

GENERAL RELATIVITY is a geometrical theory which describes all physical phenomena on scales above that where quantum mechanics is required!

1) Geometry and notation of GR. In GR model describable world as a 4 dimensional Pseudo-Riemannian MANIFOLD.  $\{$  A Manifold is any set that can be continuously parameterised. # independent parameters needed to uniquely specify a point in the set = Dimension of manifold ; parameters themselves are manifold's coordinates  $\}$ . Hypersurfaces are subsets of a given manifold with dimensions are less than the manifold. These are described by an equation of the form  $f(x^1, x^2, x^3, \dots, x^N) = 0$  for N dimensional manifold

An M dimensional surface is  $\therefore$  described by a system of equations (N-M of them)  $f_a(x^1, x^2, x^3, \dots, x^N) = 0$  where  $a = \{1, \dots, N-M\}$ .

SCALARS are values of points on a given manifold. VECTORS are geometrical objects defined in local cartesian tangent spaces to a point on a given manifold. They are intrinsically dependent on a particular manifold point and can  $\therefore$  only have algebraic properties (such as addition) when one refers to locally similar manifold tangent spaces. (For a cartesian manifold all tangent spaces are coincident and  $\therefore$  effectively local). TENSORS are linear mappings of T vectors to  $\mathbb{R}$ . T = Rank of the tensor. (Note T=0  $\Rightarrow$  scalar, T=1  $\Rightarrow$  vector).

NOTE SYMMETRIC:  $g_{ab} = g_{ba}$

Given coordinates  $x^a$ : a pseudo Riemannian manifold has LINE ELEMENT  $ds^2 = g_{ab} dx^a dx^b$   $ds =$  invariant line element (independent of coordinates).  $\|g_{ab}\| =$  metric.

Note  $\sum$  convention. "sum over repeated indices"   
 i.e.  $g_{ab} dx^a dx^b = \sum_a \sum_b g_{ab} dx^a dx^b$ . LENGTHS, AREAS and VOLUMES are PRM intrinsic properties of a manifold. (Assume from now on a Pseudo Riemannian)

LENGTH  $L_{AB} = \int_A^B ds = \int_A^B |g_{ab} dx^a dx^b|^{1/2}$ . Usually specify some curve between A and B i.e.  $x^a(t)$ .

AREA  $dA = \sqrt{|g_{11} g_{22}|} dx^1 dx^2$  (1,2 are labels of any 2 coordinate subset of manifold coordinates)

$\rightarrow$   $d^N V = \sqrt{|g_{11} g_{22} \dots g_{NN}|} dx^1 dx^2 \dots dx^N$  (generalised) VOLUME. (element).

These assume metric is DIAGONAL  $\Rightarrow$  ORTHOGONAL coordinates. in general VOLUME is  $d^N V = \sqrt{|g|} dx^1 dx^2 \dots dx^N$  where  $g = \det \|g_{ab}\|$ .

A vector  $\underline{v}$  can be described in terms of its contravariant components and basis vectors or equivalently covariant components and dual basis vectors.

i.e.  $\underline{v} = v^a \underline{e}_a = v_a \underline{e}^a$ .  $\underline{e}_a = \lim_{\delta x^a \rightarrow 0} \frac{\delta \underline{s}}{\delta x^a}$  where  $\delta \underline{s}$  is the vector displacement (in the tangent space of point P) between point P and nearby point Q.  $\Rightarrow \underline{ds} = \underline{e}_a dx^a$ .

NOTE: "CO VECTORS BELOW" (HOBSON 2001).

Hence:  $ds^2 \equiv \underline{ds} \cdot \underline{ds} = (\underline{e}_a \cdot \underline{e}_b) dx^a dx^b \Rightarrow g_{ab} = \underline{e}_a \cdot \underline{e}_b$ .

Hence:  $\underline{w} \cdot \underline{v} = g_{ab} w^a v^b$ . (Definition of dot product of vectors in PRM).

