**Continued fractions** are a representation of numbers in terms of 'nested' integer and fractional parts. *Rational* numbers (which can be expressed as a fraction of integers) will have a finite number of 'levels', whereas irrationals like $\pi$, $e$ or the golden ratio $\phi$ will have an *infinite* continued fraction. It is the latter representation which is useful, as a truncation of the process will yield a rational approximation to these numbers.

## Examples using rational numbers

$$\frac{654}{321} = 2 + \cfrac{1}{26 + \cfrac{1}{1 + \cfrac{1}{2+1}}} = 2\tfrac{4}{107} = 2.03738317757...$$

$$\frac{9876}{5432} = 1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{122+1}}}} = 1\tfrac{1111}{1358} = 1.81811487482...$$

## Examples using irrational numbers

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1+...}}}}}} = \boxed{3.14159265359...}$$

*Truncations* of this continued fraction yield **rational approximations**

$$\pi \approx 3 + \frac{1}{7} = \boxed{\frac{22}{7}} = \boxed{3.142}85714286...$$

$$\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15+1}} = \boxed{\frac{335}{113}} = \boxed{3.141592}92035...$$

$$\pi \approx 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{293}}}} = \boxed{\frac{104,348}{33,215}} = \boxed{3.141592653}92...$$

**Algorithm** for finding a continued fraction of the form

$$\boxed{x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + ...}}}}}$$

$$a_0 = \operatorname{floor}(x)$$
$$r_0 = x - a_0$$
$$a_1 = \operatorname{floor}(1/r_0)$$
$$r_1 = 1/r_0 - a_1$$
$$a_2 = \operatorname{floor}(1/r_1)$$
$$r_2 = 1/r_1 - a_2$$

....

**Iteration**, i.e. a *loop*

$$a_{n+1} = \operatorname{floor}(1/r_n)$$
$$r_{n+1} = 1/r_n - a_{n+1}$$

Stop if $r$ becomes less than a set small value e.g. $10^{-13}$, or if $n$ exceeds a desired maximum number of levels (e.g. 10)

**floor(x)** means 'round down to the nearest integer'

**Example:** $x = \sqrt{2}$

$$\sqrt{2} = 1.41421356237..$$
$$a_0 = 1$$
$$1/r_0 = 2.41421356237...$$
$$a_1 = 2$$
$$1/r_1 = 2.41421356237...$$
$$a_2 = 2$$

.....

$$\boxed{\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2+...}}}} \approx \frac{1393}{985}}$$

*Note:*

$$a_0 = 1$$

$$1/r_0 = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)} = \frac{1+\sqrt{2}}{1} = 1+\sqrt{2} = \boxed{2+\sqrt{2}-1}$$

$$\therefore a_1 = 2$$

$$1/r_2 = \frac{1}{\sqrt{2}-1}$$

$$\therefore \boxed{a_n = 2 \quad n > 0}$$

The identical values for the continued fraction numbers $a_1, a_2, a_3, a_4$.... occurs because the remainder $r$ is always the *same* for the square root of 2.

## Algorithm for evaluating a continued fraction numerically

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \frac{1}{a_4 + \dots}}}}$$

$$\mathbf{a} = (a_0, a_1, a_2, a_3 \dots, a_N)$$

$$\begin{aligned}
y_0 &= a_N \\
y_1 &= a_{N-1} + 1/y_0 \\
y_2 &= a_{N-2} + 1/y_1 \\
&\dots \\
y_n &= a_{N-n} + 1/y_{n-1} \\
&\dots \\
y_N &= a_0 + 1/y_{N-1}
\end{aligned}$$

## More continued fractions of irrational numbers

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \dots}}}}}} = 2.71828182846\dots \approx \frac{1457}{536}$$

$$\mathbf{a} = (2,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12\dots)$$

$$\sqrt{42} = 6 + \cfrac{1}{2 + \cfrac{1}{12 + \cfrac{1}{2 + \frac{1}{12 + \frac{1}{2 + \frac{1}{12 + \dots}}}}}} = 6.48074069841\dots \approx \frac{4206}{649}$$

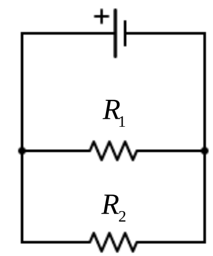
$$\mathbf{a} = (6,2,12,2,12,2,12,2,12\dots)$$

$$\phi = \tfrac{1}{2}\left(1 + \sqrt{5}\right) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}}}} = 1.61803398875\dots \approx \frac{1597}{987}$$

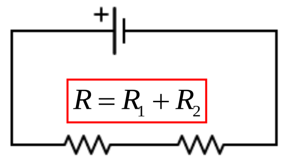
The *Golden Ratio* $\phi$ is clearly the most beautiful continued fraction, as all numbers which construct it are unity!

