GENERAL RELATIVITY is a geometrical theory which describes all physical phenomena on scales above that where quantum mechanics is required!

1) Geometry and notation of GR. In GR model observable world as a 4 dimensional pseudo-Riemannian MANIFOLD. { A manifold is any set that can be continuously parameterised. # independent parameters needed to uniquely specify a point in the set = Dimension of manifold. parameters themselves are manifold's coordinates }. Hypersurfaces are subsets of a given manifold with dimensions one less than the manifold. These are described by an equation of the form

$$f(x^1, x^2, x^3 \dots x^N) = 0 \qquad \text{for } N \text{ dimensional manifold}$$

An M dimensional surface is ∴ described by a system of equations (N−M of them)

$$f_a(x^1, x^2, x^3 \dots x^N) = 0 \qquad \text{where } a = \{1, \dots N-M\}.$$

SCALARS are values of points on a given manifold. VECTORS are geometrical objects defined in local cartesian tangent spaces to a point on a given manifold. They are intrinsicly dependent on a particular manifold point and can ∴ only have algebraic properties (such as addition) when one refers to locally similar manifold tangent spaces. (For a cartesian manifold all tangent spaces are coincident and ∴ effectively local).

TENSORS are linear mappings of T vectors to $\mathbb{R}$. T = Rank of the tensor. (Note T=0 ⇒ scalar, T=1 ⇒ vector).

NOTE SYMMETRIC : $g_{ab} = g_{ba}$

Given coordinates x^a : a pseudo Riemmannian manifold has LINE ELEMENT

$$ds^2 = g_{ab}\, dx^a dx^b$$

ds = invariant line element (independent of coordinates).

$\| g_{ab} \|$ = metric.

Note ∑ convention. " sum over repeated indicies"

i.e $g_{ab}\, dx^a dx^b \equiv \sum_a \sum_b g_{ab}\, dx^a dx^b$.

LENGTHS, AREAS and VOLUMES are intrinsic properties of a manifold. (Assume from now on Pseudo Riemannian) PRM

LENGTH $\quad L_{AB} = \int_A^B ds = \int_A^B |g_{ab}\, dx^a dx^b|^{1/2}$. Usually specify some curve between A and B i.e $x^a(u)$.

AREA $\quad dA = \sqrt{|g_{11} g_{22}|}\, dx^1 dx^2$ (1,2 are labels of any 2 coordinate subset of manifold coordinates)

→ $d^N V = \sqrt{|g_{11} g_{22} \cdots g_{NN}|}\, dx^1 dx^2 \dots dx^N$ (generalised) VOLUME (element).

These assume metric is DIAGONAL ⇒ ORTHOGONAL coordinates. In general (PRM) VOLUME is $d^N V = \sqrt{|g|}\, dx^1 dx^2 \dots dx^N$ where $g = \det \| g_{ab} \|$.

A vector $\underline{v}$ can be described in terms of its contravariant components and basis vectors or equivalently covariant components and dual basis vectors.

i.e $\quad \underline{v} = v^a \underline{e}_a = v_a \underline{e}^a$. $\quad \underline{e}_a = \lim_{\delta x^a \to 0} \frac{\delta \underline{s}}{\delta x^a}$ where $\delta \underline{s}$ is the vector displacement (in the tangent space of point P) between point P and nearby point Q. ⇒ $d\underline{s} = \underline{e}_a dx^a$.

NOTE: "Co GOES BELOW" (HOBSON 2001).

Hence: $\quad ds^2 \equiv d\underline{s} \cdot d\underline{s} = (\underline{e}_a \cdot \underline{e}_b)\, dx^a dx^b \Rightarrow g_{ab} = \underline{e}_a \cdot \underline{e}_b$.

Hence: $\quad \underline{w} \cdot \underline{v} = g_{ab} w^a v^b$. (Definition of dot product of vectors in PRM).

Define DUAL BASIS $\underline{e}^a$ to have the property $\underline{e}^a \cdot \underline{e}_b = \delta^a_b \equiv \begin{cases} 1 & a=b \\ 0 & a \neq b \end{cases}$

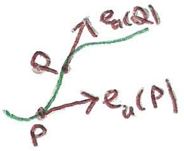
∴ $d\underline{s} = \underline{e}_a dx^a \Rightarrow \underline{e}^b \cdot d\underline{s} = dx^b$. Define $g^{ab} = \underline{e}^a \cdot \underline{e}^b$.

Using $\underline{v}$ and $\underline{w}$ in contra/basis, co/dual forms and evaluating all four possible forms of $\underline{v} \cdot \underline{w}$ ⇒ $w_a = g_{ab} w^b$; $v^a = g^{ab} v_b$ ← Same for $\underline{e}^a, \underline{e}_a$

RAISE AND LOWER INDICIES BY METRIC

Now consider $\delta^a_c v^c$. $\delta^a_c v^c = v^a$. $v^a = g^{ab} v_b = g^{ab} g_{bc} v^c \Rightarrow g^{ab} g_{bc} = \delta^a_c$.
Hence $\|g_{ab}\|$ and $\|g^{ab}\|$ are inverses.
Transform coordinates via $dx^a = \frac{\partial x^a}{\partial x'^b} dx'^b$. Since $d\underline{s} = dx^a \underline{e}_a = dx'^a \underline{e}_{a'}$

$\Rightarrow \underline{e}_{a'} = \frac{\partial x^b}{\partial x'^a} \underline{e}_b$. Similarly: $\underline{e}'^a = \frac{\partial x'^a}{\partial x^b} \underline{e}^b$.

