Derivation of Kepler's laws - concern binary system of gravitating point masses.

Kepler's laws are:

I. Orbits are conic sections i.e circles, ellipses, parabola, hyperbola.

II. line connecting two masses in mutual orbit sweeps out equal areas in equal times i.e rate of area swept out by this line = constant.

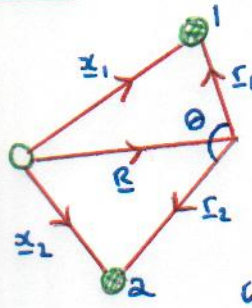
III. If P = orbital period of one mass about another (assumes closed orbit i.e circles, ellipses) and a = semi-major axis of ellipse (with one mass at a particular focus)

$$\left(\frac{P}{2\pi}\right)^2 = \frac{a^3}{G(m_1+m_2)}$$

where m_1, m_2 are respective masses.

[Classical mechanics assumed throughout proof].

Proof/ consider fixed origin O and positions of gravitating masses displaced from O by vectors $\mathbf{x}_1, \mathbf{x}_2$.



Applying Newton's 2nd law and law of gravity

$$m_1\ddot{\mathbf{x}}_1 = \frac{Gm_1m_2}{|\mathbf{x}_2-\mathbf{x}_1|^3}(\mathbf{x}_2-\mathbf{x}_1) \quad (1) \qquad m_2\ddot{\mathbf{x}}_2 = \frac{Gm_1m_2}{|\mathbf{x}_1-\mathbf{x}_2|^3}(\mathbf{x}_1-\mathbf{x}_2) \quad (2)$$

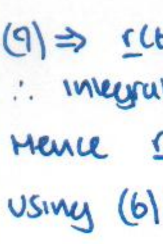
Define $\mathbf{R} = \dfrac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1+m_2}$ and $M = m_1+m_2$ ($\mathbf{R}$ is centre of mass vector).

consider mass, energy and angular momentum of system to be conserved. $\therefore$ $\ddot{\mathbf{R}} = \frac{m_1}{M}\ddot{\mathbf{x}}_1 + \frac{m_2}{M}\ddot{\mathbf{x}}_2$ using mass conservation and also NO MASS TRANSFER. ($\dot{m}_1 = \dot{m}_2 = 0$).

using (1) and (2) $\ddot{\mathbf{R}} = \frac{1}{M}\frac{Gm_1m_2}{|\mathbf{x}_1-\mathbf{x}_2|^3}\left\{\mathbf{x}_2-\mathbf{x}_1+\mathbf{x}_1-\mathbf{x}_2\right\} = 0 \Rightarrow \dot{\mathbf{R}} = \text{constant}$ (3)

By (3) $\Rightarrow$ point $\mathbf{R}$ moves at constant velocity relative to O. $\therefore$ frame where $\dot{\mathbf{R}}=0$ is an inertial frame and $\therefore$ (1), (2) still hold exclusively, but with vectors centered on point $\mathbf{R}$ rather than O. i.e continue analysis with vectors $\mathbf{r}_1, \mathbf{r}_2$

conversion: $\mathbf{x}_1 = \mathbf{R} + \mathbf{r}_1$ (4) $\mathbf{x}_2 = \mathbf{R} + \mathbf{r}_2$ (5) (i.e can consider dynamics of $\mathbf{R}$; $\mathbf{r}_1, \mathbf{r}_2$ seperately - as long as $\ddot{\mathbf{R}}=0$).



Define vector $\mathbf{r}$ from mass 2 to mass 1.

$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ (6) Applying Newton's law just like for (1), (2)

$$\Rightarrow m_1\ddot{\mathbf{r}}_1 = -\frac{Gm_1m_2\mathbf{r}}{r^3} \quad (7) \qquad m_2\ddot{\mathbf{r}}_2 = \frac{Gm_1m_2\mathbf{r}}{r^3} \quad (8)$$

From (6) $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2$

$\therefore$ using (7), (8) $\Rightarrow \ddot{\mathbf{r}} = -\frac{Gm_2\mathbf{r}}{r^3} - \frac{Gm_1\mathbf{r}}{r^3} \Rightarrow \ddot{\mathbf{r}} = -\frac{GM\mathbf{r}}{r^3}$ (9)

(9) $\Rightarrow$ $\mathbf{r}(t)$ can be found.