Define DUAL BASIS  $\underline{e}^a$  to have the property  $\underline{e}^a \cdot \underline{e}_b = \delta^a_b \equiv \begin{cases} 1 & a=b \\ 0 & a \neq b \end{cases}$

$\therefore \underline{ds} = \underline{e}_a dx^a \Rightarrow \underline{e}^b \cdot \underline{ds} = dx^b$ . Define  $g^{ab} = \underline{e}^a \cdot \underline{e}^b$ .

Using  $\underline{v}$  and  $\underline{w}$  in contra/basis, w dual forms and evaluating all four possible forms of  $\underline{v} \cdot \underline{w} \Rightarrow w_a = g_{ab} w^b$ ;  $v^a = g^{ab} v_b$  (Same for  $\underline{e}^a$  i.e. GR ①)

RAISE AND LOWER INDICES BY METRIC

Now consider  $\delta^a_b \cdot v^b = v^a = \delta^a_b \cdot v^b = v^a = g^a_b v^b = g^a_b v^c = \delta^a_c v^c = \delta^a_c v^c$

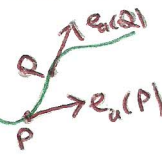
Hence  $\|g_{ab}\|$  and  $\|g^{ab}\|$  are inverses.

Transform coordinates via  $dx^a = \frac{\partial x^a}{\partial x'^b} dx'^b$ . Since  $ds = dx^a e_a = dx'^a e'_a$

$\Rightarrow e'_a = \frac{\partial x^b}{\partial x'^a} e_b$ . Similarly:  $e'^a = \frac{\partial x'^a}{\partial x^b} e^b$ .

The tangent vector to a curve  $x^a(u)$  is  $\underline{t} = \frac{dx^a}{du} e_a$ .  $|\underline{t}| = |g_{ab} \frac{dx^a}{du} \frac{dx^b}{du}|^{1/2}$   
 $= |\frac{ds}{du}|$ .  $\therefore$  For a null curve  $ds=0 \iff \underline{t}$  is zero at all points on the curve.

Note for null curve cannot let  $u=s$  i.e. use  $s$  as a parameter.  
 if points  $P$  and  $Q$  are such that their tangent spaces are coincident



Define AFFINE CONNECTION  $\Gamma^b_{ac}$  by  $\frac{\delta e_a}{\delta x^c} = \lim_{\delta x^c \rightarrow 0} \frac{\delta e_a}{\delta x^c} = \Gamma^b_{ac} e_b$ .  
 $\delta e_a = \Gamma^b_{ac} e_b \delta x^c$ . So  $\partial_c e_a = \Gamma^b_{ac} e_b$

using  $e^a \cdot e_b = \delta^a_b \Rightarrow \Gamma^b_{ac} = e^b \cdot \partial_c e_a$  where  $\partial_c \equiv \frac{\partial}{\partial x^c}$ .

Noting  $\partial_c (e^a \cdot e_b) = 0 \Rightarrow \partial_c e^a = -\Gamma^a_{bc} e^b$ . Affine connection is not a tensor.

However, present  $T^b_{ac} \equiv \Gamma^b_{ac} - \Gamma^b_{ca}$  (TORSION) is. In standard GR basis is not  $(\Rightarrow \partial_c e_a = \partial_a e_c)$

Noting  $\Gamma^a_{bc} = e^a \cdot \frac{\partial e_b}{\partial x^c}$  we show  $\Gamma^a_{bc} = \frac{\partial x'^a}{\partial x^d} \frac{\partial x'^b}{\partial x^c} \frac{\partial x'^d}{\partial x'^e} \Gamma^e_{fg} - \frac{\partial x^d}{\partial x'^b} \frac{\partial x'^f}{\partial x'^c} \frac{\partial^2 x'^a}{\partial x'^d \partial x'^f}$

Now  $\partial_c g_{ab} = \partial_c (e_a \cdot e_b) = \Gamma^d_{ac} g_{db} + \Gamma^d_{bc} g_{ad}$  [1] (cyclically permuting a,b,c)

$\Rightarrow \partial_b g_{ca} = \Gamma^d_{cb} g_{da} + \Gamma^d_{ab} g_{cd}$  [2]  $\partial_a g_{bc} = \Gamma^d_{ba} g_{dc} + \Gamma^d_{ca} g_{bd}$  [3].