The tangent vector to a curve $x^a(u)$ is $\underline{t} = \frac{dx^a}{du} \underline{e}_a$. $|\underline{t}| = \left| g_{ab} dx^a dx^b \right|^{\frac{1}{2}} / du$
$= \left|\frac{ds}{du}\right|$. $\therefore$ for a NULL curve $ds = 0 \Leftrightarrow \underline{t}$ is zero at all points on the curve.
Note for null curve cannot let $u = s$ ie use s as a parameter.

If points P and Q are local s.t. their tangent spaces are coincident
$\underline{e}_a(Q) = \underline{e}_a(P) + \delta \underline{e}_a$. Define AFFINE CONNECTION Γ^b_{ac} by
$\delta \underline{e}_a = \Gamma^b_{ac} \underline{e}_b \delta x^c$. $\frac{\partial \underline{e}_a}{\partial x^c} = \lim_{\delta x^c \to 0} \frac{\delta \underline{e}_a}{\delta x^c} = \Gamma^b_{ac} \underline{e}_b$.
So $\partial_c \underline{e}_a = \Gamma^b_{ac} \underline{e}_b$

using $\underline{e}^a \cdot \underline{e}_b = \delta^a_b \Rightarrow \Gamma^b_{ac} = \underline{e}^b \cdot \partial_c \underline{e}_a$ where $\partial_c \equiv \frac{\partial}{\partial x^c}$.
Noting $\partial_c(\underline{e}^a \cdot \underline{e}_b) = 0 \Rightarrow \partial_c \underline{e}^a = -\Gamma^a_{bc} \underline{e}^b$. Affine connection is not a tensor.
However, $T^b_{ac} \equiv \Gamma^b_{ac} - \Gamma^b_{ca}$ (TORSION) is. In standard GR torsion is not present $\Rightarrow \Gamma^b_{ac} = \Gamma^b_{ca}$. $(\Rightarrow \partial_c \underline{e}_a = \partial_a \underline{e}_c)$

Noting $\Gamma'^a_{bc} = \underline{e}'^a \cdot \frac{\partial \underline{e}'_b}{\partial x'^c}$ can show $\Gamma'^a_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x^f}{\partial x'^b} \frac{\partial x^g}{\partial x'^c} \Gamma^d_{fg} - \frac{\partial x^d}{\partial x'^b} \frac{\partial x^f}{\partial x'^c} \frac{\partial^2 x'^a}{\partial x^d \partial x^f}$

Now $\partial_c g_{ab} = \partial_c(\underline{e}_a \cdot \underline{e}_b) = \Gamma^d_{ac} g_{db} + \Gamma^d_{bc} g_{ad}$ [1] cyclically permuting a, b, c
$\Rightarrow \partial_b g_{ca} = \Gamma^d_{cb} g_{da} + \Gamma^d_{ab} g_{cd}$ [2] $\partial_a g_{bc} = \Gamma^d_{ba} g_{dc} + \Gamma^d_{ca} g_{bd}$ [3].
[1] + [2] − [3] $= 2\Gamma^d_{cb} g_{ad}$*. Multiplying by g^{ea} and noting $g^{ea} g_{ad} = \delta^e_d$
(and relabelling indices) $\Rightarrow \Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})$
in this case "connection"
* Using $\Gamma^a_{bc} = \Gamma^a_{cb}$ define as METRIC CONNECTION ($\Rightarrow$ torsion $= 0$).
if metric connection = affine !!

Define $\Gamma_{abc} \equiv g_{ad} \Gamma^d_{bc}$. Note $\partial_c g_{ab} = \Gamma_{abc} + \Gamma_{bac}$.

$\partial_b \underline{e}_a = \Gamma^c_{ba} \underline{e}_c$

Now covariant derivative of a vector is $\partial_b \underline{v} \equiv (\nabla_b v^a) \underline{e}_a$.
$\partial_b \underline{v} = \partial_b(v^a \underline{e}_a) = \underline{e}_a \partial_b v^a + v^a \partial_b \underline{e}_a = \underline{e}_a(\partial_b v^a + (\Gamma^c_{ba} \underline{e}_c) v^a)$
$= \underline{e}_a(\partial_b v^a + v^c \Gamma^a_{bc}) \Rightarrow \nabla_b v^a = \partial_b v^a + v^c \Gamma^a_{bc}$
swap a, c
using $\underline{v} = v_a \underline{e}^a$ and $\partial_c \underline{e}^a = -\Gamma^a_{bc} \underline{e}^b \Rightarrow \nabla_b v_a = \partial_b v_a - \Gamma^c_{ab} v_c$.