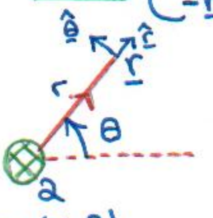
$\therefore$ integrating (7) and (8) $\Rightarrow$ $\mathbf{r}_1(t) = -Gm_2\int^t\!\!\int^{t'}\frac{\mathbf{r}(t'')}{r^3(t'')}dt''$ and $\mathbf{r}_2(t) = Gm_1\int^t\!\!\int^{t'}\frac{\mathbf{r}(t'')}{r^3(t'')}dt''$

Hence $\mathbf{r}_1 \parallel \mathbf{r}_2$ ($\Rightarrow \theta = \pi$) and $-\mathbf{r}_1/m_2 = \mathbf{r}_2/m_1 \Rightarrow \mathbf{r}_2 = -\mathbf{r}_1 m_1/m_2$ (10)

using (6) $\Rightarrow$ $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2$ are all $\parallel$ and $\mathbf{r} = \mathbf{r}_1\left(1+\frac{m_1}{m_2}\right) = \frac{\mathbf{r}_1 M}{m_2}$ or $\mathbf{r} = -\mathbf{r}_2\left(1+\frac{m_2}{m_1}\right)$

$$\Rightarrow \mathbf{r} = \begin{cases} \mathbf{r}_1 M/m_2 \\ -\mathbf{r}_2 M/m_1 \end{cases} \quad (11)$$

Since (9) describes dynamics of entire system and since $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2$ are all $\parallel$ we can describe $\mathbf{r}$ in 2D polar coordinates. (well $z=0$ cyl. polars)



$\mathbf{r} = (r, \theta)$ with unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}$

Note: $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$ (12)

$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$ (13)

Angular momentum of entire system is $\mathbf{J}$. $\mathbf{J}$ = constant by conservation statement.

From definition of angular momentum ($\underline{j} = \underline{r} \times \underline{p}$ ← momentum) ; in $\underline{R}$ centered frame:

$\Rightarrow \underline{J} = m_1(\underline{r}_1 \times \dot{\underline{r}}_1) + m_2(\underline{r}_2 \times \dot{\underline{r}}_2)$ using (11) $\Rightarrow \underline{J} = m_1\left(\frac{m_2}{M}\right)^2(\underline{r}\times\dot{\underline{r}}) + m_2\left(\frac{m_1}{M}\right)^2(\underline{r}\times\dot{\underline{r}})$

$\Rightarrow \underline{J} = \frac{m_1 m_2(\underline{r}\times\dot{\underline{r}})}{M^2}\{m_2 + m_1\} \Rightarrow \underline{J} = \frac{m_1 m_2}{M}(\underline{r}\times\dot{\underline{r}})$ (14) Now using (12), (13)

$\underline{r}\times\dot{\underline{r}} = r^2\dot{\theta}(\hat{\underline{r}}\times\hat{\underline{\theta}})$

$\therefore |\underline{J}| = \frac{m_1 m_2}{M} r^2\dot{\theta}$ Since $\underline{J}$ = constant $\Rightarrow r^2\dot{\theta}$ = constant. Define $h = r^2\dot{\theta}$ (15)

$J = \frac{m_1 m_2}{M} h$ (16)

Applying (12), (13) ; (9) becomes $(\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\underline{\theta}} = -\frac{GM}{r^2}\hat{\underline{r}}$

$\Rightarrow \ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$ (17) and $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$ (18). Desire $r(\theta)$ so:

Note $\dot{r} = \frac{dr}{d\theta}\dot{\theta} \Rightarrow \ddot{r} = \frac{dr}{d\theta}\ddot{\theta} + \frac{d^2r}{d\theta^2}\dot{\theta}^2$. From (18) $\ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r}$ and from (15) $\dot{\theta} = \frac{h}{r^2}$

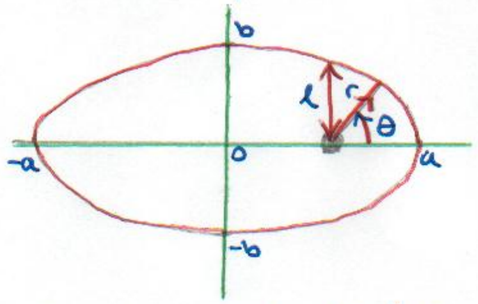
$\therefore \ddot{r} = \frac{-2\dot{r}\dot{\theta}}{r}\frac{dr}{d\theta} + \frac{h}{r^2}\frac{d^2r}{d\theta^2}$. From $\dot{r} = \frac{dr}{d\theta}\dot{\theta} \Rightarrow \dot{r}\dot{\theta} = \frac{dr}{d\theta}\dot{\theta}^2$ and using (15)

