Algebra with matrices

Matrices are a way of collecting numbers together in form of grid or 'array' of R rows and C columns. Matrices themselves have a algebra (i.e. rules of combination) which is similar to the algebra of numbers in many ways, but different in others.

Matrix algebra is very useful as it enables many calculations to be performed simultaneously (i.e. 'in parallel') with a computer, resulting in major speed improvements. Matrices are used to move shapes around a screen (e.g. rotations, reflections, enlargements etc), so you will be performing matrix calculations every time you use Windows, Android, IOS etc, or play a computer game!

Different matrices

$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ 2 rows, 1 columns. This is called a *column vector*

$\begin{pmatrix} 1 & 0 & -5 \end{pmatrix}$ 1 rows, 3 columns. This is called a *row vector*

$\begin{pmatrix} 4 \end{pmatrix}$ 1 rows, 1 columns. This is called a *scalar* i.e. a single number.

$\begin{pmatrix} 2 & 3 \\ a & 7 \end{pmatrix}$ 2 rows, 2 columns.

$\begin{pmatrix} -k & 12 & 3 \\ 1 & 0 & b \end{pmatrix}$ 2 rows, 3 columns.

Rules of algebra with numbers

$a+b=b+a$ e.g. $2+3=3+2=5$

$a-b=-b+a$ e.g. $2-3=-3+2=-1$

$k(a+b)=ka+kb$ e.g. $3(2+3)=6+9=15$

$ab=ba$ e.g. $2\times 3=3\times 2$

This is called *commutation.*

Rules of algebra with matrices

NOTE: matrices *must* be the **same size** to add or subtract

Addition, subtraction and multiplication by a scalar are **'element-wise'**. i.e. we perform arithmetic on the numbers individually based upon their location within the matrix.

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A} \quad \text{e.g.} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}+\begin{pmatrix} -5 & 2 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} -4 & 4 \\ 3 & 5 \end{pmatrix}$$

$$\mathbf{A}-\mathbf{B}=-\mathbf{B}+\mathbf{A} \quad \text{e.g.} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}-\begin{pmatrix} -5 & 2 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 6 & 0 \\ 3 & 3 \end{pmatrix}$$

$$k(\mathbf{A}+\mathbf{B})=k\mathbf{A}+k\mathbf{B} \quad \text{e.g.}\ 2\left\{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}+\begin{pmatrix} -5 & 2 \\ 0 & 1 \end{pmatrix}\right\}=\begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}+\begin{pmatrix} -10 & 4 \\ 0 & 2 \end{pmatrix}=\begin{pmatrix} -8 & 8 \\ 6 & 10 \end{pmatrix}$$

$$\mathbf{C}(\mathbf{A}+\mathbf{B})=\mathbf{CA}+\mathbf{CB}$$

$$\mathbf{AB}\neq\mathbf{BA} \quad \text{e.g.} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -5 & 2 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} -5 & 4 \\ -15 & 10 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -5 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}=\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$$

But matrix multiplication is **different** and, in general, *non-commutative*.

Matrix multiplication

They may seem rather strange, but the rules of matrix multiplication are *why* matrices have such useful applications.
As an aide-memoir, *'go along the rows and down the columns.'*

$$\underset{2\times 2}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} = \underset{2\times 3}{\begin{pmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{pmatrix}}$$

$$\underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}\underset{1\times 3}{\begin{pmatrix} a & b & c \end{pmatrix}} = \underset{3\times 3}{\begin{pmatrix} a & b & c \\ 2a & 2b & 2c \\ 3a & 3b & 3c \end{pmatrix}}$$

Dimensions of each matrix
i.e. rows x columns

$$\underset{1\times 3}{\begin{pmatrix} a & b & c \end{pmatrix}}\underset{3\times 1}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} = \underset{1\times 1}{a+2b+3c}$$

$$\underset{3\times 2}{\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & -2 \end{pmatrix}}\underset{2\times 5}{\begin{pmatrix} 1 & 0 & 2 & -1 & 3 \\ 0 & -1 & 0 & 2 & 2 \end{pmatrix}} = \underset{3\times 5}{\begin{pmatrix} 1 & -1 & 2 & 1 & 5 \\ 2 & 0 & 4 & -2 & 6 \\ -1 & 2 & -2 & -3 & -7 \end{pmatrix}}$$

