0$ since one assumes $\rho(r)$ approaches ρ_c by ever decreasing amounts.

$$\therefore M = 4\pi\left[\frac{(n+1)}{4\pi G}K\right]^{\frac{3}{2}}\rho_c^{\frac{3-n}{2n}}\left[-\psi_1^2\left.\frac{d\theta}{d\psi}\right|_{\psi_1}\right] \quad \text{(S34).}$$

Write $R = \Lambda_1\rho_c^{\frac{1-n}{2n}}$, $M = \Lambda_2\rho_c^{\frac{3-n}{2n}}$. $\rho_c^{\frac{1}{2n}} = \left(\frac{R}{\Lambda_1}\right)^{\frac{1}{1-n}}$ ∴ $M = \Lambda_2\left(\frac{R}{\Lambda_1}\right)^{\frac{3-n}{1-n}} \Rightarrow M \propto R^{\frac{3-n}{1-n}}$ (S35)

∴ $M = C(n,K)R^{\frac{n-3}{n-1}}$ (S36) so all (polytropic) stars of given n, K will have this M, R relation.

Define mean density $\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3}$. Can ∴ work out CENTRAL CONDENSATION $\bar{\rho}/\rho_c$.

$$\bar{\rho} = \frac{\Lambda_2 \left(\frac{1}{\Lambda_1}\right)^{\frac{3-n}{1-n}} R^{\frac{2n}{1-n}}}{\frac{4}{3}\pi} \Rightarrow \bar{\rho} = \frac{3\Lambda_2\left(\frac{1}{\Lambda_1}\right)^{\frac{3-n}{1-n}}}{4\pi}\left[\Lambda_1^{\frac{1}{1-n}}\rho_c^{\frac{1}{2n}}\right]^{2n} \Rightarrow \frac{\bar{\rho}}{\rho_c} = \frac{3\Lambda_2}{4\pi}\Lambda_1^{\frac{3n-3}{1-n}} = \frac{3\Lambda_2}{4\pi}\Lambda_1^{-3}$$

$$\Rightarrow \bar{\rho}/\rho_c = 3\psi_1^{-3}\left[-\psi_1^2 \frac{d\theta}{d\psi}\Big|_{\psi_1}\right] \quad (S36)$$

n	0	1	2	3	4	5
ψ_1	2.4	3.1	4.3	6.9	15.0	∞
$-\psi_1^2 \frac{d\theta}{d\psi}\big\vert_{\psi_1}$	4.9	3.1	2.4	2.0	1.8	1.7
$\rho_c/\bar{\rho}$	1	3.3	11.4	54.2	622	∞

* Boundary conditions

I - zero b.c.'s: $T = P = 0$ at $r = R$. $[T = T_c,\ P = P_c$ at $r = 0]$

II - Convective atmosphere: $r \sim R$: $\Delta\nabla T = 0$, $\gamma = \frac{5}{3} \Rightarrow [4]$ becomes $\frac{d\ln P}{d\ln T} = \frac{\gamma}{\gamma - 1} = \frac{5}{2} \Rightarrow P = CT^{2.5}$.

Can define physical surface is $r = R$ where $L = 4\pi R^2 \sigma T_e^4$ i.e where $T = T_e$.

* Optical depth # m.free paths from a radial coordinate r to ∞. $\tau = \int_r^\infty \kappa\rho\, dr$ or $\frac{d\tau}{dr} = \kappa\rho$. (S37)

See QSII Q7 for use. (usually set $\tau = \frac{2}{3}$ to yield R).

* Main sequence stars - Homology of radiative stars. For radiative stars structure equations [1], [2], [3], [5] apply. Also note

High temp: $\varepsilon = \varepsilon_{CN}\rho T^{16}$, $\kappa = \kappa_0$ ← electron scattering

low temp: $\varepsilon = \varepsilon_{pp}\rho T^4$, $\kappa = \kappa_0 \rho T^{-3.5}$ ← Kramers opacity

Also assume primarily ideal gas.

$\Rightarrow P = \rho RT/\mu$. Can write [1], [2], [3], [5] + eq. of state in terms of m:

i.e (i) $\frac{dr}{dm} = \frac{1}{4\pi r^2\rho}$ (ii) $\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$ (iii) $\frac{dL}{dm} = \varepsilon$ (iv) $\frac{dT}{dm} = -\frac{3\kappa L}{64\pi^2 ac r^4 T^3}$ (v) $P = R\rho T/\mu$

Now define $x = \frac{m}{M}$ and let $r(m) = M^{a_1} r_0(x)$, $P(m) = M^{a_2}P_0(x)$, $L(m) = M^{a_3}L_0(x)$, $T(m) = M^{a_4}T_0(x)$ and $\rho(m) = M^{a_5}\rho_0(x)$. Substitution of these into (i) → (v) yields differential equations for x and multiplicative factors of M. Choose index of M's s.t. these equations are independent of M.

i.e (ii) → $\frac{M^{a_2}}{M}\frac{dP_0(x)}{dx} = \frac{-GxM}{4\pi r_0^4(x) M^{4a_1}}$. ∴ independent of M if $a_2 - 1 = 1 - 4a_1 \Rightarrow 4a_1 + a_2 = 2$

If $\varepsilon = \varepsilon_0 \rho T^\eta$ and $\kappa = \kappa_0 \rho^\alpha T^{-\beta}$

yield:
$4a_1 + a_2 = 2$
$3a_1 + a_5 = 1$
$a_3 = 1 + a_5 + \eta a_4$
$4a_1 + (\beta + 4)a_4 = \alpha a_5 + a_3 + 1$
$a_2 = a_5 + a_4$