$\Rightarrow \dot{r}\dot{\theta} = \frac{h^2}{r^4}\frac{dr}{d\theta}$. Hence $\ddot{r} = -\frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{h}{r^2}\frac{d^2r}{d\theta^2}$. $\therefore$ using $\dot{\theta}^2 = \frac{h^2}{r^4}$ again

$\Rightarrow$ (17) becomes $-\frac{h^2}{r^3} - \frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{h}{r^2}\frac{d^2r}{d\theta} = -\frac{GM}{r^2} \Rightarrow \frac{GM}{h^2} = \frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^2}\frac{d^2r}{d\theta^2} + \frac{1}{r}$ (19)

let $u = \frac{1}{r}$ $\therefore \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} \Rightarrow \frac{d^2u}{d\theta^2} = -\frac{1}{r^2}\frac{d^2r}{d\theta^2} + \frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2$ using this substitution (19) becomes:

$\Rightarrow \frac{GM}{h^2} = u + \frac{d^2u}{d\theta^2}$ (20) closed solution of this (periodic with θ) is $u(\theta) = \frac{GM}{h^2} + A\cos(\theta + \alpha)$. Assume $\frac{1}{r}$ is maximised when $\theta = 0 \Rightarrow \alpha = 0$. Hence $\frac{1}{r} = \frac{GM}{h^2} + A\cos\theta$. (21)



Now an ellipse has r, θ equation $r = \frac{l}{1 + e\cos\theta}$

Eccentricity $e = \sqrt{1 - b^2/a^2}$

Semi latus rectum $l = a(1 - e^2)$

(21) can be written as $r = \frac{1}{\frac{GM}{h^2} + A\cos\theta} = \frac{h^2/GM}{1 + \frac{h^2 A}{GM}\cos\theta}$ i.e. it indeed describes an ellipse with

$l = \frac{h^2}{GM}$ (22) $e = \frac{h^2}{GM}A$ (23). THUS PROVING KEPLER'S FIRST LAW.

Now since $e^2 = 1 - \frac{l}{a}$, using (22) $\Rightarrow e = \sqrt{1 - \frac{h^2}{GMa}}$ (24)

$\therefore$ using (23) $\Rightarrow A = \frac{GM}{h^2}\sqrt{1 - \frac{h^2}{GMa}}$ (25) Hence $\frac{1}{r} = \frac{GM}{h^2} + \frac{GM}{h^2}\sqrt{1 - \frac{h^2}{GMa}}\cos\theta$ (26)

Note $r(\theta)$ is defined by 3 independent parameters h, M, a.

Now Energy of system $E = \frac{1}{2}m_1|\dot{\underline{r}}_1|^2 + \frac{1}{2}m_2|\dot{\underline{r}}_2|^2 - Gm_1m_2/r$ (27)

Applying (11) $\Rightarrow E = \frac{1}{2}m_1\left(\frac{m_2}{M}\right)^2|\dot{\underline{r}}|^2 + \frac{1}{2}m_2\left(\frac{m_1}{M}\right)^2|\dot{\underline{r}}|^2 - Gm_1m_2/r$

$\Rightarrow E = \frac{1}{2}\frac{m_1m_2}{M}|\dot{\underline{r}}|^2 - \frac{Gm_1m_2}{r}$ (28). Using (12) $|\dot{\underline{r}}|^2 = \dot{r}^2 + r^2\dot{\theta}^2$.

Since $\dot{r} = \frac{dr}{d\theta}\dot{\theta} \Rightarrow \dot{r}^2 = \left(\frac{dr}{d\theta}\right)^2\dot{\theta}^2$

Hence using (15) $\Rightarrow \dot{\theta} = \frac{h}{r^2} \Rightarrow |\dot{\underline{r}}|^2 = \left(\frac{dr}{d\theta}\right)^2 \frac{h^2}{r^4} + \frac{h^2}{r^2}$ using $u = \frac{1}{r} \Rightarrow \frac{du}{d\theta} = \frac{-1}{r^2}\frac{dr}{d\theta}$

$\Rightarrow |\dot{\underline{r}}|^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$ using (26) and (28) $\left\{ (26) \Rightarrow \left(\frac{du}{d\theta}\right)^2 = \frac{G^2M^2}{h^4}\left(1 - \frac{h^2}{GMa}\right)\sin^2\theta \right\}$

$$\Rightarrow E = \frac{m_1 m_2}{M}\left\{ \frac{1}{2}h^2\left[\frac{G^2M^2}{h^4}\left(1-\frac{h^2}{GMa}\right)\sin^2\theta + \frac{G^2M^2}{h^4} + \frac{2G^2M^2}{h^4}\sqrt{1-\frac{h^2}{GMa}}\cos\theta + \frac{G^2M^2}{h^4}\left(1-\frac{h^2}{GMa}\right)\cos^2\theta\right] - \frac{G^2M^2}{h^2}\left[1 + \sqrt{1-\frac{h^2}{GMa}}\cos\theta\right]\right\}$$

