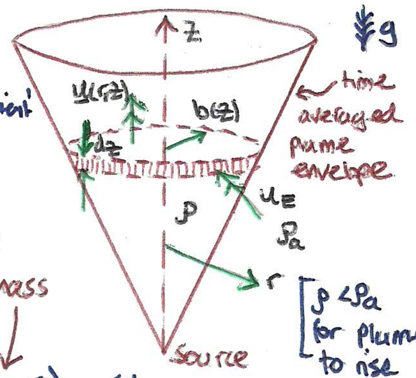
# NATURAL AND FORCED FLOWS INVOLVING BUOYANCY

– fluid flows which occur in a natural environment or industrial setting and involve the interaction of fluids of different density. Density contrasts may arise from differences in temperature, concentration of a particular quantity.... – High Reynolds # flows (Plumes, gravity currents) – Low Reynolds # flows (gravity driven flows and plumes in porous media).

**Turbulent, high Re plumes:** $Re = \frac{\rho u L}{\eta} > 10^3 - 10^4$ [We will consider axisymmetric plumes, derivation is analogous for line plumes, though results different]

A plume is a vertical buoyancy driven flow, so named because of a conical time averaged geometry and bulbous head (if finite height). Consider an axisymmetric plume emanating from $z=0$ consisting of isotropically mixed, fully turbulent fluid of density $\rho(z)$. Let external/ambient fluid outside the time averaged plume envelope $b(z)$ ← (radius) be of density $\rho_a(z)$ and be radially entrained into plume at $b(z)$ at velocity $u_E(z)$. Let plume velocity field (time averaged) be $u(r,z)\hat{z}$ for $0 < r \leq b(z)$.




[$\rho < \rho_a$ for plume to rise]

C has units ? / unit mass

Relate plume density to ambient and difference in some conserved quantity C. (i.e. salt, particles, bubbles, heat.....)

$$\rho_a - \rho = \rho_a \beta (C_a - C) \quad (1)$$

Define useful quantities: Flowrate $Q = \int_0^{b(z)} u(r,z) \cdot 2\pi r \, dr$ (2) $\quad g' = g\left(\frac{\rho - \rho_a}{\rho_a}\right)$ (3)

Buoyancy flux $B = g'Q$ (4)

Assert 'seperable' velocity field $u(r,z) = v(r) w(z)$ (5). (i.e. $v(r)$ Gaussian $v(r) = e^{-\frac{r^2}{2b^2}}$; Top hat $v(r) = \begin{cases} 1 & r \leq b \\ 0 & r > b \end{cases}$).

To characterise the plume would like to find explicit z dependance of $w(z)$, $b(z)$. Use CONSERVATION LAWS to find this. Consider thin section of plume illustrated above.

MASS CONSERVATION $\int_0^{b(z)} \rho(z) u(r,z) \cdot 2\pi r \, dr = \int_{z'=0}^{z} u_E(z) \rho_a(z) \cdot 2\pi b(z) \, dz + Q_0 \rho_0$ (6) ↑source term

" Flux of mass is balanced by fluid entrained at boundary b at velocity $u_E$"

MOMENTUM CONSERVATION $d\left(\int_0^{b(z)} \rho(z) u^2(r,z) \cdot 2\pi r \, dr\right) = (\rho_a - \rho) \cdot \pi b^2 g \, dz$ (7)

" rate of change of momentum = imbalance of Archimedian upthrust and gravitational force".

"C" CONSERVATION $\int_0^{b} \rho(z) u(r,z) C(z) \cdot 2\pi r \, dr = \int_{z'=0}^{z} u_E(z) \rho_a(z) C_a(z) \cdot 2\pi b(z) \, dz + C_0 \rho_0 Q_0$ (8)

Same for "C"

For closure hypothesis relate entrainment velocity $u_E(z)$ to plume velocity at $b(z) = r$. i.e. $u_E(z) = \varepsilon u(b,z)$ (†). $\varepsilon$ is entrainment coefficient. (Typically $\sim 0.1$).

Using a Top Hat velocity field i.e. $u(r,z) = \begin{cases} w(z) & r \leq b(z) \\ 0 & r > b(z) \end{cases}$, conservation laws become: (in differential form)

⇒ MASS $\frac{d}{dz}(\rho w b^2) = 2\varepsilon w b \rho_a$ (9)

MOM. $\frac{d}{dz}(w^2 b^2 \rho) = \rho_a \beta (C_a - C) b^2 g$ (10)

"C". $\frac{d}{dz}(\rho w b^2 C) = 2\varepsilon w b \rho_a C_a$ (11)

Note closure hypothesis (†) may be extended for certain "Non-Boussinesq" plumes to $u_E = \varepsilon w(z) (\rho/\rho_a)^{\gamma}$. (12)

Now from (9) $C_a \frac{d}{dz}(\rho w b^2) = 2\varepsilon w b \rho_a C_a$. (12).

Consider $\frac{d}{dz}(\rho w b^2 [C_a - C]) = -\frac{d}{dz}(\rho w b^2 C) + \frac{d}{dz}(\rho w b^2 C_a) = -2\varepsilon w b \rho_a C_a + C_a \frac{d}{dz}(\rho w b^2) + \rho w b^2 \frac{d}{dz} C_a$.
(11)↑

From (12) ⇒ $\frac{d}{dz}(\rho w b^2 [C_a - C]) = \rho w b^2 \frac{dC_a}{dz}$ (13)

Now using (1, 13); $B = -g\beta(C_a - C) Q$. Using top hat velocity field: $Q = w\pi b^2$ (14)

∴ $B = -\pi g \beta (C_a - C) w b^2$ (15). Application of "Boussinesq approximation", that is: $\rho \approx \rho_a$ in all terms apart from those involving $\rho - \rho_a$ (i.e. (1) is used) ⇒ (9), (10), (13) become ≈ constant

$$\frac{d}{dz}(wb^2) = 2\varepsilon w b \quad (16) \qquad \frac{d}{dz}(w^2 b^2) = \beta g (C_a - C) b^2 \quad (17) \qquad \frac{d}{dz}(wb^2(C_a - C)) = wb^2 \frac{dC_a}{dz} \quad (18)$$

(16) – (18) can be recast in terms of variables $Q = \pi w b^2$, $M = w^2 b^2 \pi$ and $B = \pi g \beta w b^2 (C_a - C)$.

(16): $\frac{dQ}{dz} = 2\pi\varepsilon M^{\frac{1}{2}}$ (17): $\frac{dM}{dz} = \frac{-1}{\pi^2}\frac{BQ}{M}$ (18): $\frac{dB}{dz} = -Qg\beta\frac{dC_a}{dz}$ let $N^2 = g\beta\frac{dC_a}{dz}$

(N = Brunt-Vaisala frequency)

∴ Plume equations are: (I) $\frac{dQ}{dz} = 2\ \varepsilon M^{\frac{1}{2}}$

(II) $\frac{dM}{dz} = -\ \frac{BQ}{M}$ (III) $\frac{dB}{dz} = -N^2 Q$