Similarly INTRINSIC derivative of a vector is $\frac{d\underline{v}}{du}$ where u is curve parameterisation ($x^a(u)$ describes curve). $\underline{v} = v^a(u) \underline{e}_a(u)$.
$\frac{d\underline{v}}{du} = \underline{e}_a \frac{dv^a}{du} + v^a \frac{d\underline{e}_a}{du}$. By the chain rule $d\underline{e}_a = \frac{\partial \underline{e}_a}{\partial x^c} dx^c = \Gamma^b_{ac} \underline{e}_b dx^c$

$\Rightarrow \frac{d\underline{v}}{du} = \underline{e}_a \frac{dv^a}{du} + v^a \frac{dx^c}{du} \Gamma^b_{ac} \underline{e}_b = \underline{e}_a \left(\frac{dv^a}{du} + v^b \Gamma^a_{bc} \frac{dx^c}{du} \right)$ $(a \Leftrightarrow b)$

A vector is $\parallel$ transported along a curve C if $\frac{d\underline{v}}{du} = 0 \Rightarrow \frac{dv^a}{du} = -v^b \Gamma^a_{bc} \frac{dx^c}{du}$.

[using $\underline{v} = v_a \underline{e}^a$ and $\partial_c \underline{e}^a = -\Gamma^a_{bc} \underline{e}^b$ find $\frac{d\underline{v}}{du} = \left(\frac{dv_a}{du} - \Gamma^b_{ac} v_b \frac{dx^c}{du} \right)$].

Geodesics. Define u to be some AFFINE parameter s.t. $\frac{d\underline{t}}{du} = 0$ where $\underline{t}$ is the tangent vector to some curve $x^a(u)$. From above result for $\frac{d\underline{v}}{du}$ and $\underline{t} = \frac{dx^a}{du} \underline{e}_a$
$\Rightarrow$ GEODESIC EQUATION: $\frac{d^2 x^a}{du^2} + \Gamma^a_{bc} \frac{dx^b}{du} \frac{dx^c}{du} = 0$

can show that change of coordinates $x^a \to x'^a$ does not change form of geodesic equation. Let $u = u(\lambda)$. Using chain rule: $du = \frac{du}{d\lambda} d\lambda$ can show:

Note geodesic eq. can be written as: $\dot{u}_\mu = \Gamma^{\nu}_{\mu\sigma} u_\nu u^\sigma$, $u^\mu = \dot{x}^\mu = \frac{dx^\mu}{ds}$

$$\frac{d^2 x^a}{d\lambda^2} + \Gamma^a_{bc} \frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = \left(\frac{d^2u/d\lambda^2}{du/d\lambda}\right)\frac{dx^a}{d\lambda}$$

i.e. linearly related

Hence if u is an affine parameter so is any $\lambda = au + b$ (a, b constants).

Lagrangian procedure is a neat way of determining Γ^a_{bc} and geodesic equations from the metric very efficiently, indirectly.

Consider $L = g_{ab}\dot{x}^a\dot{x}^b$ where $\dot{} \equiv \frac{d}{du}$ (u some affine parameter).

Substitution into Euler Lagrange equation $\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0$ (*)

$\Rightarrow \ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$. So writing down L and substitution into (*) generates geodesic equation and allows connection coefficients to be picked off.

Note for null geodesics $ds = 0 \Rightarrow ds^2 = 0 \Rightarrow \left(\frac{ds}{du}\right)^2 = 0 \Rightarrow L = 0$.

For non null geodesics choose $u = s \Rightarrow |g_{ab}\dot{x}^a\dot{x}^b| = 1$.

Force term pushes particles off geodesics

classical lagrangian

Note if $L = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b - V(x)$, E.L equation $\Rightarrow \ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = -g^{ab}\partial_b V$.

Extension of above vector calculus discussion to Tensor calculus on Manifolds.

(Start with rank 2 tensors and results illustrate the progression to higher orders).

If $\underline{t}$ is a rank 2 tensor can write $\underline{t} = t^{ab}\underline{e}_a \otimes \underline{e}_b$ where $\underline{e}_a \otimes \underline{e}_b$ is the outer product of basis vectors $\underline{e}_a, \underline{e}_b$. (Note Σ convention). As for vectors define covariant derivative by $\partial_c \underline{t} = (\nabla_c t^{ab})\underline{e}_a \otimes \underline{e}_b$. Using product rule and $\partial_c \underline{e}_a = \Gamma^b_{ca}\underline{e}_b$ can show

$\nabla_c t^{ab} = \partial_c t^{ab} + \Gamma^a_{dc}t^{db} + \Gamma^b_{dc}t^{ad}$. Now can write $\underline{t} = t^a{}_b\,\underline{e}_a \otimes \underline{e}^b$

and $\underline{t} = t_{ab}\underline{e}^a \otimes \underline{e}^b$. Using $\partial_c \underline{e}^a = -\Gamma^a_{bc}\underline{e}^b$ and $\partial_c \underline{e}_a = \Gamma^b_{ca}\underline{e}_b$ can show

$$\nabla_c t^a{}_b = \partial_c t^a{}_b + \Gamma^a_{dc}t^d{}_b - \Gamma^d_{bc}t^a{}_d \;;\; \nabla_c t_{ab} = \partial_c t_{ab} - \Gamma^d_{ac}t_{db} - \Gamma^d_{bc}t_{ad}$$

Result of this is one can show $\nabla_c g_{ab} = \nabla_c g^{ab} = 0$. i.e. covariant derivative of metric is zero for any coordinate basis. This result $\Rightarrow$ ∇_c and raising/lowering operators ($\times g^{ab}$ or g_{ab}) commute without affecting the result. i.e. $\nabla_c(g^{ab}t^d{}_b) = g^{ab}\nabla_c t^d{}_b$.