In Physics, the total **resistance** (*R*) of two resistors in *parallel* is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

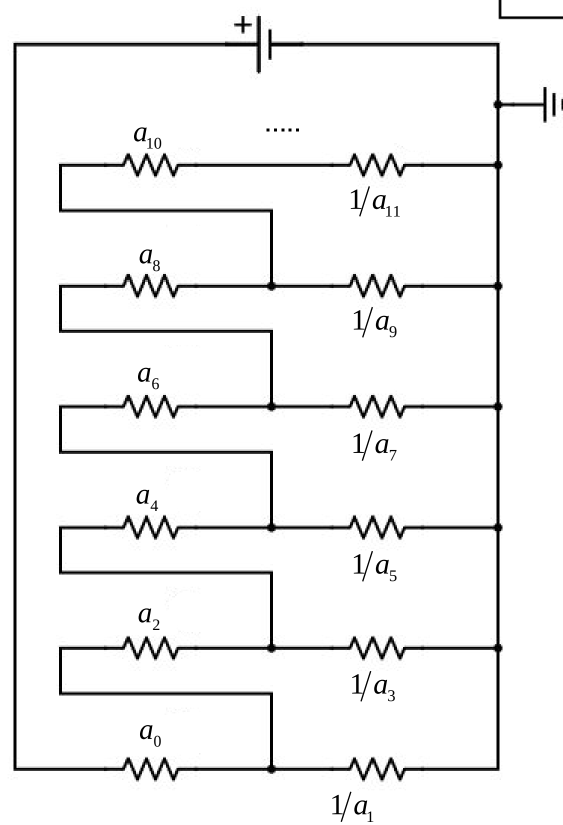



whereas resistances in the same loop (i.e. in *series*) are additive.

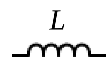


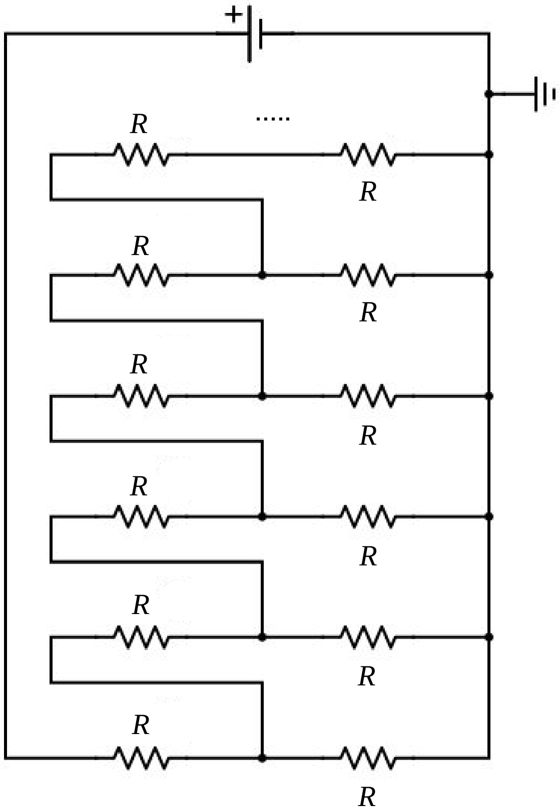

We can use this fact to construct *circuits* which have resistances given by continued fractions.

$$R = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \frac{1}{a_4 + \dots}}}}$$