If matrix **A** has dimensions $R_A \times C_A$

and matrix **B** has dimensions $R_B \times C_B$

AB is *only allowed if* $C_A = R_B$

The resulting matrix has dimensions $R_A \times C_B$

Note the (2 x 2) **identity** matrix is defined as:

$$\mathbf{AA}^{-1} = \mathbf{I}$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

'Matrix division' i.e. use of the inverse

$$\mathbf{A} \div \mathbf{B} \quad \text{means} \quad \mathbf{AB}^{-1}$$

$ad - bc$ is called the **determinant** of the 2 x 2 matrix. If it equals zero there can be no inverse. A matrix with a determinant of zero is called a *singular* matrix.

For a 2 x 2 matrix: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

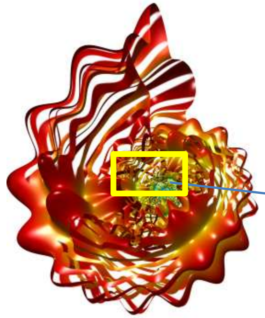
e.g. $$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}^{-1} = \frac{1}{(1)(4)-(2)(-3)}\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix} = \tfrac{1}{10}\begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{pmatrix}$$

There are more complicated recipes for working out the inverses of matrices of higher dimensions. (e.g. 3 x 3 ..... N x N).
Note to have an inverse, the matrix must be *square*. i.e. number of rows = number of columns.

(Some) applications of matrices

Storage of information

A 12 mega-pixel digital photograph of aspect ratio 3:4 consists of three matrices of 3000 x 4000 square pixels. Each matrix contains a intensity for Red, Green or Blue colou
The intensity is typically a number between 0 and 255. Hence 'pure red' has a red value of 255, a green value of 0 and a blue value of 0.
There are 28 = 256 possible intensities for red, green or blue, so (28)3 = 224 = 16.7 million possible colours overall. ('24 bit colour depth').



Zooming shows the pixel detail

Solving systems of simultaneous equations

$$x + 2y - z = 5$$
$$2x - 3y + z = -1$$
$$-x + y + 2z = 3$$

We can write a set of simultaneous linear equations as a matrix multiplication, which can be solved by multiplying by the inverse.

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$$

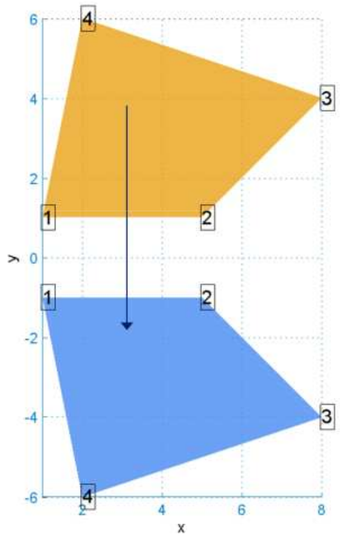
This was performed using MATLAB's recipe for computing the inverse of the 3 x 3 matrix

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ -1 & 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2.0625 \\ 2.1875 \\ 1.4375 \end{pmatrix}$$

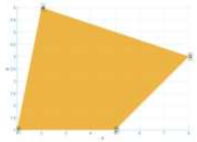
MATLAB ('MATrix LABoratory') is a computer programming language which is based upon matrix algebra.

MATLAB

Transforming 2D shapes via matrix multiplication



$$\mathbf{V} = \begin{pmatrix} 1 & 5 & 8 & 2 \\ 1 & 1 & 4 & 6 \end{pmatrix}$$



In this case a reflection in the x axis

$$\mathbf{W} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 5 & 8 & 2 \\ 1 & 1 & 4 & 6 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 5 & 8 & 2 \\ -1 & -1 & -4 & -6 \end{pmatrix}$$

Multiplying by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

means 'reflect in the x axis', regardless of the number of vertices of the shape.