(Note $\sin^2\theta + \cos^2\theta = 1$)

$$\Rightarrow E = \frac{m_1 m_2}{M}\left\{ \frac{1}{2}\frac{G^2M^2}{h^2}\left(1 - \frac{h^2}{GMa}\right) + \frac{1}{2}h^2\frac{G^2M^2}{h^4} - \frac{G^2M^2}{h^2}\right\}$$

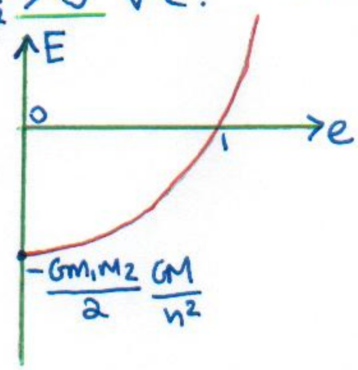
$$\Rightarrow E = -\frac{m_1 m_2}{M}\frac{1}{2}\frac{G^2M^2}{h^2}\frac{h^2}{GMa} \Rightarrow E = -\frac{Gm_1 m_2}{2a} \quad (29)$$

Now using (24) $e^2 = 1 - \frac{h^2}{GMa} \Rightarrow \frac{1}{a} = \frac{GM}{h^2}(1-e^2)$ $\therefore$ $E = -\frac{Gm_1m_2}{2}\cdot\frac{GM}{h^2}(1-e^2)$ (30)

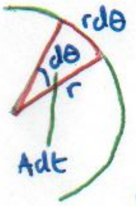
i.e. $E(e)$ is a parabola. Note $\frac{dE}{de} = Gm_1m_2\frac{GM}{h^2}e \Rightarrow \frac{d^2E}{de^2} > 0 \ \forall e$.

Hence minimum E when $e=0$. i.e when $\frac{dE}{de} = 0$.

Hence most bound orbit is for $e=0$ i.e CIRCULAR. This explains tendency of orbits who have $J \neq 0$ (well negative as well) to 'circularise'.



Now if rate of area swept out by $\underline{r}$ is (consider time interval dt) A: $A\,dt = \frac{1}{2}r^2 d\theta \Rightarrow A = \frac{1}{2}r^2\dot{\theta}$



Since (15) $h = r^2\dot{\theta} \Rightarrow A = \frac{h}{2}$ = constant. (31)

THUS PROVING KEPLER'S SECOND LAW.

Noting above, period P is $\therefore$ area of ellipse$/A = \frac{\pi ab}{A}$

using $e^2 = 1 - \frac{b^2}{a^2} \Rightarrow b^2 = a^2(1-e^2)$

Now $l = a(1-e^2) \Rightarrow (1-e^2) = \frac{l}{a}$ $\therefore$ $b^2 = al \Rightarrow b = \sqrt{la}$.

Now using $A = \frac{h}{2}$ and (22) $l = \frac{h^2}{GM} \Rightarrow P = \frac{\pi a}{\frac{1}{2}h}\sqrt{\frac{ah^2}{GM}}$

$$\Rightarrow \left(\frac{P}{2\pi}\right)^2 = \frac{a^2}{h^2}\frac{ah^2}{GM} \Rightarrow \left(\frac{P}{2\pi}\right)^2 = \frac{a^3}{G(m_1+m_2)} \quad (32)$$

THUS PROVING KEPLERS THIRD LAW.

Now if P, a, M, e are known (and m_1, m_2) it is useful to calculate J directly. From $P = \frac{\pi ab}{A} \Rightarrow h = \frac{2\pi ab}{P}$

$\Rightarrow h = ab\Omega$ where Ω is angular frequency $\frac{2\pi}{P}$.

using $b^2 = a^2(1-e^2) \Rightarrow h = a^2(1-e^2)^{\frac{1}{2}}\Omega$ $\therefore$ Since (16) $\Rightarrow J = \frac{m_1m_2}{M}h$

$$\Rightarrow J = \frac{m_1m_2}{M}a^2(1-e^2)^{\frac{1}{2}}\Omega \quad (33)$$

Note for circular orbits $J = \frac{m_1m_2}{M}a^2\Omega$ (34)