[1] + [2] - [3] =  $2\Gamma^d_{cb} g_{ad}$  \*. Multiplying by  $g^{ea}$  and using  $g^{ea} g_{ad} = \delta^e_d$   
 (and relabelling indices)  $\Rightarrow \Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})$  in this case "connection"

\* Using  $\Gamma^a_{bc} = \Gamma^a_{cb}$  define as METRIC CONNECTION ( $\Rightarrow$  torsion = 0).  
 if metric connection = affine !!

Define  $\Gamma_{abc} \equiv g_{ad} \Gamma^d_{bc}$ . Note  $\partial_c g_{ab} = \Gamma_{abc} + \Gamma_{bac}$ .

Now covariant derivative of a vector is  $\nabla_b \underline{v} \equiv (\nabla_b v^a) e_a$ .  
 $\nabla_b \underline{v} = \partial_b (v^a e_a) = e_a \partial_b v^a + v^a \partial_b e_a = e_a (\partial_b v^a + (\Gamma^c_{ba} e_c) v^a)$

$= e_a (\partial_b v^a + v^c \Gamma^a_{bc}) \Rightarrow \nabla_b v^a = \partial_b v^a + v^c \Gamma^a_{bc}$   
 using  $\underline{v} = v_a e^a$  and  $\partial_c e^a = -\Gamma^a_{bc} e^b \Rightarrow \nabla_b v_a = \partial_b v_a - \Gamma^c_{ab} v_c$ .

Similarly INTRINSIC derivative of a vector is  $\frac{d\underline{v}}{du}$  where  $u$  is curve parameterization ( $x^a(u)$  describes curve).

$\frac{d\underline{v}}{du} = e_a \frac{dv^a}{du} + v^a \frac{de_a}{du}$ . By the chain rule  $de_a = \frac{\partial e_a}{\partial x^c} dx^c = \Gamma^b_{ac} e_b dx^c$

$\Rightarrow \frac{d\underline{v}}{du} = e_a \frac{dv^a}{du} + v^a \frac{dx^c}{du} \Gamma^b_{ac} e_b = e_a (\frac{dv^a}{du} + v^b \Gamma^a_{bc} \frac{dx^c}{du})$  (a $\leftrightarrow$ b)

A vector is parallel transported along a curve  $C$  if  $\frac{d\underline{v}}{du} = 0 \Rightarrow \frac{dv^a}{du} = -v^b \Gamma^a_{bc} \frac{dx^c}{du}$ .

using  $\underline{v} = v_a e^a$  and  $\partial_c e^a = -\Gamma^a_{bc} e^b$  find  $\frac{d\underline{v}}{du} = (\frac{dv_a}{du} - \Gamma^b_{ac} v_b \frac{dx^c}{du}) e^a$ .

Geodesics. Define  $u$  to be some AFFINE parameter s.t.  $\frac{dt}{du} = 0$  where  $\underline{t}$  is the tangent vector to some curve  $x^a(u)$ . From above result for  $\frac{d\underline{v}}{du}$  and  $\underline{t} = \frac{dx^a}{du} e_a$

$\Rightarrow$  GEODESIC EQUATION:  $-\frac{d^2 x^a}{du^2} + \Gamma^a_{bc} \frac{dx^b}{du} \frac{dx^c}{du} = 0$  GR (2)

one show that change of coordinates  $x^a \rightarrow x'^a$  does not change form of geodesic equation. let  $u = u(\lambda)$ . using chain rule:  $du = \frac{du}{d\lambda} d\lambda$  can show:

Note geodesic eq. can be written as:  $\ddot{x}^\mu + \Gamma^\mu_{\nu\sigma} \dot{x}^\nu \dot{x}^\sigma = 0$

$$\frac{d^2 x^a}{d\lambda^2} + \Gamma^a_{bc} \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda} = \left( \frac{d^2 u / d\lambda^2}{du/d\lambda} \right) \frac{dx^a}{d\lambda}$$

i.e. linearly related  $\downarrow$

Hence if  $u$  is an affine parameter so is any  $\lambda = au + b$  ( $a, b$  constants).

