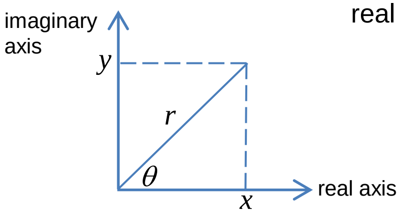
Complex numbers are a 'geometric' extension to the number system. If a real number can be represented on a horizontal number line, a complex number has an *additional vertical part*. This is known as the *imaginary axis*. A general complex number z can therefore be represented as an *(x,y)* coordinate on the *Complex Plane*, or *Argand Diagram.*



The fundamental difference between a complex number and a two dimensional vector is *the algebraic relationship* between the real and imaginary parts of z

$$\boxed{\begin{aligned} z &= x+iy \\ i^2 &= -1 \end{aligned}}$$

Note the algebra of complex numbers is the same as real numbers. They are not *non-commutative* like matrices

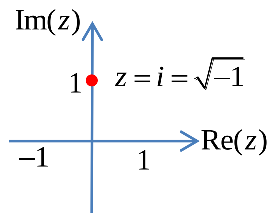
$z_1 \times z_2 = z_2 \times z_1$ Commutation

$z_1 \times (z_2 + z_3) = z_1 \times z_2 + z_1 \times z_3$ Distributive

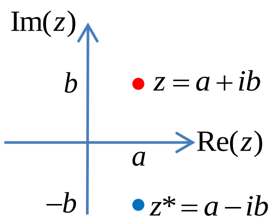
This definition enables complex numbers to represent *generalizations* of a *huge variety* of mathematical objects. The most obvious is the ability to geometrically represent the square root of minus 1.

Definitions:

$$\boxed{\begin{aligned} &z = x+iy \\ &i^2 = -1 \\ &x = \mathrm{Re}(z) \\ &y = \mathrm{Im}(z) \\ &r = \sqrt{x^2+y^2} = |z| \\ &\theta = \tan^{-1}\tfrac{y}{x} = \arg(z) \\ &\mathrm{conj}(z) = z^* = \bar{z} = x - iy \end{aligned}}$$



Conjugate. Reverse the sign of the imaginary part.



The conjugate has useful properties

$$zz^* = (a+ib)(a-ib)$$

$$zz^* = a^2 + iab - iab - i^2b^2$$

$$zz^* = a^2 + b^2$$

$$\boxed{zz^* = |z|^2}$$

$$\boxed{\frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2}}$$

$$\therefore \boxed{\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}}$$

e.g. $\dfrac{1}{4-3i} = \dfrac{4+3i}{4^2+3^2} = \boxed{\tfrac{1}{25}(4+3i)}$

This means we can now represent the general solution to a quadratic equation even when the *discriminant* $b^2 - 4ac < 0$

$$az^2 + bz + c = 0$$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Assume all coefficients *a,b,c* are real numbers

$$b^2 - 4ac < 0$$

$$\Delta = -(b^2 - 4ac) \quad \therefore \Delta > 0$$

$$\therefore z = \frac{-b \pm \sqrt{\Delta}\sqrt{-1}}{2a} \Rightarrow \boxed{z = \frac{-b \pm i\sqrt{\Delta}}{2a}}$$

So if the discriminant is zero then *roots of the quadratic come in conjugate pairs*.

This is a special case of *The Fundamental Theorem of Algebra* that is, the roots of an n^{th} order polynomial come in *n*/2 *conjugate pairs* if *n* is even, and *conjugate pairs and one real root* if *n* is odd.

Note for the quadratic, if the discriminant is greater than zero, the conjugate pair simply means a +/- pair of real roots.

