Mathematical induction is a powerful technique for *proving* statements which involves propositions involving the natural numbers (i.e. positive integers 1,2,3,4,...) The recipe is as follows:

Let *P(n)* be a *proposition* involving **positive integer** *n*. e.g. the sum of the first *n* square numbers has the formula $P(n)=\sum_{k=1}^{n}k^2=\frac{1}{6}n(n+1)(2n+1)$

1. **Is *P(1)* true?** $P(1)=1^2=\frac{1}{6}(1)(1+1)(2(1)+1)=1$ so yes in our case!

2. **If *P(1)* true, *assume P(n) is true*. *Then work out whether this means P(n+1) is true***. If it is, then the proposition is true, if not then the proposition is false. This last process is called the *inductive step*. It works since *n +1* could be *any* positive integer if *n* is also a positive integer.

In our example:

Work out *P(n+1)* if *P(n)* is true

$$P(n+1)=\tfrac{1}{6}(n+1)(n+2)(2(n+1)+1)$$

$$\boxed{P(n+1)=\tfrac{1}{6}(n+1)(n+2)(2n+3)}$$

Now work out *P(n+1)* directly, assuming *P(n)* is true

$$P(n+1)=\sum_{k=1}^{n+1}k^2=P(n)+(n+1)^2$$

$$P(n+1)=\tfrac{1}{6}n(n+1)(2n+1)+(n+1)^2$$

$$P(n+1)=\tfrac{1}{6}(n+1)\{n(2n+1)+6n+6\}$$

$$P(n+1)=\tfrac{1}{6}(n+1)\{2n^2+7n+6\}$$

$$\boxed{P(n+1)=\tfrac{1}{6}(n+1)(n+2)(2n+3)}$$

Since both approaches yield the same answer, we can conclude that indeed

$$\boxed{P(n)=\sum_{k=1}^{n}k^2=\tfrac{1}{6}n(n+1)(2n+1)}$$

Example: Prove that *P(n)* is divisible by 7, where *n* is a positive integer $P(n)=2^{n+2}+3^{2n+1}$

$$\therefore P(n)=4\times 2^n+3\times 3^{2n}$$

$$P(1)=2^{1+2}+3^{2+1}$$

$P(1)=8+27=35=5\times 7$ ***So P(1)* is true**

$$P(n+1)=2^{n+3}+3^{2n+3}$$

$$P(n+1)=8\times 2^n+27\times 3^{2n}$$

$$P(n+1)=2\times(4\times 2^n+3\times 3^{2n})+21\times 3^{2n}$$

$$P(n+1)=2\times P(n)+7\times 3^{2n+1}$$

Now ***if*** *P(n)* is divisible by 7, $P(n)=7k$ where *k* is a positive integer

$$\therefore P(n+1)=2\times 7k+7\times 3^{2n+1}$$

$$\Rightarrow P(n+1)=7\times(2k+3^{2n+1})$$

Hence ***P(n+1)* is divisible by 7 if *P(n)* is.**
Since *P(1)* is true this means **the proposition must be true for all *n*.**

$$P(n)=2^{n+2}+3^{2n+1}$$

$$P(2)=2^4+3^5=7\times 37$$

$$P(5)=2^7+3^{11}=7\times 25,325$$

$$P(10)=2^{12}+3^{21}=7\times 1,494,336,757$$