Define $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} \Rightarrow \underline{\underline{M}}\,\underline{a} = \underline{b}$ where $\underline{\underline{M}} = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\eta & -1 \\ 4 & 0 & -1 & \beta+4 & -\alpha \\ 0 & 1 & 0 & -1 & -1 \end{pmatrix}$

and $\underline{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

Now from (S1) $L = 4\pi R^2 \sigma T_e^4$. At stellar surface $x = 1$ ← only regard M dependence

So $L/R^2 \propto T_e^4 \Rightarrow T_e \propto M^{\frac{a_3 - 2a_1}{4}}$

For Theoretical HR diagram $\log L$ vs $\log T_e$

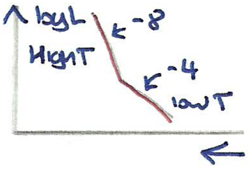
$\underline{a} = \underline{\underline{M}}^{-1}\underline{b}$

Solutions for $\underline{a}$ are:

LOW T	1/13	~4	~5.5	12/13	3/4
High T	15/19	~8	3	4/19	~26/19
	a_1	a_2	a_3	a_4	a_5

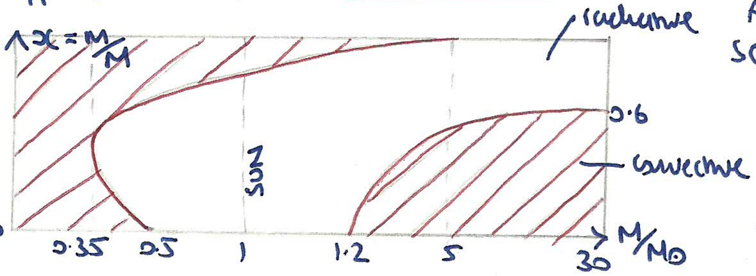
$$\frac{d\log L}{d\log T_e} = \frac{4a_3\, d\log M}{(a_3 - 2a_1)\, d\log M} = \frac{4a_3}{a_3 - 2a_1}$$

So $\frac{d\log L}{d\log T_e} = \begin{cases} \text{High } T & 8.44 \\ \text{low } T & 4.12 \end{cases}$



- Explains general trend of Main sequence stars.

Deviations from above homological approach occur when radiation pressure is significant and/or core or surface convection occurs.



Application of stellar models outlined above plus Schwarzschild criterion yield figure on the left.

- get surface convection due to presence of ionization zones (which change μ and ∴ create layers of varying density). IDEAL GAS ⇒ $\frac{\rho}{\mu} = $ [illegible]
- core convection due to v. sensitive T dependence for high mass stars of ε. i.e $\varepsilon_{CN} \propto T^{16}$.

* Eddington's Quartic. $\beta = \frac{P_{gas}}{P} \Rightarrow P = P_{rad} + \beta P \Rightarrow P_{rad} = P(1-\beta). \Rightarrow P_{rad} = P_{gas}(1-\beta)/\beta$.

$\Rightarrow \frac{\rho RT}{\mu} = \frac{\beta}{1-\beta}\frac{1}{3}aT^4 \Rightarrow T = \left[\frac{3\rho R}{\mu}\frac{1-\beta}{\beta}\frac{1}{a}\right]^{\frac{1}{3}}$ ∴ $P = \frac{P_{gas}}{\beta} = \frac{1}{\beta}\frac{R}{\mu}\rho\left[\frac{3\rho R}{\mu}\frac{1-\beta}{\beta}\frac{1}{a}\right]^{\frac{1}{3}}$

Hence $\beta^4\left[\frac{P^3}{\rho^4}\right]=\left(\frac{R}{M}\right)^4\frac{3}{a}(1-\beta)$ (S38) EDDINGTON'S QUARTIC. Hence if $\frac{P^3}{\rho^4}$ is estimated $\Rightarrow \beta$.

* Condensed objects are supported by degeneracy pressure. White Dwarfs are one type. In non relativistic limit $P_{deg} \propto N_e^{5/3}$. Now $N_e \propto \rho \Rightarrow P_{deg} \propto \rho^{5/3} \Rightarrow$ Polytrope of index $1+\frac{1}{n}=\frac{5}{3} \Rightarrow n=3/2$. As objects collapse further electrons go relativistic $\Rightarrow P_{deg} \propto N_e^{4/3} \Rightarrow n=3$

Now if one assumes degeneracy pressure = energy density (i.e. ignore radiation / gas pressure)

Total Energy of star is $VP_{deg}+\Omega=E$. $V=\frac{4}{3}\pi R^3$. For non-relativistic limit

Polytrope order 3/2

$$E=\left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{5m_e}\left(\frac{M}{m_p}\frac{1}{\frac{4}{3}\pi R^3}\right)^{5/3}\frac{4}{3}\pi R^3-\frac{6}{7}\frac{GM^2}{R} \Rightarrow E=\left(\frac{3}{8\pi}\right)^{2/3}\left(\frac{4}{3}\pi\right)^{-2/3}\frac{h^2}{5m_e}\left(\frac{M}{m_p}\right)^{5/3}\frac{1}{R^2}-\frac{6}{7}\frac{GM^2}{R}$$

Equilibrium is when $\frac{dE}{dR}=0 \Rightarrow M=\left(\frac{14h^2}{30m_eG}\right)^3\left(\frac{9}{32\pi^2}\right)^2\frac{1}{m_p^5}R^{-3}$ (S39) Note $M\uparrow \Rightarrow R\downarrow$

In relativistic limit (attained by shrinking R) $E=2^{1/3}\left(\frac{3}{8\pi}\right)^{5/6}\frac{hc}{4}m_p^{-4/3}\frac{M^{4/3}}{R}-\frac{3}{2}\frac{GM^2}{R}$

Note no equilibrium R so star unstable collapse.