Using $\underline{t} = t^{ab}\underline{e}_a \otimes \underline{e}_b$ total derivative of $\underline{t}$ along a curve $x^a(u)$ can be found:

$$\frac{d\underline{t}}{du} = \left(\frac{dt^{ab}}{du} + \Gamma^a_{dc}t^{db}\frac{dx^c}{du} + \Gamma^b_{dc}t^{ad}\frac{dx^c}{du}\right)\underline{e}_a \otimes \underline{e}_b$$

AND SIMILAR EXPRESSIONS INVOLVING $t^a{}_b$ and t_{ab}

$\underline{t}$ can be $\parallel$ transported along a curve $\Rightarrow d\underline{t}/du = 0$.

To be and not to be a tensor..... A tensor's components must satisfy the following transformation properties - this indeed is the defining property of a tensor!

* Each superscript(a) inherits a transformation matrix $\partial x'^a/\partial x^c$
* " subscript (a) " " " $\partial x^c/\partial x'^a$

i.e. $$t'^c{}_{ab} = \frac{\partial x^d}{\partial x'^a}\frac{\partial x^e}{\partial x'^b}\frac{\partial x'^c}{\partial x^f}t_{de}{}^f.$$

Why are tensors useful? TENSOR EQUATIONS i.e. $t_{ab} = S_{ab}$ are valid in all coordinate systems. So if can prove a result (like $S_{ab} = 0$) in a simple coordinate system (i.e. Cartesian) you will have proved a GENERAL result.

Quotient theorem: "Tensor contracted with test quantity = tensor $\Rightarrow$ test quantity = tensor". i.e. if a^{xy} and b_{pq} are components of tensors $\underline{a}, \underline{b}$ $a^{xy} = z^{xypq}b_{pq} \Rightarrow z^{xypq}$ are components of a 4th rank tensor $\underline{z}$

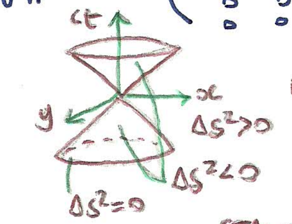
2) Special Relativity and Electromagnetism. Special relativity ⇒ a Minkowski spacetime. AXIOM

i.e. the universe has geometry characterised by a pseudo Riemannian manifold which has the following line element at every point

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu . \qquad \|\eta_{\mu\nu}\| = \begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}.$$

$x^0 \equiv t$; $x^1 \equiv \frac{x}{c}$; $x^2 \equiv \frac{y}{c}$; $x^3 \equiv \frac{z}{c}$ (time; Space $\vec r = (x,y,z)$; speed of light)

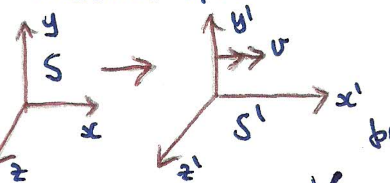
if $ds^2 > 0$: timelike interval
$ds^2 = 0$: null "
$ds^2 < 0$: spacelike interval



Note orthogonality of basis vectors in Minkowski spacetime ⇒ all tangent spaces are coincident. Hence can define position vector $\underline{R} = (ct, x, y, z)$.

Consider two inertial frames in Minkowski spacetime related by a boost $v\hat{\underline{x}}$.

STANDARD CONFIGURATION



Now since $\eta_{\mu\nu}$ has constant elements ⇒ under transformation $S \to S'$, x^μ must transform LINEARLY to preserve $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ formalism.

i.e. let $t' = At + Bx$; $x' = Dt + Ex$; $y' = y$; $z' = z$. when $x = 0$, $x' = -vt'$!

Now when $x' = 0$, $x = vt$. ∴ $Dt + Evt = 0 \Rightarrow D = -Ev$.

$\Rightarrow -vt' = Dt \Rightarrow t' = A\left(\frac{-vt'}{D}\right) \Rightarrow D = -Av$. Hence $E = A$.

∴ $t' = At + Bx$; $x' = A(x - vt)$; $y' = y$; $z' = z$. Now for a photon interval $ds^2 = 0$. ∴ using metric: $dt'^2 - \frac{1}{c^2}(dx'^2 + dy'^2 + dz'^2) = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$

Substituting above results: $A^2dt^2 + B^2dx^2 + 2AB\,dt\,dx - \frac{1}{c^2}(A^2dx^2 + A^2v^2dt^2 - 2A^2v\,dx\,dt) = dt^2 - \frac{1}{c^2}dx^2$. Comparing coefficients of dt^2: $A^2 - \frac{A^2v^2}{c^2} = 1 \Rightarrow A = \left(1 - \frac{v^2}{c^2}\right)^{-\frac12}$

Comparing coefficients of $dx\,dt$: $2AB + \frac{1}{c^2}2A^2v = 0 \Rightarrow B = -\frac{Av}{c^2}$.

Defining $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac12} \Rightarrow t' = \gamma\left(t - \frac{vx}{c^2}\right)$ $x' = \gamma(x - vt)$ $y' = y$ $z' = z$.

This is the LORENTZ TRANSFORM. Can write in matrix form

$$\underline{R}' = \begin{pmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0&0&1&0\\0&0&0&1\end{pmatrix}\underline{R}$$

where $\beta = v/c$. {Note rapidity $\psi = \tanh^{-1}\beta$ – often used}.