Complex numbers can enable any (integer) polynomial to be written in factorized form. Consider a cubic with roots

$x = a, b+ic, b-ic$ i.e. with only one real root

$$y = (x-a)(x-b-ic)(x-b+ic) = x^3 - Ax^2 + Bx - C$$

a,b,c are defined to be real numbers i.e. $a,b,c \in \mathbb{R}$

	x	$-b$	$-ic$
x	x^2	$-bx$	$-icx$
$-b$	$-bx$	b^2	ibc
ic	icx	$-ibc$	$-i^2c^2$

$$\therefore y = (x-a)(x^2 + b^2 + c^2 - 2bx)$$

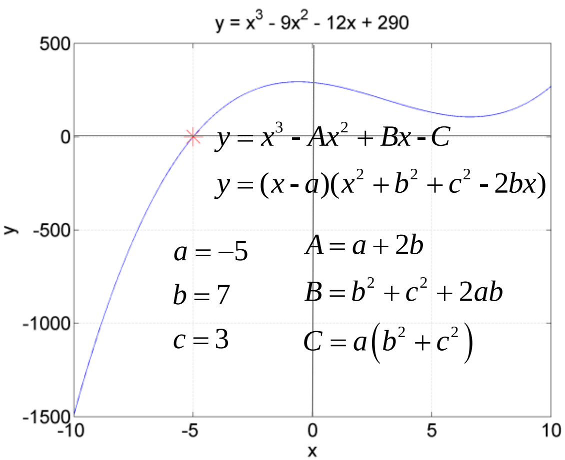
$$y = x^3 - x^2(a+2b) + x(b^2 + c^2 + 2ab) - a(b^2 + c^2)$$

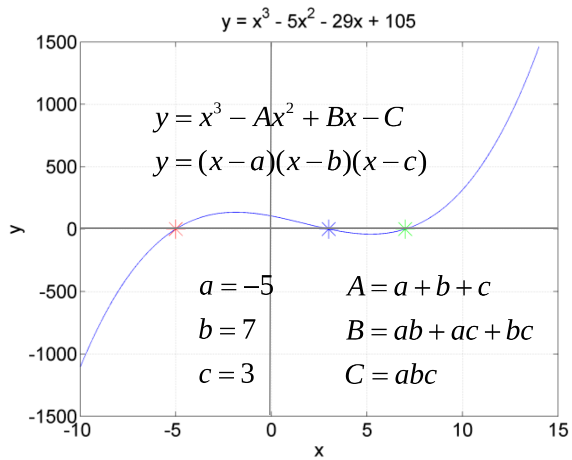
So *all coefficients A,B,C of the polynomial are real* if two of the roots are in conjugate pairs. Can we find *a,b,c* in terms of *A,B,C* ? If so we have a *cubic formula for solving y = 0.*

$$\boxed{\begin{aligned} A &= a + 2b \\ B &= b^2 + c^2 + 2ab \\ C &= a(b^2 + c^2) \end{aligned}}$$

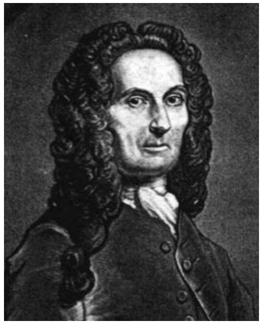
Unfortunately to solve these equations requires firstly solving a cubic in one of the unknowns *a,b,c*, so this *does not represent a good method for solving a cubic*. We also need to know *a priori* that the cubic only has one real root! However, if we *know a,b,c* then we can *construct* a cubic with one real root.

[Continued on next page...]



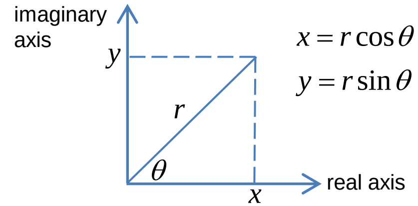


De Moivre's Theorem, powers of complex numbers, trigonometry & exponentials



Abraham de Moivre
1667-1754

Consider writing a complex number z in terms of the polar angle θ and range r in the Argand Diagram. Note the polar angle is in *radians*.