Set $E=0 \Rightarrow M=\sqrt{2}\left(\frac{3}{8\pi}\right)^{5/4}\left(\frac{2}{12}\right)^{3/2}\left(\frac{hc}{G}\right)^{3/2}\frac{1}{m_p^2}=0.2M_\odot$ (S40). This is the CHANDRASEKHAR MASS. The value $0.2M_\odot$ is actually an underestimate because we have neglected the true composition (of mostly C, O) and approximated just ionized H. ($\Rightarrow N_e=\frac{M}{m_p}\frac{1}{\frac{4}{3}\pi R^3}$). → Though this should reduce M....... ? (electron : nucleon ratio decreases).

Correct $M_{ch}\sim 1.4M_\odot$. $M>M_{ch} \Rightarrow$ unstability.

→ Finding K from degeneracy pressure $P=K\rho^{1+\frac{1}{n}}$ relation and using $M=M_{ch}$ $\Rightarrow$(Lane Emden)* $R_{WD}\sim R_\oplus$. (* know $\bar{\rho}/\rho_c$ .....

Neutron Stars are another type of condensed object. These form when density is such that neutrons almost touch (well nuclei). From (S39) $M_{WD}\propto\frac{1}{m_e^3R^3}$. If Neutron stars have $M_N\sim M_{WD}$ $\Rightarrow m_nR_n\sim m_eR_{WD}$. If $R_{WD}\sim R_\oplus \Rightarrow R_n\sim\frac{m_e}{m_n}R_\oplus\sim 10^4$m.

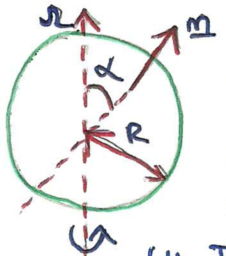
{# neutrons $\sim 1.4M_\odot/m_n$

∴ Space / neutron $\sim\frac{4}{3}\pi R_n^3\frac{m_n}{1.4M_\odot}$

$\Rightarrow$ Separation $\sim\left[\frac{4}{3}\pi\left(\frac{m_e}{m_n}\right)^3R_\oplus^3\frac{m_n}{1.4M_\odot}\right]^{1/3}$ $\sim 10^{-15}$m.

→ So nuclei touching!}

Some Neutron Stars are called PULSARS because they are often rotating and magnetised - Model as a rotating magnetic dipole.



Emitted power $P=\frac{2}{3c^3}\left(\frac{\mu_0}{4\pi}\right)|\ddot{m}|^2$ ← See Relativity and EM notes.

where $m=\frac{B_pR^3}{2}\left(\frac{4\pi}{\mu_0}\right)$ B_p is magnetic field strength at polar surface.

Now $|\ddot{m}|=m\Omega^2\sin\alpha$ and $\dot{E}_{rot}=-P=-\frac{\pi}{\mu_0}.2B_p^2R^6\Omega^4\sin^2\alpha/3c^3$ (S41)

Now if rotational energy $E_{rot}=\frac{1}{2}I\Omega^2$ ($I=$ moment of inertia) $\Rightarrow \dot{E}_{rot}=I\Omega\dot{\Omega}$ (if $I=$ const). Define spin down time $\tau=-\frac{\Omega}{\dot{\Omega}}\Big|_0=-\frac{I\Omega^2}{\dot{E}_{rot}}=\frac{3Ic^3\mu_0}{2\pi B_p^2R^6\sin^2\alpha\,\Omega_0^2}$ (S42)

Define BRAKING INDEX n as $n=\frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2}$.

For (theoretical) pulsar: $\Omega\dot{\Omega}\propto\Omega^4 \Rightarrow \Omega\ddot{\Omega}+\dot{\Omega}^2\propto 4\Omega^3\dot{\Omega} \Rightarrow \frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2}=A4\frac{\Omega^3}{\dot{\Omega}}-1$ (A constant of proportionality)

$\Rightarrow n=4A/A-1=3$ (using $\frac{\dot{\Omega}}{\Omega^3}=A$). observationally $n\sim 2.5$ (crab pulsar).

Note electric fields at surface of pulsar $\Rightarrow E\sim|v\times B|\sim\Omega RB$. For typical B, $ER\sim 10^{11}$/Period ↙ ? which is enough to strip electrons (well charges) off the surface $\Rightarrow$ currents which make things horribly non linear.

* Eddington limit. Peak luminosity of a star is when radiation pressure balances the gravitational pressure on surface layers. If radiation pressure force is $F_{rad}=\frac{L}{4\pi R^2h\nu}\cdot\frac{h\nu}{c}\cdot\sigma_T$ ↖ Thomson cross section

and gravitational force $F_g=\frac{GMm_p}{R^2}$ (on proton in surface)

↑ # particles (well photons!) passing through unit surface / s.

↑ photon momentum $E=pc$

$\sigma_T=\frac{\mu_0^2e^4}{6\pi m_e^2}$

$F_g=F_{rad} \Rightarrow L=\frac{24\pi^2Gm_pm_e^2c}{\mu_0^2e^4}M$ (S43) This is Eddington limit.

Can use this to infer maximum mass of stars given highest L observed. $\Rightarrow M_{max}\sim 100M_\odot$. (Not including Black Holes.....)

[Note Red Giants are missing from discussion - these will appear in Dynamic Stellar model section].

8) Stellar Accretion and Shocks. Accretion is the attraction of interstellar matter to a star. Orbits form if matter has angular momentum. (And forms accretion disks). If accreted matter orbits with velocity v define ACCRETION RADIUS to be that at which v is the escape velocity.

↑ or gets sucked into star

i.e. $\frac{1}{2}v^2 = \frac{GM}{R_A} \Rightarrow R_A = \frac{2GM}{v^2}$ (S44). Mass flux into star system is ρv where ρ is density of accreted matter. $\therefore \rho v = \frac{\dot{M}}{4\pi R_A^2}$ (This assumes no Angular momentum but we will ignore this.)