Now POINCARÉ transformations are generalisations of Lorentz and involve a shift of the origin. One may also consider Lorentz transformations between frames moving with relative 3 velocity $\vec\beta c$. (3 vector notation)

Again relate quantities $t, \vec r, \vec\beta$ by a linear transformation. ($t, \vec r, \vec\beta$ only available quantities)

$\Rightarrow ct' = a_1 ct + a_2 \vec\beta\cdot\vec r$ and $\vec r' = b_1\vec r + b_2 ct\vec\beta + b_3(\vec\beta\cdot\vec r)\vec\beta$ most general forms.

Now if $\vec\beta = (\beta, 0, 0)$ must generate L.T. derived above. $\Rightarrow a_1 = \gamma$. $a_2 = -\gamma$.
$b_1 = 1$. $b_2 = -\gamma$. $b_3 = \frac{\gamma - 1}{\beta^2}$

So general Lorentz transformation matrix is: $\left\{\alpha = \frac{\gamma-1}{\beta^2}\right\}$

$$\begin{pmatrix}ct'\\ \vec r'\end{pmatrix} = \begin{pmatrix}\gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z\\ -\gamma\beta_x & 1+\alpha\beta_x^2 & \alpha\beta_x\beta_y & \alpha\beta_x\beta_z\\ -\gamma\beta_y & \alpha\beta_y\beta_x & 1+\alpha\beta_y^2 & \alpha\beta_y\beta_z\\ -\gamma\beta_z & \alpha\beta_z\beta_x & \alpha\beta_z\beta_y & 1+\alpha\beta_z^2\end{pmatrix}\begin{pmatrix}ct\\ \vec r\end{pmatrix}$$

or: $ct' = \gamma(ct - \vec\beta\cdot\vec r)$
$\vec r' = \vec r - \gamma ct\vec\beta + \frac{\gamma - 1}{\beta^2}(\vec\beta\cdot\vec r)\vec\beta$

From Lorentz transform physical effects such as length contraction, time dilation can be described. Use L.T. to test whether a 4 component geometrical entity is a 4-vector. i.e. All 4 vectors obey L.T. Define proper time τ s.t. $ds = d\tau$. (rest frame of particle – $|d\vec r| = 0$). From metric $d\tau^2 = dt^2 - \frac{\vec u\cdot\vec u\, dt^2}{c^2} \Rightarrow \frac{d\tau}{dt} = \frac{1}{\gamma}$ ($\vec u = \frac{d\vec r}{dt}$). Given this result one can generate 4 velocity $\underline{u} = d\underline{R}/d\tau$ and 4 acceleration $\underline{a} = d\underline{u}/d\tau$. {useful ex. is to apply L.T to $\underline{u}, \underline{a}$ to get correct velocity, acceleration addition formulae. Note by construction $\underline{u}, \underline{a}$ are 4 vectors}.

Define 4-momentum $\underline{P} = m_0\underline{u}$ (m_0 rest mass – assume Lorentz invariant).

and 4-force $\underline{f}$. $\underline{f} = \frac{d\underline{p}}{d\tau}$ is the relativistic equation of motion. When $\underline{f} = 0$ $\underline{p}$ = constant. Newtonian limit of components of $\underline{p}$ imply $\underline{p} = (E/c, \vec{p})$ where E = energy of system described by $\underline{p}$ {can include potential terms} and $\vec{p}$ the 3 momentum. SEE RELATIVITY AND ELECTRODYNAMICS COURSE!

Now for any two 4 vectors $\underline{a}, \underline{b}$ can show $\underline{a}\cdot\underline{b} = \eta_{\mu\nu}a^\mu b^\nu$ is Lorentz invariant. Idea is to evaluate · products in frames where components are mostly zeros.

$\underline{u}\cdot\underline{u} = c^2$ since $\underline{u} = (\gamma_u c, \vec{u}\gamma)$. (Pick frame where $\vec{u} = 0 \Rightarrow \gamma = 1$.) $\underline{p}\cdot\underline{p} = m_0^2c^2$ if no potential energy terms. ∴ $E^2 - p^2c^2 = m_0^2c^4$ ENERGY MOMENTUM INVARIANT.

Four Force $\underline{f} = \gamma_u(\vec{f}\cdot\vec{u}/c, \vec{f})$. Note $\vec{f}\cdot\vec{u} = \frac{dE}{dt}$. $\vec{f} = \frac{d\vec{p}}{dt}$ ← CLASSICAL MECHANICS

Note $\underline{k} = (\frac{2\pi}{\lambda}, \vec{k})$ is another handy 4 vector → Doppler effect, Aberration

$\Rightarrow \gamma_u \frac{\vec{f}\cdot\vec{u}}{c} = \frac{dE}{d\tau}\frac{1}{c}$; $\gamma_u\vec{f} = \frac{d\vec{p}}{d\tau}$ $\Rightarrow$ $\underline{f} = d\underline{p}/d\tau$ as required.

