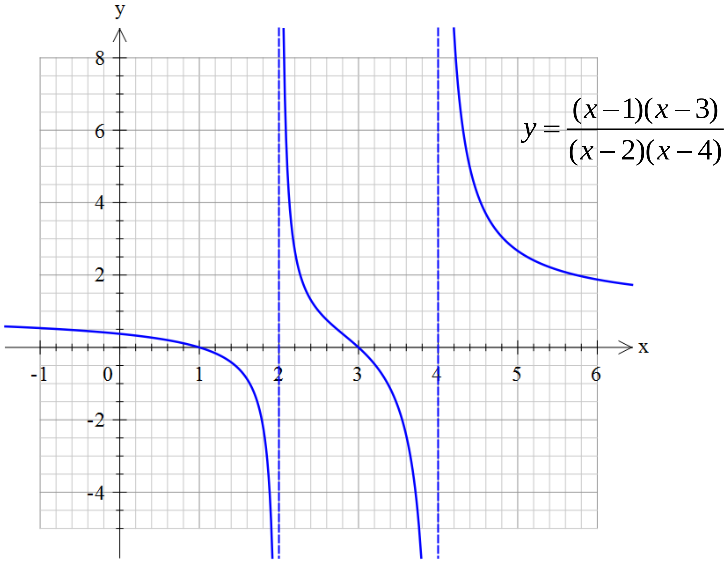
Rational functions, partial fractions and point discontinuities

A rational function is a fraction of *polynomials P(x), Q(x)*

$$f(x)=\frac{P(x)}{Q(x)}$$

For example: $f(x)=\frac{x^2-4x+3}{x^2-6x+8}=\frac{(x-1)(x-3)}{(x-2)(x-4)}$

If both numerator and denominator can be *factorized*, this is very useful as one can immediately determine any *zeros* or any points of *discontinuity*.



In this case, zeros are at x =1,3 and vertical asymptotes at x =2,4. The y axis intercept is at y = 3/8.

The zeros and vertical asymptotes can help enable one to sketch the curve. However, it is often difficult to know which side of the axes the curve is between these points.

By splitting up the rational function into *partial fractions* we can overcome this issue.

$$\frac{Ax+B}{(ax-b)(cx-d)}=\frac{\alpha}{ax-b}+\frac{\beta}{cx-d}$$

Partial fractions decomposition

e.g. $\frac{2x+1}{(3x-1)(4x+1)}=\frac{\alpha}{3x-1}+\frac{\beta}{4x+1}=\frac{\alpha(4x+1)+\beta(3x-1)}{(3x-1)(4x+1)}$

$\therefore 2x+1=\alpha(4x+1)+\beta(3x-1)=x(4\alpha+3\beta)+\alpha-\beta$

Comparing coefficients of powers of x

$x^1: \quad 2=4\alpha+3\beta \qquad (1)$

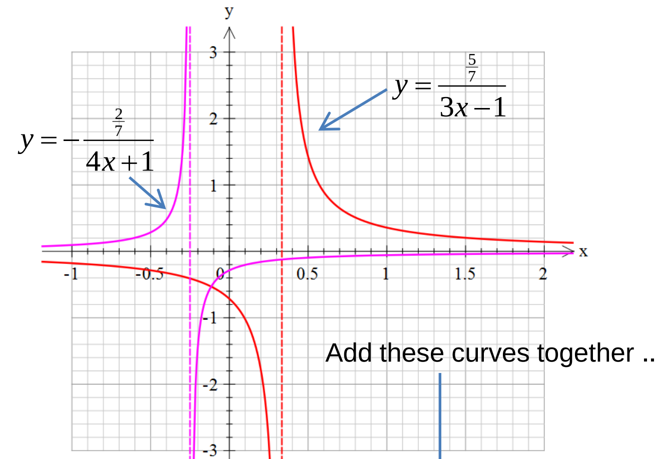
$x^0: \quad 1=\alpha-\beta \qquad (2)$

$5=4\alpha+3\alpha+3\beta-3\beta \qquad (1)+3(2)$

$\therefore \frac{5}{7}=\alpha$

In (2): $\beta=\alpha-1=-\frac{2}{7}$

$$\therefore \frac{2x+1}{(3x-1)(4x+1)}=\frac{\frac{5}{7}}{3x-1}-\frac{\frac{2}{7}}{4x+1}$$