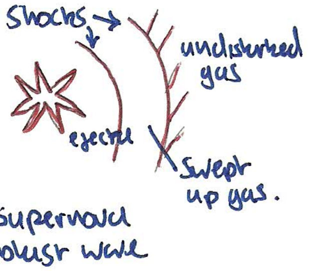
Now matter accreted onto magnetized Neutron stars or white Dwarfs will have a RAM pressure ρv^2. This is balanced in the MAGNETOSPHERE by the pressure exerted by the stars' magnetic field. $P_B = \frac{B^2}{2\mu_0}$ (energy density) From above $B^2 = (2^{-1}\mu_0 m \pi^{-1}/R_A^3)^2 = \frac{\mu_0^2 m^2}{4\pi^2 R_A^6}$

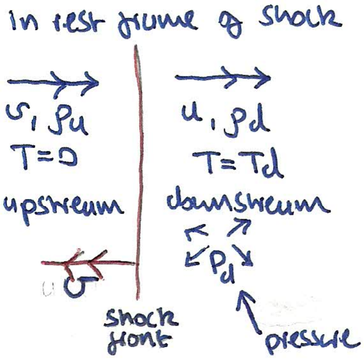
$\Rightarrow P_B = \frac{\mu_0 m^2}{8\pi^2 R_A^6}$. Now using (S44) $\Rightarrow \rho v^2 = \frac{\dot{M}}{4\pi R_A^2}\sqrt{\frac{2GM}{R_A}}$ $\therefore$ if $\rho v^2 = P_B \Rightarrow R_A \propto m^{4/7}\dot{M}^{-2/7}M^{-1/7}$

Now if accretion results in luminosity $L = \frac{G\dot{M}M}{R_*}$ (R_* = star radius) and for white Dwarf M, R_* vary little about $M_\odot, R_\odot$ $\Rightarrow R_A \propto m^{4/7}L^{-2/7}$. (S45)

(R_A should really be R_M for magnetosphere in this case).

When accreted matter collides with compact object it creates a SHOCK. Shocks are also produced in supernovae. Shocks are waves with speed > sound speed in a medium.





Assume matter streams into shock at temperature $T \ll T_d$ and flows out at temperature T_d. Hence $T \simeq 0$ relative to T_d. Per unit area of shock front we conserve mass, momentum and energy in a unit time interval.

mass: $\rho_u v = \rho_d u$ momentum: $\rho_u v^2 = P_d + \rho_d u^2$ ← pressure effectively

energy $v(\frac{1}{2}\rho_v v^2) - u(\frac{1}{2}\rho_d u^2) - u(\frac{3}{2}P_d) = uP_d$

[uP_d is work done / unit area / unit time at shock]

Think of last step as the work done $P_d u$ by shock in passing = change in internal energy across shock front. Substituting for v and P_d we get $\left(\frac{\rho_d}{\rho_u}\right)^2 u^3 - 5\frac{\rho_d}{\rho_u}u^3 + 4u^3 = 0$

$\Rightarrow \left(\frac{\rho_d}{\rho_u}-4\right)\left(\frac{\rho_d}{\rho_u}-1\right) = 0 \Rightarrow \frac{\rho_d}{\rho_u} = 4$ or 1. From mass conservation $\frac{\rho_d}{\rho_u} = \frac{v}{u} \Rightarrow \underset{\text{null}}{v = u}$ or $\underset{\text{shock}}{v = 4u}$

Now if shocked material is an ideal gas $P_d = \rho_d R T_d/\mu$. momentum conservation + $v = 4u$ + $\rho_d = 4\rho_u$ $\Rightarrow P_d = \frac{3}{16}\rho_d v^2$

Hence $T_d = \frac{3}{16}\frac{\mu}{R}v^2$ (S46) (R = molar gas constant).

Note as an aside $P_d = \frac{3}{4}\rho_u v^2 \Rightarrow P_d \propto \rho_u \dot{r}^2$ if spherical shock front of radius r.

Now $\frac{E}{\frac{4}{3}\pi r^3} \cong P_d \Rightarrow \frac{E}{\rho_u} \propto r^3\dot{r}^2 \Rightarrow \dot{r}r^{3/2} \propto \left(\frac{E}{\rho_u}\right)^{1/2} \Rightarrow r^{5/2} \propto t\left(\frac{E}{\rho_u}\right)^{1/2}$

energy

$\Rightarrow r \propto t^{2/5}\left(\frac{E}{\rho_u}\right)^{1/5}$. If $r(t)$ is known and ρ_u guessed $\Rightarrow E$. c.f. G.I. Taylors calculation of energy of first atomic bomb.

4) Dynamic Stellar Models – Stellar Evolution

- modify structure equations to include time dependence. i.e. [2] + [1] $\rightarrow \frac{1}{4\pi r^2}\frac{\partial^2 r}{dt^2} = -\frac{\partial P}{\partial m} - \frac{Gm}{4\pi r^4}$ (Newton's 2nd law mass·acc = Σ forces).
- Need to include ε_{grav} (S21) in [5] - Need to model model nuclear reaction rates to see how μ changes with varying composition
- use timescales (K.Helmholtz, free fall, acoustic, leakout ← see ESI Q7 for details of f.fall / acoustic) to help assess relative rates of processes.

⇒ useful theorem for analysing pulsations / small dynamic perturbations as well as proving static relations is the VIRIAL THEOREM.