Lagrangian procedure is a neat way of determining  $\Gamma^a_{bc}$  and geodesic equations from the metric very efficiently, indirectly.

Consider  $L = g_{ab} \dot{x}^a \dot{x}^b$  where  $\dot{x}^a \equiv \frac{d}{du}$  ( $u$  some affine parameter).

Substitution into Euler Lagrange equation  $\frac{d}{du} \left( \frac{\partial L}{\partial \dot{x}^a} \right) - \frac{\partial L}{\partial x^a} = 0$  (\*)

$$\Rightarrow \ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0$$

So writing down  $L$  and substitution into (\*) generates geodesic equation and allows connection coefficients to be picked off.

Note for null geodesics  $ds = 0 \Rightarrow ds^2 = 0 \Rightarrow \left( \frac{ds}{du} \right)^2 = 0 \Rightarrow L = 0$ .

For non null geodesics choose  $u = s \Rightarrow |g_{ab} \dot{x}^a \dot{x}^b| = 1$ . Force term pushes particles off geodesics

Note if  $L = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b - V(x)$ , E.L equation  $\Rightarrow \ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = -g^{ab} \partial_b V$ .

Extension of above vector calculus discussion to Tensor Calculus on Manifolds. (Start with rank 2 tensors and results illustrate progression to higher orders).

If  $\underline{\underline{t}}$  is a rank 2 tensor can write  $\underline{\underline{t}} = t^{ab} \underline{e}_a \otimes \underline{e}_b$  where  $\underline{e}_a \otimes \underline{e}_b$  is the outer product of basis vectors  $\underline{e}_a, \underline{e}_b$ . (Note of convention). As for vectors define covariant derivative by  $\nabla_c \underline{e}_a = \Gamma^b_{ca} \underline{e}_b$  can show

$$\nabla_c t^{ab} = \partial_c t^{ab} + \Gamma^a_{dc} t^{db} + \Gamma^b_{dc} t^{ad}$$

Now can write  $\underline{\underline{t}} = t^a_b \underline{e}_a \otimes \underline{e}^b$

and  $\underline{\underline{t}} = t_{ab} \underline{e}^a \otimes \underline{e}^b$ . using  $\nabla_c \underline{e}^a = -\Gamma^a_{bc} \underline{e}^b$  and  $\nabla_c \underline{e}_a = \Gamma^b_{ca} \underline{e}_b$  can show

$$\nabla_c t^a_b = \partial_c t^a_b + \Gamma^a_{dc} t^d_b - \Gamma^d_{bc} t^a_d ; \nabla_c t_{ab} = \partial_c t_{ab} - \Gamma^d_{ac} t_{db} - \Gamma^d_{bc} t_{ad}$$

Result of this is one can show  $\nabla_c g_{ab} = \nabla_c g^{ab} = 0$ . i.e. covariant derivative of metric is zero for any coordinate basis. This result  $\Rightarrow \nabla_c$  and raising/lowering operators' ( $\times g^{ab}$  or  $g_{ab}$ ) commute without affecting the result. i.e.  $\nabla_c (g^{ab} t^d_b) = g^{ab} \nabla_c t^d_b$

using  $\underline{\underline{t}} = t^{ab} \underline{e}_a \otimes \underline{e}_b$  total derivative of  $\underline{\underline{t}}$  along a curve  $x^a(u)$  can be found:

AND SIMILAR EXPRESSIONS INVOLVING  $t^a_b$  and  $t_{ab}$

$$\frac{d\underline{\underline{t}}}{du} = \left( \frac{dt^{ab}}{du} + \Gamma^a_{dc} t^{db} \frac{dx^c}{du} + \Gamma^b_{dc} t^{ad} \frac{dx^c}{du} \right) \underline{e}_a \otimes \underline{e}_b$$

$\underline{\underline{t}}$  can be // transported along a curve  $\Rightarrow \frac{d\underline{\underline{t}}}{du} = 0$ .

To be and not to be a tensor.... A tensor's components must satisfy the following transformation properties - this indeed is the defining property of a tensor!