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$z = x + iy$$
$$\therefore \boxed{z = r\left(\cos\theta + i\sin\theta\right)}$$

Constructing a cubic with only one real root

$$\boxed{\begin{aligned} &y = (x-a)(x-b-ic)(x-b+ic) \\ &y = x^3 - Ax^2 + Bx - C \end{aligned}}$$

So if $c = 0$ we have a *two real roots, with one 'repeated'*

$$\boxed{\begin{aligned} &A = a + 2b \\ &B = b^2 + 2ab \\ &C = ab^2 \\ &y = (x-a)(x-b)^2 \\ &y = x^3 - (a+2b)x^2 + (b^2 + 2ab)x - ab^2 \end{aligned}}$$

For *three real roots a,b,c* we need an alternative formulation

$$\boxed{y = (x-a)(x-b)(x-c)}$$
$$y = (x-a)(x^2 - (b+c)x + bc)$$
$$y = x^3 - x^2(a+b+c) + x(ab+ac+bc) - abc$$

i.e. $\boxed{y = x^3 - Ax^2 + Bx - C}$ $\boxed{\begin{aligned} &A = a+b+c \\ &B = ab+ac+bc \\ &C = abc \end{aligned}}$

Now consider the following *Maclaurin Expansions* of cosine and sine functions

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ...$$
$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + ...$$

Noting these results, consider the Maclaurin Expansion of $e^{i\theta}$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{3!} + \frac{(i\theta)^5}{3!}...$$
$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{2!} - ...\right)$$
$$\therefore \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Hence we can write *any* complex number in *exponential* form. Note the *Euler Relation* when $\theta = \pi$

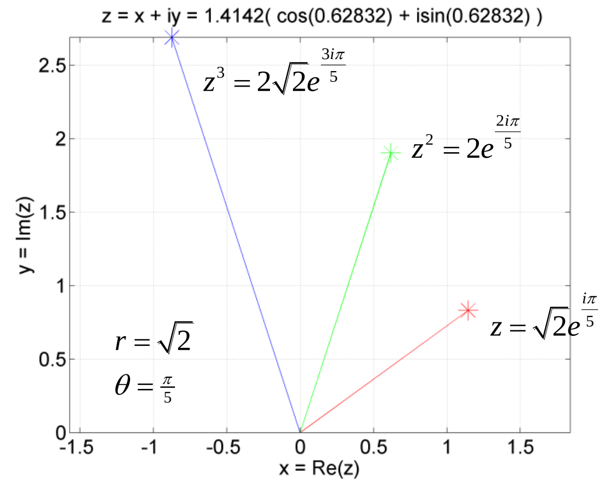
$$\boxed{e^{i\pi} + 1 = 0}$$

$$\boxed{\begin{aligned} &z = x + iy \\ &z = re^{i\theta} \\ &r = |z| \\ &\theta = \arg(z) \end{aligned}}$$

Exponential form means we can readily evaluate (and geometrically interpret) **powers** of complex numbers

$$\boxed{z^n = r^n e^{in\theta} = r^n\left(\cos n\theta + i\sin n\theta\right)}$$

Raising a complex number by power n therefore increases its *magnitude* |z| by power n, and *multiplies* its *polar angle* θ by n



De Moivre's Theorem using calculus

$z = \cos\theta + i\sin\theta \quad \therefore \frac{dz}{d\theta} = -\sin\theta + i\cos\theta$

$iz = i\cos\theta - \sin\theta$

$\therefore \frac{dz}{d\theta} = iz \;\Rightarrow \int \frac{1}{z} dz = i\int d\theta$

$\ln z = i\theta + c \quad \Rightarrow z = Ae^{i\theta}$

$|z| = 1 \;\; \therefore A = 1$

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Complex roots

Exponential form enables us to generalize the meaning of the n^{th} roots of positive real numbers.

i.e. solutions to the equation $\boxed{z^n = k}$ where $n, k \in \mathbb{R}^+$

To fully realize this, let us note that the *periodicity* of sine and cosine functions mean we can write any complex number in the form

$z = re^{i(\theta+2N\pi)}$

$r = |z|$ So r is positive

$\theta = \arg z$

$N \in \mathbb{Z}$ N is any integer

Since, if $N \in \mathbb{Z}$

$\cos(\theta + 2\pi N) = \cos\theta$

$\sin(\theta + 2\pi N) = \sin\theta$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore \boxed{e^{i(i\theta+2\pi N)} = e^{i\theta}}$

Hence: $\boxed{\sqrt[n]{z} = z^{\frac{1}{n}} = \sqrt[n]{r}\, e^{i\left(\frac{\theta+2\pi N}{n}\right)}}$