- consider element of mass in star dm at position $\underline{r}(t)$. Equation of motion is $\ddot{\underline{r}} = \underline{F} - \text{div}\underline{\underline{P}}/\rho$

$\underline{F}$ is force per unit mass (externally) and $\underline{\underline{P}}$ is the stress Tensor. $(\text{div}\underline{\underline{P}})_j = \frac{\partial}{\partial x_i}P_{ij}$. [For gas $P_{ij} = P\delta_{ij}$]

Using $\int dm = \int_V \rho dV \Rightarrow \underset{(I)}{\int_M \underline{r}\cdot\ddot{\underline{r}}\,dm} = \underset{(II)}{\int_M \underline{r}\cdot\underline{F}\,dm} - \underset{(III)}{\int_V \underline{r}\cdot\text{div}\underline{\underline{P}}\,dV}$ For (I) $\underline{r}\cdot\ddot{\underline{r}} = \frac{d}{dt}(\underline{r}\cdot\dot{\underline{r}}) - \dot{\underline{r}}^2 = \frac{1}{2}\frac{d^2}{dt^2}(\underline{r}\cdot\underline{r}) - \dot{\underline{r}}^2$

$\Rightarrow \int_M \underline{r}\cdot\ddot{\underline{r}}\,dm = \frac{1}{2}\ddot{I} - 2T$

where I = moment of inertia $\int_M \underline{r}^2 dm$, T = kinetic energy $\int_M \frac{1}{2}\dot{\underline{r}}^2 dm$.

For (II) Just consider gravity (In general $\int_M \underline{F}\cdot\underline{r}\,dm$ is called the VIRIAL OF CLAUSIUS). Let $\underline{F} = -\nabla\phi$ where $\phi(r) = -G\int_M \frac{dm'}{|\underline{r}-\underline{r}'|}$. (Solution to Poisson's equation for gravity $\nabla^2\phi = +4\pi G\rho$).

ravitational energy $\Omega = \int_V \frac{1}{2}\rho\phi\,dV = \frac{1}{2}\int_m \phi\,dm = -\frac{1}{2} G\iint_M \frac{dm\,dm'}{|\underline{r}-\underline{r}'|}$ // {Note $\frac{1}{2}$ to avoid 'double counting'}

∴ $\int_M \underline{r}\cdot\underline{F}\,dm = -\int_M \underline{r}\cdot\nabla\phi\,dm = G\iint_M (\underline{r}\cdot\nabla)\frac{dm'}{|\underline{r}-\underline{r}'|}\,dm$ (Note subtle interchange of $(\underline{r}\cdot\nabla)$ and $\int$ ..... ?

Swap dm ⇔ dm' (dummy variables) ⇒ $\int_M \underline{r}\cdot\underline{F}\,dm = \frac{1}{2} G\iint_M \left\{(\underline{r}\cdot\nabla)\frac{1}{|\underline{r}-\underline{r}'|} + \underline{r}'\cdot\nabla'\frac{1}{|\underline{r}'-\underline{r}|}\right\}dm\,dm'$

$= -\frac{1}{2} G\iint_M\left\{\frac{\underline{r}\cdot(\underline{r}-\underline{r}')}{|\underline{r}-\underline{r}'|^3} + \frac{\underline{r}'\cdot(\underline{r}'-\underline{r})}{|\underline{r}-\underline{r}'|^3}\right\}dm\,dm' = -\frac{1}{2} G\iint_M\frac{dm\,dm'}{|\underline{r}-\underline{r}'|} = \Omega$ //

For (III) Noting $\underline{r} = (x_1,x_2,x_3)$ $(\underline{r}\cdot\text{div}\,\underline{\underline{P}}) = x_j\frac{\partial}{\partial x_i}P_{ij} = \frac{\partial}{\partial x_j}[x_j P_{ij}] - P_{ij}\delta_{ij} = \text{div}(\underline{r}\cdot\underline{\underline{P}}) - \text{tr}\,\underline{\underline{P}}$

Define $P_{ii} = 3\bar{P}$. ∴ $-\int_V \underline{r}\cdot\text{div}\,\underline{\underline{P}}\,dV = 3\int_V\bar{P}\,dV - \int_V \text{div}(\underline{r}\cdot\underline{\underline{P}})\,dV$ = (Divergence theorem) $3\int_V \bar{P}\,dV - \int_S(\underline{r}\cdot\underline{\underline{P}})\cdot d\underline{S}$

$\underline{r}\cdot\underline{\underline{P}} = \underline{\underline{P}}_S = P_S\underline{\underline{I}}$ (P_S constant on surface). ∴ $\int_S(\underline{r}_S\cdot\underline{\underline{P}}_S)\,dS = P_S\int_S\underline{r}\cdot d\underline{S}$ $(\underline{r}\cdot\underline{\underline{I}} = \underline{r})$ $= P_S\int \text{div}\,\underline{r}\,dV$

$= 3P_SV$. Hence all together Virial theorem states $\frac{1}{2}\ddot{I} = 2T + 3\int_V\bar{P}\,dV - 3P_SV + \Omega$ (S47)

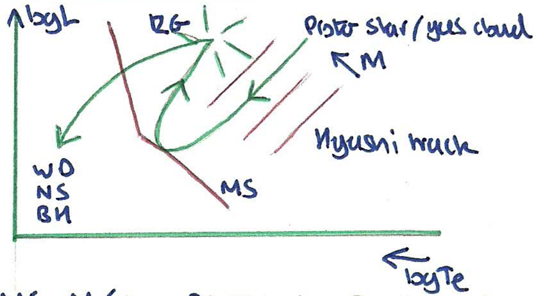
ase of stellar structure zero BC ⇒ $P_S = 0$. Also if ideal gas, radiation $P = (\gamma-1)u$ ← energy density

⇒ $\int_V \bar{P}\,dV = \langle\gamma-1\rangle U$ ↑ total energy internally. If γ = constant ⇒ $\frac{1}{2}\ddot{I} = 2T + 3(\gamma-1)U + \Omega$ (S48)

Note for static star in eq. $\ddot{I} = T = 0$. ∴ $\Omega = -3(\gamma-1)U$ Total energy $E = U+\Omega = \Omega - \frac{\Omega}{3(\gamma-1)}$

⇒ $E = \frac{3\gamma-4}{3(\gamma-1)}\Omega$. (S49) If $\Omega < 0$ ⇒ Star bound if $3\gamma > 4$ ⇒ $\gamma > \frac{4}{3}$. For radiation $\gamma = \frac{4}{3}$. So as $\beta \to 0$, $\gamma \to \frac{4}{3}$ ⇒ Star becomes less bound.