- \* Each superscript inherits a transformation matrix  $\frac{\partial x'^a}{\partial x^c}$
- \* " Subscript (a) " " " "  $\frac{\partial x^c}{\partial x'^a}$

i.e.  $t'^c_{ab} = \frac{\partial x^d}{\partial x'^a} \frac{\partial x^e}{\partial x'^b} \frac{\partial x^c}{\partial x'^f} t_{def}$

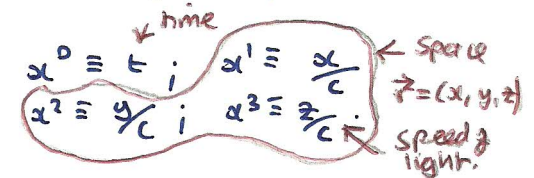
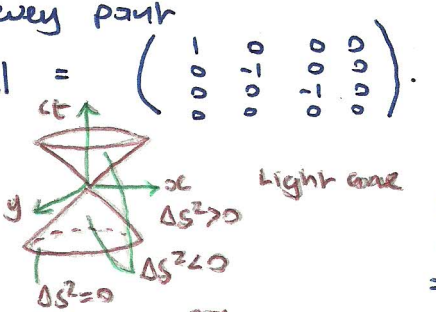
Quotient theorem: "Tensor contracted with test quantity = tensor"  $\Rightarrow$  test quantity = tensor. i.e. if  $a^{\alpha\gamma}$  and  $b_{\beta\delta}$  are components of tensors  $\underline{a}, \underline{b}$   $a^{\alpha\gamma} b_{\beta\delta} = z^{\alpha\gamma p q} b_{p q} \Rightarrow z^{\alpha\gamma p q}$  are components of a 4th rank tensor  $\underline{z}$

Why are tensors useful? TENSOR EQUATIONS  $\underline{t}_{ab} = S_{ab}$  are valid in all coordinate systems. So if can prove a result (like  $S_{ab} = 0$ ) in a simple coordinate system (i.e. cartesian) you will have proved a GENERAL result.

Special Relativity and Electrodynamics.  $\Rightarrow$  a Minkowski spacetime.  $\Rightarrow$  a Riemannian manifold which has the following line element at every point

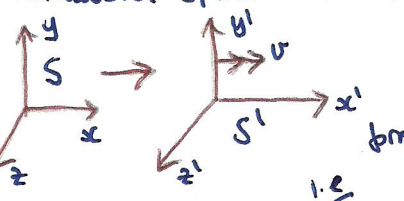
$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad \|\eta_{\mu\nu}\| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- if  $ds^2 > 0$  : timelike interval
- $ds^2 = 0$  : null "
- $ds^2 < 0$  : spacelike interval



Note orthogonality of basis vectors in Minkowski spacetime  $\Rightarrow$  all tangent spaces are coincident. Hence can define position vector  $\underline{R} = (ct, 0, 0, z)$ .

Consider two inertial frames in Minkowski spacetime related by a boost  $v$  along  $x$ .



Now since  $\eta_{\mu\nu}$  has constant elements  $\Rightarrow$  under transformations  $S \rightarrow S'$ ,  $x^\mu$  must transform LINEARLY to preserve  $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$  formalism.

i.e. let  $t' = At + Bx$  ;  $x' = Dt + Ex$  ;  $y' = y$  ;  $z' = z$ .

Now when  $x' = 0$ ,  $x = vt$ .  $\therefore Dt + Evt = 0 \Rightarrow D = -Ev$ . when  $x=0$ ,  $x' = -vt'$   
 $\Rightarrow -vt' = Dt \Rightarrow t' = A(-vt'/0) \Rightarrow 0 = -Av$ . Hence  $E = A$ .