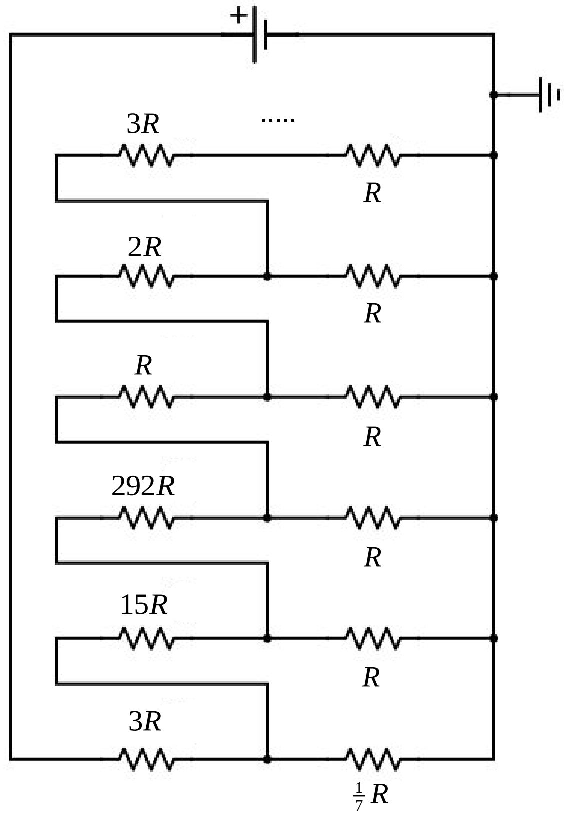



Since the combination of *inductors* (with no mutual inductance) follows the same rules as resistors, we can construct continued fraction inductances in exactly the same way



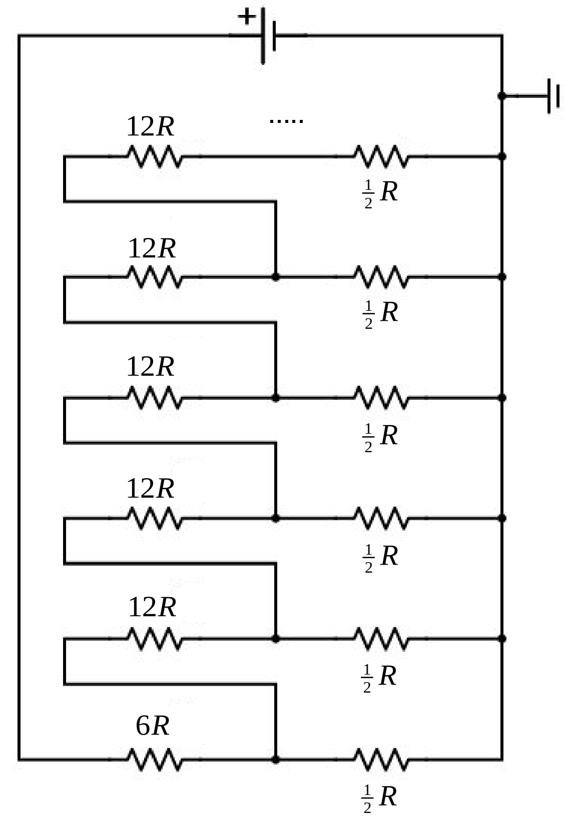



$$R_{total} \to \tfrac{1}{2}\left(1+\sqrt{5}\right)R$$




$$R_{total} \to \pi R$$

The Circuit of Deep Thought (!)




$$R_{total} \to \sqrt{42}R$$

$$\phi = \tfrac{1}{2}\left(1+\sqrt{5}\right) = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+...}}}}}} = 1.61803398875...$$

$$\mathbf{a} = (1,1,1,1,1,1,1,1..)$$

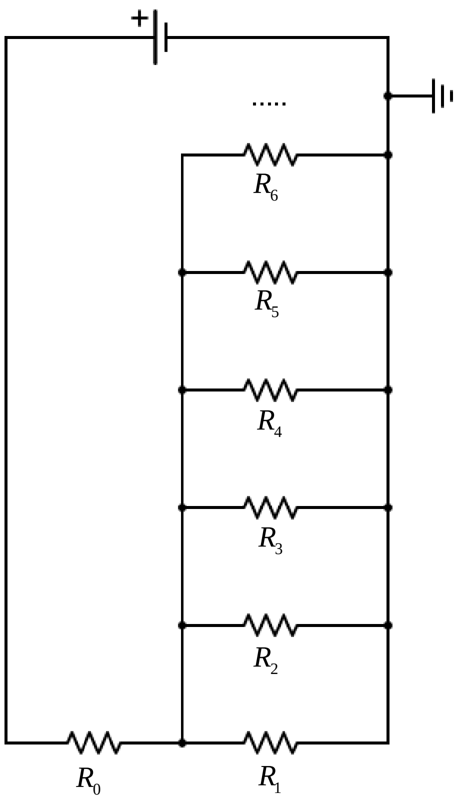
$$\pi = 3 + \cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+...}}}}}} = 3.14159265359...$$