* Evolution of a 'typical' star



Stars form from gas cloud coalescence. These descend down the 'Hayashi track' of the theoretical H.R diagram until main sequence is reached. After a time of relative stability on the MS (M↑ ⇒ time spent on MS ↓) Star expands to form a red giant. (ERYTHROGIGANTISM) (After brief pause as He ignites) Depending on the main sequence mass the R.G will shed mass and the resulting core will collapse to form a WD. If mass of MS is high enough iron core will collapse and a mighty fusion reaction occurs releasing vast energies and leaving a Neutron star / Naked He star or perhaps even a black hole.

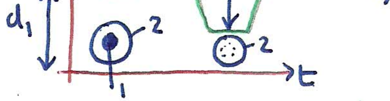
MS $M/M_\odot$	Remnant	Envelope lost
≤ 0.9	He WD < $0.4M_\odot$	AGB
0.9 - 8	C/O WD	AGB
8 - 8.3	O/Ne/Na WD	AGB
8.3 - 40	NS	SN
> 40	NAKED HE	SN
> 50	BH	SN

AGB - Asymptotic Giant Branch
RGB - Red Giant Branch
SN - Supernova

[Note most of the results of this section are discussed in ESIV]

10) Binary systems. ~50% of observable stars are in binary systems. (20% single). Binaries classified by what information one can observe.

VISUAL - orbits observed (astrometric ⇒ only one orbit)
SPECTROSCOPIC - Get radial velocities as well (Doppler shifts)
ECLIPSING - Get temperatures as well if observe with two wavebands.



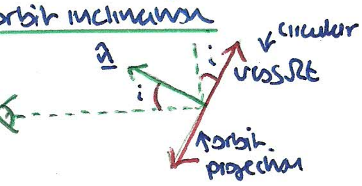
(assume $R_2 > R_1$, $T_1 > T_2$)

If $L_0 = d_1+d_2$: $d_1 = 4\pi R_1^2\sigma T_1^4$ $d_2 = 4\pi R_2^2\sigma T_2^4$

⇒ $\left(\frac{d_1}{d_2}\right) = \left(\frac{T_1}{T_2}\right)^4$. Measure two independent $\frac{d_1}{d_2}$ ratios (for L_U, L_B say) ⇒ T_1, T_2. [See ESIV Q1].

* Kepler's laws. - See "Derivation of Kepler's laws" - Summary of results listed below.

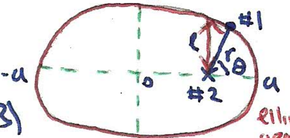
↑ i.e. definition of orbit inclination



I - Closed binary orbits are ellipses (in general conic sections)
II - rate of area swept out by line joining masses in binary = constant
III - If P = period, a = semi major axis of mass #1 orbit about mass #2 then if $M = m_1+m_2$ $\left(\frac{P}{2\pi}\right)^2 = \frac{a^3}{GM}$ (S50)

Orbit equation is $\frac{1}{r} = \frac{GM}{h^2} + \frac{GM}{h^2}\cos\theta\sqrt{1-\frac{h^2}{GMa}}$ (S51)

with $J = \frac{M_1M_2}{M}h$ (Total angular momentum) (S52) $h = r^2\dot{\theta}$ (S53)



eccentricity ↙
$e^2 = 1 - \frac{b^2}{a^2}$
$\ell = a(1-e^2)$
↑ ellipse

semi-latus rectum and eccentricity of ellipse of general form $r = \frac{\ell}{1+e\cos\theta}$ (554)
are $\ell = \frac{h^2}{GM}$ (555) and $e = \sqrt{1-\frac{h^2}{GMa}}$ (556). Note orbit is defined by 3 parameters a, M, h.

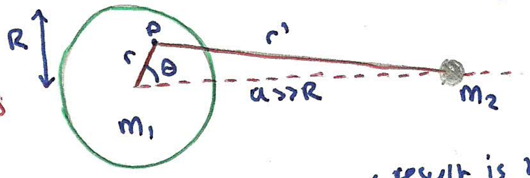
Energy of orbit $E = -\frac{Gm_1m_2}{2a}$ and is minimised for circular orbits. (557)

can write $J = \frac{m_1m_2}{M}a^2(1-e^2)^{\frac{1}{2}}\Omega$ $\Rightarrow$ for circular orbits $J = \frac{m_1m_2}{M}a^2\Omega$. (558) $\Omega = \frac{2\pi}{P}$.

* **Tidal forces** - spherical mass distribution of gaseous stars can be deformed by gravitational forces of binary pair.

Potential due to * 2 at P is $\phi_2 = \frac{GM_2}{r'}$

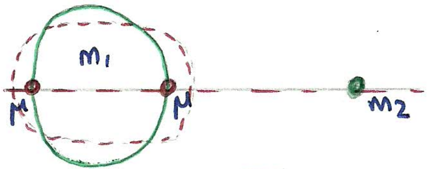
$= \frac{Gm_2}{(a^2+r^2+2ar\cos\theta)^{\frac{1}{2}}} = \frac{GM_2}{a}\sum_{n=0}^{\infty}P_n(\cos\theta)\left(\frac{r}{a}\right)^n$ (559) legendre polynomials



$n=0$. Just balances orbit. $n=1$. No torque - apsidal motion. $n=2$ $\phi_2 = \frac{Gm_3}{a}\left(\frac{r}{a}\right)^2\frac{1}{4}(1+3\cos2\theta)$ ↙ result is tidal bulge.