Even more information about curve shape can be found if we can find *stationary points*

$$y=\frac{2x+1}{(3x-1)(4x+1)}=\frac{2x+1}{12x^2-x-1}$$

Quotient rule

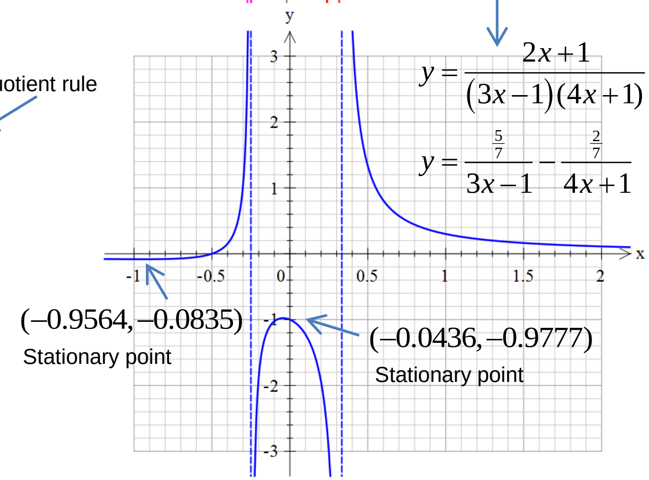
$$\frac{dy}{dx}=\frac{(12x^2-x-1)(2)-(2x+1)(24x-1)}{(3x-1)^2(4x+1)^2}$$

$$\frac{dy}{dx}=0 \Rightarrow 24x^2-2x-2-48x^2-22x+1=0$$

$$24x^2+24x+1=0$$

$$x=\frac{-24\pm\sqrt{24^2-4(24)(1)}}{48}=-\frac{1}{2}\pm\frac{1}{12}\sqrt{30}$$

$$x\approx-0.9564,-0.0436$$



The partial fractions decomposition is particularly useful in evaluating *integrals* of rational functions

$$\int \frac{2x+1}{(3x-1)(4x+1)}dx = \int\left(\frac{\frac{5}{7}}{3x-1} - \frac{\frac{2}{7}}{4x+1}\right)dx$$

$$= \tfrac{5}{7}\tfrac{1}{3}\int \frac{3}{3x-1}dx - \tfrac{2}{7}\tfrac{1}{4}\int \frac{4}{4x+1}dx \quad \leftarrow \quad \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$= \boxed{\tfrac{5}{21}\ln|3x-1| - \tfrac{1}{14}\ln|4x+1| + c}$$

The partial fractions decomposition can be extended to when

$f(x) = \dfrac{P(x)}{Q(x)}$ where *P(x)* is one polynomial order less than *Q(x)*

$$\frac{Ax^2+Bx+C}{(ax-b)(cx-d)(ex-f)} = \frac{\alpha}{ax-b} + \frac{\beta}{cx-d} + \frac{\gamma}{ex-f}$$

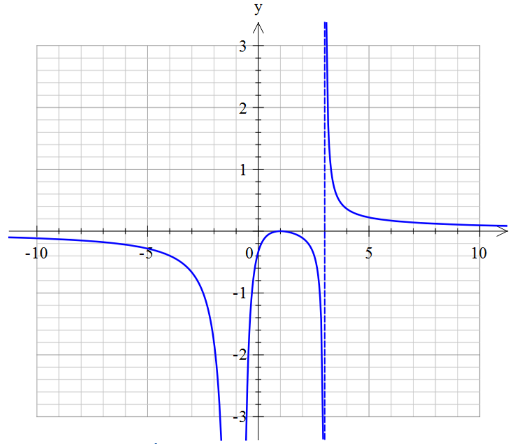
$$\frac{Ax^3+Bx^2+Cx+D}{(ax-b)(cx-d)(ex-f)(gx-h)} = \frac{\alpha}{ax-b} + \frac{\beta}{cx-d} + \frac{\gamma}{ex-f} + \frac{\delta}{gx-h}$$

Care must be taken to include enough terms when there are *powers* in the denominator factors

$$\frac{Ax^2+Bx+c}{(ax-b)^2(cx-d)} = \frac{\alpha}{(ax-b)^2} + \frac{\beta}{ax-b} + \frac{\gamma}{cx-d}$$

$$\frac{Ax^4+Bx^3+Cx^2+dx+E}{(ax-b)^3(cx-d)^2} = \frac{\alpha}{(ax-b)^3} + \frac{\beta}{(ax-b)^2} + \frac{\gamma}{ax-b} + \frac{\delta}{(cx-d)^2} + \frac{\varepsilon}{cx-d}$$