Consider $\underline{u}\cdot\underline{f} = \underline{u}\cdot\frac{d\underline{p}}{d\tau} = \underline{u}\cdot\left(\frac{dm_0}{d\tau}\underline{u} + m_0\frac{d\underline{u}}{d\tau}\right) = c^2\frac{dm_0}{d\tau} + m_0\,\underline{u}\cdot\frac{d\underline{u}}{d\tau}$

$\frac{d}{d\tau}(\underline{u}\cdot\underline{u}) = 2\,\underline{u}\cdot\frac{d\underline{u}}{d\tau} \Rightarrow \underline{u}\cdot\frac{d\underline{u}}{d\tau} = \frac{1}{2}\frac{d}{d\tau}(\underline{u}\cdot\underline{u}) = \frac{1}{2}\frac{d}{d\tau}(c^2) = 0$. ∴ $\underline{u}\cdot\underline{f} = c^2\frac{dm_0}{d\tau}$.

A PURE FORCE is one where m_0 is invariant with proper time $\Rightarrow \underline{u}\cdot\underline{f} = 0$. In this case $\underline{f} = m_0\underline{a}$ describes dynamics. Can write in form $f^\nu\underline{e}_\nu = m_0\underline{a}$

$\Rightarrow \underline{a} = \frac{f^\nu}{m_0}\underline{e}_\nu$ Now $\underline{a} = \frac{d\underline{u}}{d\tau} = \underline{e}_\nu\left(\frac{du^\nu}{d\tau} + u^b\Gamma^\nu_{bc}\frac{dx^c}{d\tau}\right) = \underline{e}_\nu\left(\frac{du^\nu}{d\tau} + \Gamma^\nu_{bc}u^bu^c\right)$

$$\Rightarrow \underbrace{\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau}}_{\text{geodesic equation}} = \frac{f^\nu}{m_0}.$$

i.e forces push particles off geodesics.

Now ELECTROMAGNETISM effectively results in a particular 4 force $\underline{f}$. In classical mechanics the Lorentz force $\vec{f} = q(\vec{E} + \vec{u}\times\vec{B})$ describes the force resulting on a particle with LORENTZ INVARIANT charge q in the presence of electric and magnetic fields $\vec{E}$ and $\vec{B}$ while moving with 3 velocity $\vec{u}$. Let's ∴ postulate $f^\nu = qF^\nu{}_\mu u^\mu$ where $\underline{F}$ is the EM FIELD STRENGTH TENSOR.

Now let $\underline{f}_{em}$ be a pure force $\Rightarrow \underline{u}\cdot\underline{f} = 0 \Rightarrow f_\mu u^\mu = 0 \Rightarrow qF_{\nu\mu}u^\nu u^\mu = 0$ $\Rightarrow F_{\nu\mu}$ must be antisymmetric i.e $F_{\mu\nu} = -F_{\nu\mu}$. Now if ρ_0 is rest frame charge density (assume Lorentz invariant) $\underline{j} = \rho_0\underline{u}$ is a 4-vector. With Maxwell's equations as a (known) goal aim to relate $\underline{F}$ and $\underline{j}$. Simplest relation is $\nabla\cdot\underline{F} = \mu_0\underline{j}$ (μ_0 is assigned arbitrarily but one can show it indeed is the same μ_0 as in Maxwells equations). So $\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu$. Note $\nabla_\nu\nabla_\mu F^{\mu\nu} = \mu_0\nabla_\nu j^\nu = 0$ since $\nabla_\nu\nabla_\mu F^{\mu\nu} = \frac{1}{2}\nabla_\nu\nabla_\mu F^{\mu\nu} + \frac{1}{2}\nabla_\nu\nabla_\mu F^{\mu\nu} = \frac{1}{2}\nabla_\nu\nabla_\mu F^{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu F^{\nu\mu} = 0$. (By symmetry of g_{ab}, g^{ab}: $F_{\mu\nu}$ antisymmetric $\Rightarrow F^{\mu\nu}$ antisymmetric). CONTINUITY OF CHARGE.

So: $\underbrace{\nabla_\nu j^\nu = 0}_{\text{GENERAL}}$ $\Rightarrow$ $\underbrace{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}_{\text{SR}}$ with $\underline{j} = (\rho c, \vec{j})$ (in SR.)

Now define vector potential $\underline{A}$ s.t. $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

$\Rightarrow \nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} + \nabla_\mu F_{\nu\sigma} = 0$. Now $F^\nu{}_\mu = -g^{\nu a}F_{\mu a} = g^{a\nu}F_{a\mu}$

∴ $\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu \Rightarrow \nabla_\mu(g_{a\nu}F^{\mu\nu}) = \mu_0 g_{a\nu}j^\nu \Rightarrow \nabla_\mu F^\mu{}_a = \mu_0 j_a \Rightarrow \nabla_\mu(g^{\mu\nu}F_{a\nu}) = \mu_0 j_a$

$\Rightarrow g^{\mu\nu}(-\nabla_\mu\nabla_a A_\nu + \nabla_\mu\nabla_\nu A_a) = \mu_0 j_a$. Now Lorentz gauge states $\nabla_\mu A^\mu = 0$

{if $A_\mu \to A_\mu + \theta_\mu$: $F_{\mu\nu}$ unchanged provided $\nabla_\mu\theta_\nu = \nabla_\nu\theta_\mu$. Let $\theta_\mu = \partial_\mu\psi$ (scalar field) ✓ ok choose scalar field s.t. $\nabla_\mu A^\mu = 0$} In this case $g^{\mu\nu}\nabla_\mu\nabla_\nu A_a = \mu_0 j_a$

So: $g^{\mu\nu}\nabla_\mu\nabla_\nu A_a = \mu_0 j_a \Rightarrow \nabla_\mu\nabla^\mu A_a = \mu_0 j_a$ or $\Box^2 \underline{A} = \mu_0 \underline{j}$ in vector form.