$$\mathbf{a} = (3,7,15,1,292,1,1,1,2,1,3,1..)$$

$$\sqrt{42} = 6 + \cfrac{1}{2+\cfrac{1}{12+\cfrac{1}{2+\cfrac{1}{12+\cfrac{1}{2+\cfrac{1}{12+...}}}}}} = 6.48074069841...$$

$$\mathbf{a} = (6,2,12,2,12,2,12,2,12...)$$

A similar infinite parallel circuit can be drawn, but this one *does not* result in a continued fraction, merely a summation of fractions.




$$\boxed{R = R_0 + \cfrac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \ldots}}$$

**Special case: All resistances the same in the parallel loops**

$$R = R_0 + \cfrac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \ldots + \frac{1}{R_N}}$$

$$n > 0: \;\; R_n = R_1$$

$$\therefore R = R_0 + \frac{R_1}{N}$$

$$\therefore \boxed{\lim_{N \to \infty} R \to R_0}$$

**Special case: Resistances are in a *geometric progression***

$$R = R_0 + \cfrac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \ldots + \frac{1}{R_N}}$$

$$n > 0: \;\; R_n = \frac{R_1}{r^{n-1}}$$

$$R = R_0 + \frac{R_1}{1 + r + r^2 \ldots + r^{N-1}}$$

$$R = R_0 + R_1 \frac{1-r}{1-r^N}$$

$$\boxed{\begin{array}{l} 0 < r < 1 \\ \lim_{N \to \infty} R = R_0 + R_1(1-r) \end{array}}$$

$$\boxed{\begin{array}{l} r > 1 \\ \lim_{N \to \infty} R = R_0 \end{array}}$$

The **Maclaurin expansion** for various functions enables a variety of irrational number total resistances to be set using integer (or rational value) components.

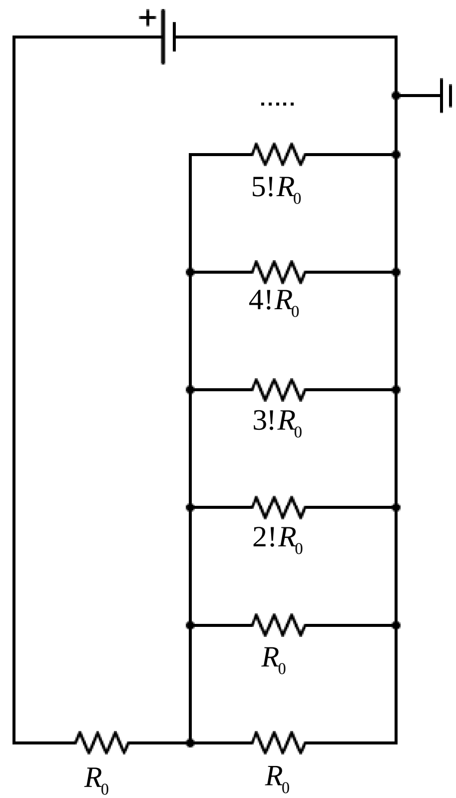
$$e^x = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \ldots \tfrac{1}{n!}x^n + \ldots$$

$$e = 1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \tfrac{1}{4!} + \tfrac{1}{5!} + \ldots$$

Define the following resistor network:

$$R = R_0 + \cfrac{1}{\frac{1}{R_0} + \frac{1}{R_0} + \frac{1}{2!R_0} + \frac{1}{2!R_0} + \frac{1}{3!R_0} + \frac{1}{4!R_0} \ldots}$$

$$\boxed{R = R_0\left(1 + \frac{1}{e}\right)}$$




$$\boxed{R = R_0 + \cfrac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \ldots}}$$