$\therefore t' = At + Bx$  ;  $x' = A(x - vt)$  ;  $y' = y$  ;  $z' = z$ . Now for a proper interval

$ds^2 = 0$ .  $\therefore$  using metric:  $dt'^2 - \frac{1}{c^2}(dx'^2 + dy'^2 + dz'^2) = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$

Substituting above results:  $A^2 dt^2 + B^2 dx^2 + 2AB dt dx - \frac{1}{c^2}(A^2 dx^2 + A^2 v^2 dt^2 - 2A^2 v dx dt)$   
 $= dt^2 - \frac{1}{c^2} dx^2$ . Comparing coefficients of  $dt^2$ :  $A^2 - \frac{A^2 v^2}{c^2} = 1 \Rightarrow A = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$

Comparing coefficients of  $dx dt$ :  $2AB + \frac{1}{c^2} 2A^2 v = 0 \Rightarrow B = -\frac{Av}{c^2}$ .

Defining  $\gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \Rightarrow t' = \gamma(t - \frac{vx}{c^2})$  ;  $x' = \gamma(x - vt)$  ;  $y' = y$  ;  $z' = z$ .

This is the LORENTZ TRANSFORM. can write in matrix form { Note rapidity  $\phi = \tanh^{-1} \beta$  - often used }.

$$\underline{R}' = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \underline{R} \quad \text{where } \beta = \frac{v}{c}$$

Now POINCARÉ transformations are generalizations of Lorentz and involve a shift of the origin. one may also consider 3 vector notation.

Lorentz transformations between frames moving with relative velocity  $\beta c$ .  $(t, \vec{r}, \vec{\beta})$  only available quantities (most general forms).  
 $\Rightarrow ct' = a_1 ct + a_2 \vec{\beta} \cdot \vec{r}$  and  $\vec{r}' = b_1 \vec{r} + b_2 ct \vec{\beta} + b_3 (\vec{\beta} \cdot \vec{r}) \vec{\beta}$

Now if  $\vec{\beta} = (\beta, 0, 0)$  must generate L.T. derived above.  $\Rightarrow a_1 = \gamma$ ,  $a_2 = -\gamma$ ,  $b_1 = 1$ ,  $b_2 = -\gamma$ ,  $b_3 = \frac{\gamma-1}{\beta^2}$

So general Lorentz transformation matrix is:  $\{\alpha = \frac{\gamma-1}{\beta^2}\}$   

$$\begin{pmatrix} ct' \\ \vec{r}' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+\alpha\beta_x^2 & \alpha\beta_x\beta_y & \alpha\beta_x\beta_z \\ -\gamma\beta_y & \alpha\beta_y\beta_x & 1+\alpha\beta_y^2 & \alpha\beta_y\beta_z \\ -\gamma\beta_z & \alpha\beta_z\beta_x & \alpha\beta_z\beta_y & 1+\alpha\beta_z^2 \end{pmatrix} \begin{pmatrix} ct \\ \vec{r} \end{pmatrix}$$
 or:  $ct' = \gamma(ct - \vec{\beta} \cdot \vec{r})$   
 $\vec{r}' = \vec{r} - \gamma ct \vec{\beta} + \frac{\gamma-1}{\beta^2} (\vec{\beta} \cdot \vec{r}) \vec{\beta}$

From Lorentz transform physical effects such as length contraction, time dilation can be described. Use L.T. to test whether a 4 component geometrical entity is a 4-vector. i.e. All 4 vectors obey L.T. Define proper time  $\tau$  s.t.  $ds = c d\tau$ . (rest frame of particle -  $|\vec{dr}'| = 0$ ). From metric  $dt^2 = d\tau^2 - \frac{\vec{u} \cdot \vec{u} d\tau^2}{c^2} \Rightarrow \frac{d\tau}{dt} = \frac{1}{\gamma}$  ( $\vec{u} = \frac{d\vec{r}}{dt}$ ). Given this result one can generate 4 velocity  $\underline{u} = \frac{d\underline{R}}{d\tau}$  and 4 acceleration  $\underline{a} = \frac{d\underline{u}}{d\tau}$ . { useful<sup>ex.</sup> is to apply L.T. to  $\underline{u}$ ,  $\underline{a}$  or get correct velocity, acceleration addition formulae. Note by construction  $\underline{u}$ ,  $\underline{a}$  are 4 vectors }.

