Proof by contradiction is a useful technique for *proving* a Mathematical proposition e.g. that $\sqrt{2}$ is *irrational*, that is, it cannot be written as a fraction of integers

Example proposition: $\sqrt{2} \neq \frac{p}{q}$ s.t. $p, q \in \mathbb{Z}$

Method of proof by contradiction

1. Assume the proposition is FALSE
2. Then show that this leads to a CONTRADICTION
3. Since the proposition can therefore not be FALSE, it must be TRUE!

"reductio ad absurdum"

$$\sqrt{2} = \frac{p}{q} \text{ s.t. } p, q \in \mathbb{Z}$$

$2q^2 = p^2$ Therefore p^2 is even

Now squares of even numbers are even numbers whereas squares of odd numbers are never even

Even: $(2n)^2 = 2 \times 2n^2, \ n \in \mathbb{Z}$

Odd: $(2n+1)^2 = 4n^2 + 4n + 1, \ n \in \mathbb{Z}$

$(2n+1)^2 = 2(2n^2 + 2n) + 1, \ n \in \mathbb{Z}$

Hence if p^2 is even, *then so is* p

$\therefore p = 2c, \ c \in \mathbb{Z}$

Substituting into $2q^2 = p^2$

$$2q^2 = (2c)^2$$

$$q^2 = 2c^2$$

So therefore q must also be even.

But if both *p* and *q* are even, then $\frac{p}{q}$ *cannot be the simplest fraction*.

i.e. we could always divide *p* and *q* by 2.

Since this process would continue *ad infinitum*
we conclude that the False proposition, that is the square root of two is rational, *must be False. Therefore the proposition is true!*

Hence the square root of 2 must be irrational.

Related example: Is a^b always irrational if both *a* and *b* are irrational?

Consider $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$

Now is *a* rational or irrational? If it *is* irrational:

$$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2$$

So a^b is rational, i.e. a **counter example** to the proposition, which means it must be **false**.

Now if $a = \sqrt{2}^{\sqrt{2}}$ *is not* irrational this *also serves to falsify the proposition*, since this is another example of a^b being rational where *a*,*b* are irrational (i.e. $\sqrt{2}$)

So although we have *not shown* whether $a = \sqrt{2}^{\sqrt{2}}$ is rational or not **BOTH** possibilities result in the proposition being false.

i.e. a^b **can be rational** if both *a* and *b* are irrational.