The ***n*^th^ root of unity** is a special case of this multiplicity of roots. i.e. solutions to the equation $\boxed{z^n = 1}$

$z = re^{i\theta}$

$z^n = 1$

$1 = e^{2\pi i N} \;\; ; \;\; N \in \mathbb{Z}$

$\therefore re^{i\theta} = e^{\frac{2\pi i N}{n}}$

$$\boxed{\begin{aligned} &\therefore r = 1 \\ &\therefore \theta = 2\pi \frac{N}{n} \end{aligned}}$$

If n is an *integer* the **roots of unity** are:

$$\boxed{\begin{aligned} &z = \omega, \omega^2, \omega^3, \ldots, \omega^n \\ &\omega = e^{\frac{2\pi i}{n}} \end{aligned}}$$

Logarithms of complex numbers are *multi-valued* for similar reasons

$$\boxed{\begin{aligned} &z = re^{i(\theta+2N\pi)} \\ &\log z = \log r + i(\theta + 2N\pi) \end{aligned}}$$

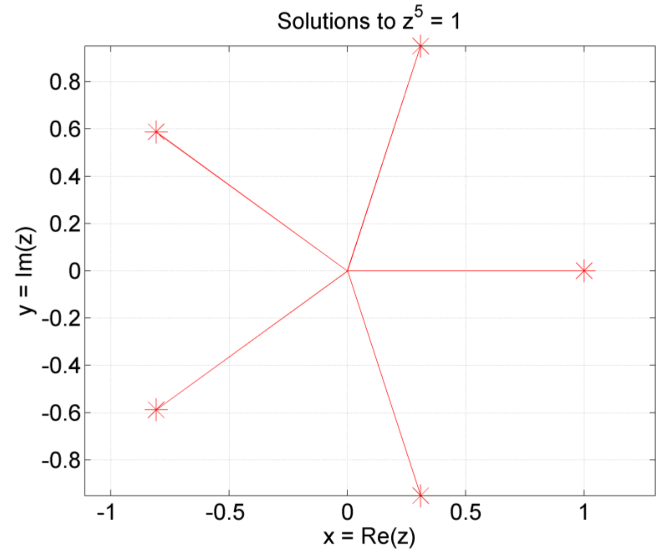
Example:

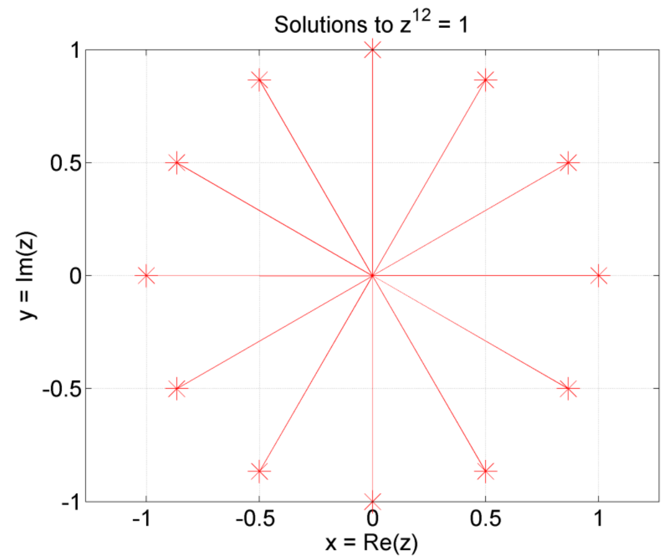
$\log(2+3i) = \log\left(\sqrt{2^2+3^2}\, \mathrm{e}^{i\left(\tan^{-1}\frac{3}{2}+2\pi N\right)}\right)$