Define 4-momentum  $\underline{p} = m_0 \underline{u}$  ( $m_0$  rest mass - assume Lorentz invariant).

and  $4 - \text{br} \underline{f} = \frac{d\underline{p}}{dt}$  is the relativistic equivalent of motion. when  $\underline{f} = 0$   
 $\underline{p} = \text{constant}$ . Newtonian limit of components of  $\underline{p}$  imply  $\underline{p} = (\frac{E}{c}, \vec{p})$  where  
 $E = \text{energy of system described by } \underline{p}$  & can include potential terms and  $\vec{p}$  the 3  
momentum.

**SEE RELATIVITY AND ELECTRODYNAMICS COURSE!!**

Now for any two 4 vectors  $\underline{a}, \underline{b}$  can show  $\underline{a} \cdot \underline{b} = \eta_{\mu\nu} a^\mu b^\nu$  is Lorentz  
invariant. Idea is to evaluate products in frames where components are mostly zeros.

$\underline{u} \cdot \underline{u} = c^2$  since  $\underline{u} = (\gamma c, \vec{u})$ . (Pick frame where  $\vec{u} = 0$ )  $\underline{p} \cdot \underline{p} = m_0^2 c^2$   
 $\Rightarrow \gamma = 1$

If no potential energy terms.  $\therefore E^2 - p^2 c^2 = m_0^2 c^4$  ENERGY-MOMENTUM INVARIANT.

Four Force  $\underline{f} = \gamma_u (\vec{f} \cdot \frac{\vec{u}}{c}, \vec{f})$ . Note  $\vec{f} \cdot \vec{u} = \frac{dE}{dt}$ .  $\vec{f} = \frac{d\vec{p}}{dt}$  ← CLASSICAL MECHANICS

$\Rightarrow \gamma_u \frac{\vec{f} \cdot \vec{u}}{c} = \frac{dE}{dt} \frac{1}{c}$ ;  $\gamma_u \vec{f} = \frac{d\vec{p}}{dt}$   $\Rightarrow \underline{f} = \frac{d\underline{p}}{dt}$  as required. Note  $\underline{R} = (\frac{2\pi}{T}, \vec{0})$   
is another handy 4 vector

Consider  $\underline{u} \cdot \underline{f} = \underline{u} \cdot \frac{d\underline{p}}{dt} = \underline{u} \cdot (\frac{dm_0}{dt} \underline{u} + m_0 \frac{d\underline{u}}{dt}) = c^2 \frac{dm_0}{dt} + m_0 \underline{u} \cdot \frac{d\underline{u}}{dt}$

$\frac{d}{dt} (\underline{u} \cdot \underline{u}) = 2 \underline{u} \cdot \frac{d\underline{u}}{dt} \Rightarrow \underline{u} \cdot \frac{d\underline{u}}{dt} = \frac{1}{2} \frac{d}{dt} (\underline{u} \cdot \underline{u}) = \frac{1}{2} \frac{d}{dt} (c^2) = 0$ .  $\therefore \underline{u} \cdot \underline{f} = c^2 \frac{dm_0}{dt}$

A PURE FORCE is one where  $m_0$  is invariant with proper time  $\Rightarrow \underline{u} \cdot \underline{f} = 0$ .  
in this case  $\underline{f} = m_0 \underline{a}$  describes dynamics. can write in form  $f^\nu \underline{e}_\nu = m_0 \underline{a}$

$\Rightarrow \underline{a} = \frac{f^\nu}{m_0} \underline{e}_\nu$  Now  $\underline{a} = \frac{d\underline{u}}{dt} = \underline{e}_\nu (\frac{du^\nu}{dt} + u^\alpha \Gamma_{bc}^\alpha \frac{dx^c}{dt}) = \underline{e}_\nu (\frac{du^\nu}{dt} + \Gamma_{bc}^\nu u^\alpha u^\alpha)$

$\Rightarrow \frac{d^2 x^\nu}{dt^2} + \Gamma_{bc}^\nu \frac{dx^b}{dt} \frac{dx^c}{dt} = \frac{f^\nu}{m_0}$ . i.e. forces push particles of geodesics.