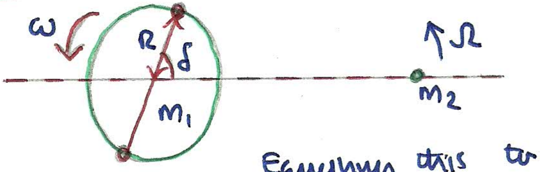
Account for bulge by adding masses to sides of *1.

can show: $\frac{\mu}{M_1} = \frac{1}{2}km_2\left(\frac{R}{a}\right)^3 << m_2$ $k = \frac{32\pi}{5MR^2}\int_0^R \rho_1 r^2 dr$ (560)



For most stars $\mu/M_1 \lesssim 0.01$.

Now suppose *1 spins at different rate to orbital angular velocity Ω

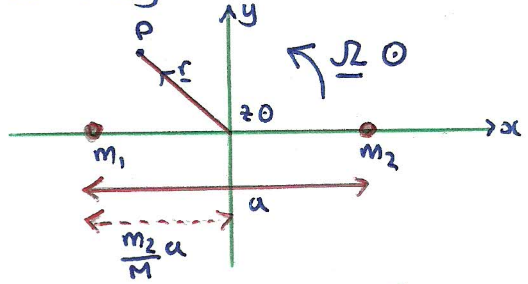


δ = tidal lag angle

can show tidal torque $T_{tide} \approx -\frac{3R^5}{a^6}Km_2^2\delta G$ (?)

Equating this to rate of change of angular momentum $\frac{d}{dt}(I\omega)$ and using $J \sim \left(\frac{M_1M_2}{M}a^2\Omega + I\omega\right)$ (561) ($\approx$ circular orbits) $\Rightarrow \frac{\dot{a}}{a} < 0$ if $\Omega > \omega$.

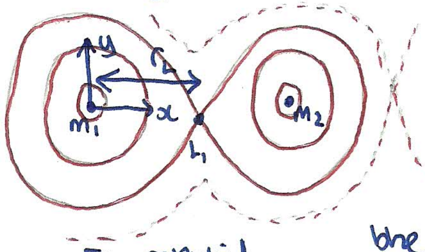
* **Mass transfer** - for v. close binaries ($a \sim R$) This is important. consider coordinate system centered on the binary c.o.m co-rotating with the binary. (assume both orbit with same period).



In inertial frame P has velocity and acceleration
$\underline{v} = \dot{\underline{r}} + \underline{\Omega}\times\underline{r}$; $\underline{a} = \ddot{\underline{r}} + 2\underline{\Omega}\times\dot{\underline{r}} + \underline{\Omega}\times(\underline{\Omega}\times\underline{r})$ (562)

From Fluid dynamics (no viscosity) Euler's equation
$\Rightarrow \underline{a} = -\frac{1}{\rho}\nabla P - \nabla\phi_g$ (P = pressure) $\nabla^2\phi_g = 4\pi G\rho$ (Poisson)

Hence $\ddot{\underline{r}} + 2\underline{\Omega}\times\dot{\underline{r}} = -\frac{1}{\rho}\nabla P - \nabla\phi_g - \underline{\Omega}\times(\underline{\Omega}\times\underline{r})$ (563) Now for co-rotating coordinates (assume circular orbits) $\dot{\underline{r}} = \ddot{\underline{r}} = 0$. i.e equilibrium is set up by the equality of pressure force $-\frac{1}{\rho}\nabla P$ with ∇(centrifugal + gravitational potential). i.e $\nabla\phi = \nabla\phi_g + \underline{\Omega}\times\underline{\Omega}\times\underline{r}$. Now $\underline{\Omega}\times\underline{\Omega}\times\underline{r}$ $= (\underline{\Omega}\cdot\underline{r})\underline{\Omega} - \Omega^2\underline{r} = -\Omega^2\underline{r}$ since $\underline{r}$ is confined to x,y plane. let $\phi_c = -\frac{\Omega^2}{2}(x^2+y^2)$. $\nabla\phi_c = -\Omega^2\underline{r}$
$\Rightarrow \nabla\phi = \nabla\phi_g + \nabla\phi_c$ $\Rightarrow$ (transforming to coordinates centered on m_1 and assuming point mass potentials)
(if $q = \frac{m_1}{m_2} \Rightarrow \frac{m_2}{M} = \frac{1}{1+q}$) $\Rightarrow$ $\phi = \frac{-Gm_1}{(x^2+y^2+z^2)^{\frac{1}{2}}} - \frac{Gm_2}{((x-a)^2+y^2+z^2)^{\frac{1}{2}}} - \frac{\Omega^2}{2}\left(\left[x - \left(\frac{a}{1+q}\right)\right]^2 + y^2\right)$ (564)



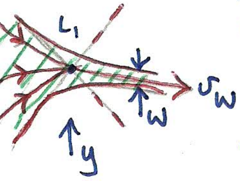
Equipotential surfaces.

Lagrangian points are where $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} = 0$ ($\nabla\phi = 0$). At first of these (L_1) shape of equipotential surface is called Roche lobe. The surface can be $\approx$ circular and $\therefore$ have an associated radius r_L. Note $r_L = r_L(q)$. {R_L defined by (Volume of roche lobe $/\frac{4}{3}\pi)^{\frac{1}{3}}$}. $\frac{r_L}{a} \approx 0.462\left(\frac{q}{1+q}\right)^{\frac{1}{3}}$ for $0 < q < 0.8$.

If roche lobe is filled with stellar matter $\Rightarrow$ mass transfer to star 2. ($-\nabla\phi$ points towards star 2 rather than 1).

Note a more exact formula for $\frac{r_L(q)}{a}$ {Eggleton 1983} is $0.49q^{2/3}/0.6q^{2/3} + \ln(1+q^{1/3})$ (S65)

mass transfer rate. At L_1 $\nabla\phi = 0$ ⇒ material at L_1 can freely expand into Roche lobe of mass 2.