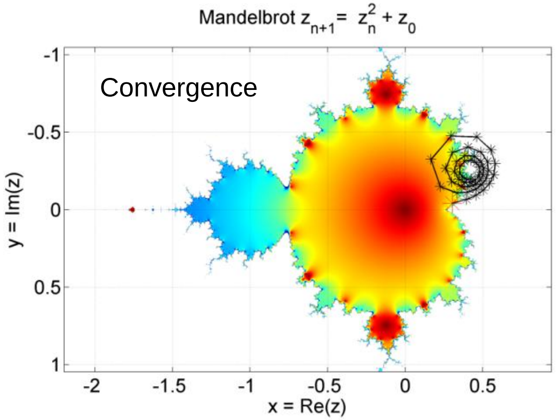
$\log(2+3i) = \log 13 + i\left(\tan^{-1}\tfrac{3}{2} + 2\pi N\right)$

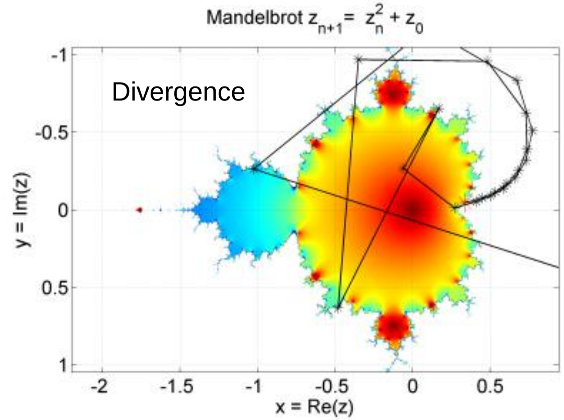
$N \in \mathbb{Z}$

Since powers of a complex number give rise to a *multiplier* of the argument (or polar angle), the integer n roots of unity create the vertices of a regular *n-gon* in the Argand Diagram









The Mandelbrot Iteration

Let z_0 describe a particular point on the Argand Diagram. Let it be subject to the following iteration:

$$\boxed{z_{n+1} = z_n^2 + z_0}$$

Amazingly, this simple formula generates *unbelievable* complexity!

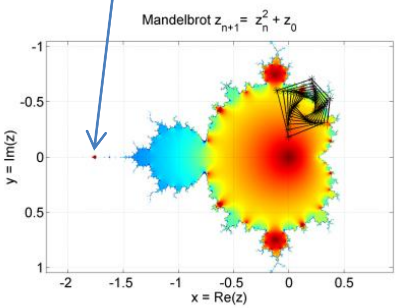
Benoit Mandelbrot
1924-2010

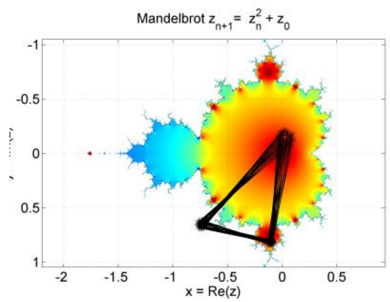
The colourful shape is produced by plotting a surface of height *h* based upon the function:

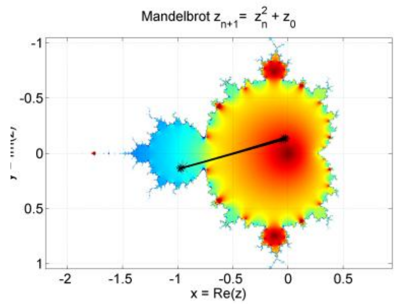
$$h = e^{-|z_{20}(x,y)|}$$
$$z_0 = x + iy$$
$$z_{n+1}(x,y) = z_n^2(x,y) + z_0(x,y)$$

If you zoom in then *more iterations* are needed to reveal the extra structure

The blue to red colours are based on the colour scale [0,1], with red corresponding to unity. The function has a very complicated boundary, beyond which the height drops away to essentially zero. (The white colour is used when $h < 0.001$).







In all the images, a sequence of the *z* values corresponding to the first 50 terms of the Mandelbrot Iteration is overlaid as a series of black stars connected by black lines. Each plot corresponds to a different z_0.

If z_0 is outside the coloured region, the iterations diverge. Inside the coloured region ("the **Mandelbrot Set**") the iterations repeat in a pattern, always returning to the coloured region.

Amazingly, if one zooms in on the boundary region (which gives rise to the most complicated of patterns) *infinitely* complicated patterns are revealed, including exact miniature replicas of the original pattern! This *self similar* structure is known as a *fractal*.