**geodesic equation**

Now ELECTROMAGNETISM effectively results in a particular

4 force  $\underline{f}$ . In classical mechanics the Lorentz force  $\vec{f} = q(\vec{E} + \vec{u} \times \vec{B})$  describes  
the force resulting on a particle with LORENTZ INVARIANT charge  $q$  in the presence  
of electric and magnetic fields  $\vec{E}$  and  $\vec{B}$  while moving with 3 velocity  $\vec{u}$ .

let's  $\therefore$  postulate  $f^\nu = q F_{\mu}^{\nu} u^\mu$  where  $\underline{F}$  is the EM FIELD STRENGTH TENSOR.

Now let  $\underline{f}_{em}$  be a pure force  $\Rightarrow \underline{u} \cdot \underline{f} = 0 \Rightarrow f_\mu u^\mu = 0 \Rightarrow q F_{\mu\nu} u^\nu u^\mu = 0$

$\Rightarrow F_{\mu\nu}$  must be antisymmetric i.e.  $F_{\mu\nu} = -F_{\nu\mu}$ . Now if  $\underline{J}_0$  is rest frame

charge density (assume Lorentz invariant)  $\underline{j} = \underline{J}_0 \underline{u}$  is a 4-vector. with  
Maxwell's equations as a (known) goal aim to relate  $\underline{F}$  and  $\underline{j}$ . Simplest relation

is  $\nabla \cdot \underline{F} = \mu_0 \underline{j}$  ( $\mu_0$  is assigned arbitrarily but one can show it indeed is the same  $\mu_0$   
as in Maxwell's equations). So  $\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu$ . Note  $\nabla_\nu \nabla_\mu F^{\mu\nu} = \mu_0 \nabla_\nu j^\nu = 0$

Since  $\nabla_\nu \nabla_\mu F^{\mu\nu} = \frac{1}{2} \nabla_\nu \nabla_\mu F^{\mu\nu} + \frac{1}{2} \nabla_\nu \nabla_\mu F^{\mu\nu} = \frac{1}{2} \nabla_\nu \nabla_\mu F^{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu F^{\mu\nu} = 0$ . (By symmetry  
of  $g_{ab}, g^{ab}$ :  $F_{\mu\nu}$  antisymmetric  $\Rightarrow F^{\mu\nu}$  antisymmetric).

**CONTINUITY OF CHARGE.**

So:  $\nabla_\nu j^\nu = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$  with  $\underline{j} = (\rho c, \vec{j})$  (in SR.)  
GENERAL SR

Now define vector potential  $\underline{A}$  s.t.  $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

$\Rightarrow \nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} + \nabla_\mu F_{\nu\sigma} = 0$ . Now  $F^\nu_\mu = -g^{\nu\alpha} F_{\mu\alpha} = g^{\alpha\nu} F_{\alpha\mu}$

$\therefore \nabla_\mu F^{\mu\nu} = \mu_0 j^\nu \Rightarrow \nabla_\mu (g^{\alpha\nu} F_{\alpha\mu}) = \mu_0 g^{\alpha\nu} j^\nu \Rightarrow \nabla_\mu F^{\mu\alpha} = \mu_0 j^\alpha \Rightarrow \nabla_\mu (g^{\mu\nu} F_{\alpha\nu}) = \mu_0 j^\alpha$

$\Rightarrow g^{\mu\nu} (-\nabla_\mu \nabla_\alpha A_\nu + \nabla_\mu \nabla_\nu A_\alpha) = \mu_0 j^\alpha$ . Now Lorentz gauge state  $\nabla_\mu A^\mu = 0$  scalar field

{ if  $A_\mu \rightarrow A_\mu + \partial_\mu \phi$ :  $F_{\mu\nu}$  unchanged provided  $\nabla_\mu \partial_\nu = \nabla_\nu \partial_\mu$ . let  $\partial_\mu = \partial_\mu \phi$ . ✓ OK  
choose scalar field s.t.  $\nabla_\mu A^\mu = 0$  } IN this case  $g^{\mu\nu} \nabla_\mu \nabla_\nu A_\alpha = \mu_0 j^\alpha$  OR ⑤