Assume stream of stellar wind passes through L_1 at speed v_W and has cross section w^2. $\dot m = \rho v_W w^2|_{L_1}$ (S66). Estimate w by comparing K.E of wind with potential in y direction. $\Delta\phi(x,y) = \phi(x,y,0) - \phi(x,0,0)$ $\sim \frac{1}{2}\frac{\partial^2\phi}{\partial y^2}\Big|_{L_1} w^2$ since $\nabla\phi = 0$ no first order terms. Now $\frac{\partial^2\phi}{\partial y^2}\Big|_{L_1} = \frac{GM_1}{x_1^3} + \frac{GM_2}{(a-x_1)^3} - \Omega^2$. Now $x_1 \sim \frac{a}{2}$ so since (S50) [Kepler III] ⇒ $\Omega^2 = \frac{GM}{a^3}$ and each term $\sim \Omega^2$ ⇒ $\frac{\partial^2\phi}{\partial y^2}\Big|_{L_1} \sim \Omega^2$. ∴ $\frac{1}{2}v_W^2 = \Delta\phi$ ⇒ $\frac{1}{2}v_W^2 \sim \frac{1}{2}\Omega^2 w^2$ ⇒ $w \sim \frac{v_W}{\Omega}$ (S67). Hence $\dot m \sim \frac{\rho v_W^3}{\Omega^2}$ (S68). An 'accepted' model for mass accretion by roche lobe overflow is BONDI-HOYLE-LYTTLETON ACCRETION

$$\dot m_2 = \frac{-\dot m_1 \alpha}{a^2 v_W^4} \frac{(GM_2)^2}{\left\{1+\left(\frac{v_{orb}}{v_W}\right)^2\right\}^{2/3}} \quad (S69)$$

v_{orb} is orbital velocity [if circular orbits $v_{orb} \simeq a\Omega$] - not sure whether v_{orb} is v about COM or about mass 1....

* Period evolution of binaries

Now if $\dot m$ is lost from system, ignoring spin of stars and assuming circular orbits

$\dot m = \dot m_1 + \dot m_2 < 0$, $J = \frac{m_1 m_2}{M} a^2\Omega$ (S58), $\Omega^2 = \frac{GM}{a^3}$ (S50) Assume $\dot J = \dot m a_1^2 \Omega$ combination of above (see ES IV Q4)

i.e A.M loss is slow compared to periodicity of binary.

⇒ $Pm^2 = \text{constant}$ ∴ if M↓ P increases. (S70)

For conservative mass transfer $\dot m = 0$, $\dot J = 0$. In this case $P(m_1 m_2)^3 = \text{constant}$. (S71)

∴ minimum P when $m_1 = m_2$. Note for STABLE mass transfer $\dot r_L > \dot R$.

*** Common envelope Evolution** (see ES IV Q7). In this case q is such that cores spiral together and envelope will remain or be blown off. i.e * cores coalesce → 1 big single star * Supernova as cores collide - system destroyed * 2 binary cores with envelope blown off. As used in ES IV Q7 a 'common envelope efficiency' is defined by α_{CE}:

$$\alpha_{CE}\{|E_{orbit\,f}| - |E_{orbit\,i}|\} = |E_{bind,i}| - |E_{bind,f}| \quad (S73)$$

with i,f being initial/final states. i.e after envelope lost, two cores orbiting/coalescing.

*** X-ray binaries.** Usually a magnetized neutron star orbiting a massive star. (i.e Red giant). Neutron star accreted matter from giant due to Roche lobe overflow. Now for Keplerian orbit $\Omega^2 = \frac{GM}{a^3}$ ∴ if angular momentum of Neutron star is $\sim a^2\dot m_N\Omega = a^2 m_N\sqrt{\frac{GM}{a^3}} = m_N\sqrt{GMa}$. If angular momentum is transferred to spin of neutron star (via process of accretion and resultant magnetic interactions) and $I_N = \text{constant}$ ⇒ $I_N\dot\omega \sim \dot m_N\sqrt{GMa}$ (assume $a \sim$ constant). ω = angular speed of neutron star. Now if $a \sim$ magnetosphere radius can use (S45) ⇒ $R_M (\simeq a) \propto \underset{\uparrow \text{magnetisation}}{m^{4/7}} L^{-2/7}$. ~~Taking ln and differentiating above~~

~~⇒~~ Now $\omega = \frac{2\pi}{P}$ ⇒ $\dot\omega = -\frac{2\pi}{P^2}\dot P$ ∴ $-\frac{2\pi I_N}{P^2}\dot P \sim \dot M_N G^{\frac{1}{2}} M^{\frac{1}{2}} m^{\frac{2}{7}} L^{-\frac{1}{7}}$

$$\Rightarrow \frac{\dot P}{P} \propto -P m^{2/7} L^{-1/7} / I_N \quad (S74)$$

← correlates well to observation
– used as a method for finding I_N
(Assume magnetisation guessed?)

*** Binary pulsars and gravitational radiation.** Binary pulsars will involve a g. relativistic treatment due to the relativistic orbital speeds and strong gravitational fields. Accelerating masses emit gravitational waves – in this case significantly. QUADRUPOLAR radiation (dipole violates momentum conservation). Model $L_{GR} = \frac{1}{5}\frac{G}{c^5}\langle \dddot{I}\dddot{I} \rangle$. Expect $\dddot{I} \sim \frac{MR^2}{T^3} \sim MR^2/(R/v)^3 \sim \frac{Mv^3}{R}$ ∴ $L_{GR} \sim \frac{1}{5}\frac{G}{c^5}\frac{M^2v^6}{R^2}$. $L_{GR} \sim \left(\frac{c^5}{G}\right)\left(\frac{R_s}{R}\right)^2\left(\frac{v}{c}\right)^6$ or $\frac{G^4M^5}{c^5a^5}$ using $\dddot{I} \sim Ma^2\Omega^3$