In SR cartesians $\underline{A} = (\phi/c, \vec{A})$ $\therefore$ $\nabla_\mu\nabla^\mu A_a = \mu_0 j_a \rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\begin{pmatrix}\phi/c \\ \vec{A}\end{pmatrix} = (\rho c, \vec{j})\mu_0$

(in Minkowski spacetime $\Gamma^a{}_{bc} = 0 \Rightarrow \nabla_\mu = \partial_\mu$. $\nabla_\mu\nabla^\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$).

$\Rightarrow \frac{\ddot{\phi}}{c^3} - \frac{\vec{\nabla}^2\phi}{c} = \rho c \mu_0 \Rightarrow \nabla^2\phi + \rho c^2\mu_0 - \frac{\ddot{\phi}}{c^2} = 0$ WAVE EQUATION FOR ϕ.

$\Rightarrow \frac{\ddot{\vec{A}}}{c^2} - \vec{\nabla}^2\vec{A} = \mu_0\vec{j} \Rightarrow \vec{\nabla}^2\vec{A} + \mu_0\vec{j} - \frac{\ddot{\vec{A}}}{c^2} = 0$ WAVE EQUATION FOR $\vec{A}$.

$[\dot{\ } \Rightarrow \frac{\partial}{\partial t}]$

These equations allow $\vec{A}, \phi$ to be found from known charge distribution.

Now $\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$ $\rightarrow$ Maxwell's equations. (See EM and SR).

Note Lorentz gauge $\nabla_\mu A^\mu = 0$ in SR becomes $\frac{1}{c}\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$. {so $\nabla_\mu j^\mu = 0$ and $\nabla_\mu A^\mu = 0$ }.

Finally $\|F_{\mu\nu}\| = \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & -B^3 & B^2 \\ -E^2/c & B^3 & 0 & -B^1 \\ -E^3/c & -B^2 & B^1 & 0 \end{pmatrix}$.

Now since $f^\nu = q F^\nu{}_\mu u^\mu$ we can write down dynamical equation of a particle with rest mass m_0 and charge q.

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\nu{}_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = \frac{q F^\nu{}_\mu}{m_0}\frac{dx^\mu}{d\tau} \quad (+)$$

Recipe for EM is:

* Find $\vec{A}, \phi$ from charge distribution (solve wave equations)
* Work out $\vec{E}, \vec{B}$
* Calculate $F^\nu{}_\mu = g^{\nu a}F_{a\mu}$

Note: Since $\vec{E}$ and $\vec{B}$ are manifestations of the same source field $\underline{A}$ they are interchangable depending on your reference frame. Applying generalised LT on $\underline{A}$ can show:

$$\vec{E}' = \gamma(\vec{E} + c\vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}) \qquad \vec{B}' = \gamma\left(\vec{B} - \vec{\beta}\times\frac{\vec{E}}{c}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B})$$

3) Machinery of GR - Einstein's Field Equations. - cannot incorporate gravity as a force term in (+) within the theoretical framework of SR. {See GAC course - Einstein tower experiment etc}. - Also why should inertial and gravitational mass be the same in (classical) mechanics? $\rightarrow$ EQUIVALENCE PRINCIPLE. "IN A FREELY FALLING (NON ROTATING) LABORATORY OCCUPYING A SMALL REGION OF SPACE, THE LAWS OF PHYSICS ARE THOSE OF SPECIAL RELATIVITY". The trick is to remove gravity as a force so ALL frames are effectively freely falling. The main point of GR is to model gravity by the CURVATURE OF SPACETIME in the presence of matter and energy. Effectively the $\Gamma^\nu{}_{bc}$ term is altered in (+) rather than the force terms. ($\Rightarrow$ metric/geometry depends on mass distribution).

If gravity is manifested as a spacetime curvature can show in weak field limit

$$g_{00} = (1 + 2\Phi/c^2) \qquad \Phi = \text{Gravitational potential}$$

results in correct Newtonian gravity. Quantify curvature (intrinsic) by the RIEMANN TENSOR. $\nabla_c\nabla_b v_a - \nabla_b\nabla_c v_a = R^d{}_{abc}v_d$

$$\text{where} \quad R^d{}_{abc} \equiv \partial_b\Gamma^d{}_{ac} - \partial_c\Gamma^d{}_{ab} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec}$$

Spacetime is flat when $R^d{}_{abc} = 0$. The Riemann tensor has some useful properties:

CYCLIC IDENTITY $R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0$ If $R_{abcd} = g_{ae}R^e{}_{bcd}$

$$R_{abcd} = \tfrac{1}{2}(\partial_d\partial_a g_{bc} - \partial_d\partial_b g_{ac} + \partial_c\partial_b g_{ad} - \partial_c\partial_a g_{bd}) - g^{ef}(\Gamma_{eac}\Gamma_{fbd} - \Gamma_{ead}\Gamma_{fbc})$$

$\Rightarrow$ $R_{abcd} = -R_{bacd}$ $\Rightarrow$ Riemann tensor has $X = N^2(N^2-1)/12$ independent components where N = dimension of manifold.

$R_{abcd} = -R_{abdc}$

$R_{abcd} = R_{cdab}$

N	1	2	3	4
X	0	1	6	20

2D manifold: Gaussian curvature $K = R_{1212}/g$

only non zero component

The Riemann tensor also satisfies the BIANCHI IDENTITY $\nabla_e R^a{}_{bcd} + \nabla_c R^a{}_{bde} + \nabla_d R^a{}_{bec} = 0$

Prove this by using $\nabla_c \nabla_b v_a - \nabla_b \nabla_c v_a = R^d{}_{abc} v_d$ to show $(\nabla_e R^a{}_{bcd})_P = (\partial_e \partial_c \Gamma^a_{bd} - \partial_e \partial_d \Gamma^a_{bc})_P$ where Point P is part of a geodesic coordinate system where $\Gamma^a{}_{bc}(P) = 0$. Cyclically permuting c, d, e and adding gives result.

Define Ricci tensor $R_{ab} \equiv R^c{}_{abc}$ and curvature scalar $R \equiv g^{ab} R_{ab}$.

From Bianchi identity and symmetry properties of Riemann tensor one can show

$$\nabla_b (R^{bc} - \tfrac{1}{2} g^{bc} R) = 0 \qquad R^{bc} - \tfrac{1}{2} g^{bc} R \equiv G^{bc}$$

EINSTEIN TENSOR.

Riemann tensor is related to ∥ transport and 'geodesic deviation'. (Two nearby geodesics, initially ∥ will diverge or converge if $R^d{}_{abc} \neq 0$).

If $\delta^a(u)$ is worldline of geodesic separation: $\dfrac{D^2 \delta^a}{Du^2} + R^a{}_{cbd}\, \delta^b \dfrac{dx^c}{du} \dfrac{dx^d}{du} = 0$

(Note $\dfrac{D x^a}{Du} = \dfrac{dx^a}{du} + \dot{u}^b \Gamma^a{}_{bc} x^c$).

Einstein's field equations of GR relate curvature to the STRESS ENERGY MOMENTUM TENSOR, $T^{\mu\nu}$.

For a perfect fluid of pressure p, $T^{\mu\nu} = (\underset{\text{density}}{\rho} + \tfrac{p}{c^2}) \underset{\text{4 velocity}}{u^\mu} u^\nu - p g^{\mu\nu}$

Conservation of energy and momentum is satisfied by $\nabla_\mu T^{\mu\nu} = 0$. (cf. in EM $\nabla_\mu j^\mu = 0$ and $\nabla_\mu A^\mu = 0$). In Newtonian limit of Minkowski spacetime this equation reduces to Euler's equation and continuity equation using above $T^{\mu\nu}$.

Most simple relationship between curvature and $T^{\mu\nu}$ is $K^{\mu\nu} = \alpha T^{\mu\nu}$. Given Newtonian limit $\alpha = \dfrac{8\pi G}{c^4}$ and $K^{\mu\nu}$ should contain linearly related 2nd order derivatives of the metric tensor. Also symmetry of $T^{\mu\nu}$ (assumed) ⇒ $K^{\mu\nu}$ symmetric.

let $K^{\mu\nu} = a R^{\mu\nu} + b R g^{\mu\nu} + a \underset{\text{constant}}{\Lambda} g^{\mu\nu}$. Now if $\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \nabla_\mu K^{\mu\nu} = 0$

Noting $\nabla_\mu (R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R) = 0 \Rightarrow b = -\tfrac{1}{2} a$. Since $\nabla_\mu g^{\mu\nu} = 0$, Λ can be any constant. For correct weak field limit $a = -1$.

$$\Rightarrow \quad R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = -\frac{8\pi G}{c^4} T^{\mu\nu} \quad *$$

Λ is the cosmological constant. Current observations ⇒ it may be small but certainly non-zero. It currently represents the poorly understood interface between GR and QM.

*Recipe for GR: Given $T^{\mu\nu}$ {energy, matter distribution} and Λ solve * for $R^{\mu\nu}$ (in terms of primitive metric perhaps). From symmetry arguments etc. suggest form of $g^{\mu\nu}$ to simplify this. Use lagrangian procedure on primitive metric to get $\Gamma^a{}_{bc}$ and compare with equation relating $R^d{}_{abc}$ and $\Gamma^a{}_{bc}$. NON LINEAR EQUATIONS! Segments

$T^{\mu\nu}$ ⟶ FIELD EQUATIONS ⟶ $R^{\mu\nu}$

* $\nabla_\mu T^{\mu\nu} = 0$

* $T^{\mu\nu}$ symmetric

guess $g^{\mu\nu}$

$\boxed{R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = -\dfrac{8\pi G}{c^4} T^{\mu\nu}}$

$0 = \dfrac{d^2 x^\mu}{du^2} + \Gamma^\mu_{ab} \dfrac{dx^a}{du} \dfrac{dx^b}{du}$

$L = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \longrightarrow \dfrac{d}{du}\left(\dfrac{\partial L}{\partial \dot{x}^\mu}\right) - \dfrac{\partial L}{\partial x^\mu} = 0$

* $\boxed{R_{\mu\nu} = \partial_\nu \Gamma^\sigma_{\mu\sigma} - \partial_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\mu\nu} \Gamma^\sigma_{\rho\sigma}}$

AIM TO FIND $g_{\mu\nu}$

Schematic of GR process