Example:



$$\frac{x^2-2x+1}{(x+1)^2(x-3)} = \frac{\alpha}{(x+1)^2} + \frac{\beta}{x+1} + \frac{\gamma}{x-3}$$

$$x^2-2x+1 = \alpha(x-3) + \beta(x+1)(x-3) + \gamma(x+1)^2$$

$$x=3 \quad \Rightarrow 9-6+1 = 16\gamma \Rightarrow \gamma = \tfrac{1}{4}$$

$$x=-1 \quad \Rightarrow 4 = -4\alpha \Rightarrow \alpha = -1$$

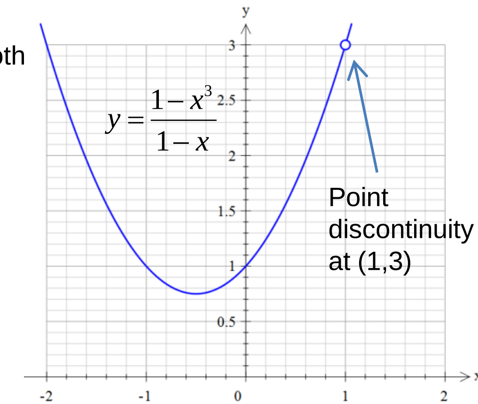
$$x=0 \quad \Rightarrow 1 = -3\alpha - 3\beta + \gamma \Rightarrow \beta = \frac{-3\times -1 + \frac{1}{4} - 1}{3} = \tfrac{3}{4}$$

$$\boxed{\frac{x^2-2x+1}{(x+1)^2(x-3)} = -\frac{1}{(x+1)^2} + \frac{\frac{3}{4}}{x+1} + \frac{\frac{1}{4}}{x-3}}$$

Curious *point discontinuities* can occur when both the numerator and denominator of the rational function have the *same factors*

$$f(x) = \frac{1-x^3}{1-x} = \frac{(1-x)(1+x+x^2)}{1-x}$$

$$f(x) = \begin{cases} 1+x+x^2 & x \neq 1 \\ \text{undefined} & x = 1 \end{cases}$$



Note using a geometric progression

$$1+x+x^2+\dots+x^{N-1} = \frac{1-x^N}{1-x}$$

$$1-x^N = (1-x)(1+x+x^2+\dots+x^{N-1})$$

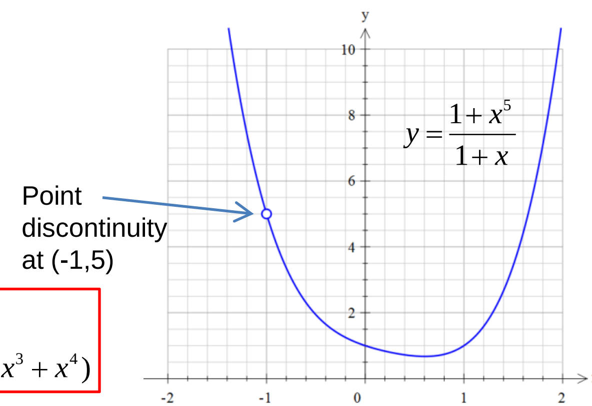
$$1-x^2 = (1-x)(1+x)$$

$$1-x^3 = (1-x)(1+x+x^2)$$

$$1-x^4 = (1-x)(1+x+x^2+x^3)$$

$$1-(-x)^3 = 1+x^3 = (1+x)(1-x+x^2)$$

$$1-(-x)^5 = 1+x^5 = (1+x)(1-x+x^2-x^3+x^4)$$



When the numerator of the rational function is of an equal or higher polynomial order than the denominator, we can split the function up into a polynomial and partial fractions.

The polynomial will become the *asymptotic behaviour* of the function when $|x| \to \infty$

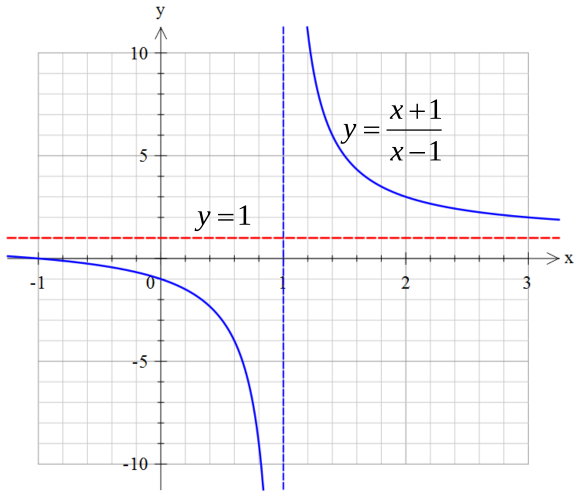
$$f(x) = \frac{x+1}{x-1} = A + \frac{B}{x-1}$$

$$\therefore x+1 = A(x-1) + B$$

$$x = 1: \quad 2 = B$$

$$x = 0: \quad 1 = -A + B \Rightarrow A = B - 1 = 1$$

$$\therefore f(x) = \frac{x+1}{x-1} = \boxed{1 + \frac{2}{x-1}}$$



In this case the asymptotic behaviour is:

$$f(x) \to 1, \ |x| \to \infty$$

$$f(x) = \frac{x^2+1}{x-1} = Ax + B + \frac{C}{x-1}$$

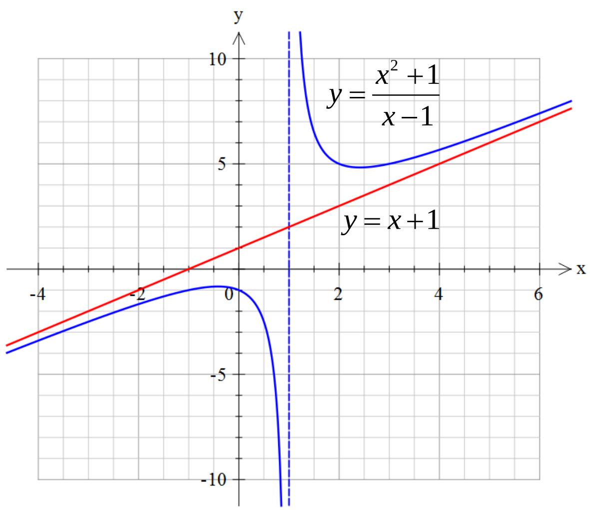
$$\therefore x^2 + 1 = (Ax + B)(x-1) + C$$

$$x = 0: \quad 1 = -B + C$$

$$x = 1: \quad 2 = C \quad \therefore B = 1$$

$$x = 2: \quad 5 = 2A + 1 + 2 \Rightarrow A = 1$$

$$\therefore f(x) = \frac{x^2+1}{x-1} = \boxed{x + 1 + \frac{2}{x-1}}$$



$y = x + 1$ in this case is called the *oblique asymptote*

$$f(x) \to x + 1, \ |x| \to \infty$$

$$f(x) = \frac{x^4+1}{(x-1)^2} = Ax^2 + Bx + C + \frac{Dx+E}{(x-1)^2}$$

Here you must be careful to note there are *five* unknowns to be found!

Find A,B,C via *polynomial division*

$$\begin{array}{r} x^2 + 2x + 3 \\ x^2 - 2x + 1 \overline{) x^4 + 1} \\ -x^4 + 2x^3 - x^2 \\ = 2x^3 - x^2 + 1 \\ -2x^3 + 4x^2 - 2x \\ = 3x^2 - 2x + 1 \\ -3x^2 + 6x - 3 \\ = 4x - 2 \end{array}$$

$$\therefore f(x) = \frac{x^4+1}{(x-1)^2} = x^2 + 2x + 3 + \frac{4x-2}{(x-1)^2}$$

$$\frac{4x-2}{(x-1)^2} = \frac{\alpha}{(x-1)^2} + \frac{\beta}{x-1} \Rightarrow 4x - 2 = \alpha + \beta(x-1)$$

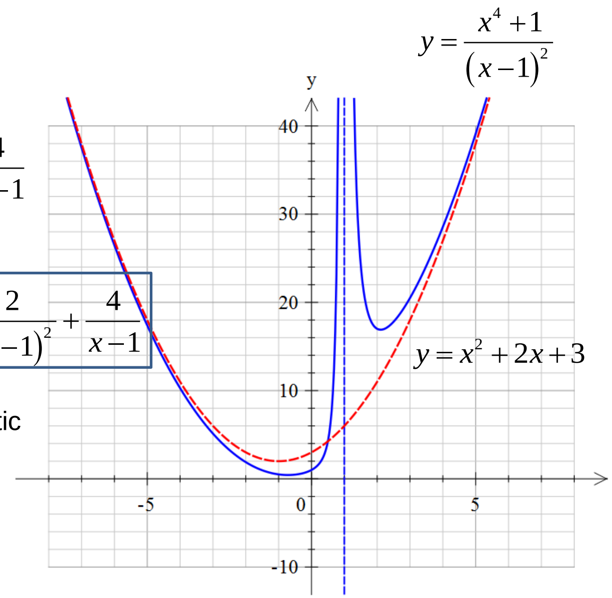
$$x = 1: \quad 2 = \alpha$$

$$x = 0: \quad -2 = \alpha - \beta$$

$$\Rightarrow \beta = \alpha + 2 = 4$$

$$\therefore \frac{4x-2}{(x-1)^2} = \frac{2}{(x-1)^2} + \frac{4}{x-1}$$

$$\therefore \boxed{\frac{x^4+1}{(x-1)^2} = x^2 + 2x + 3 + \frac{2}{(x-1)^2} + \frac{4}{x-1}}$$



In this case, the asymptotic curve is:

$$y = x^2 + 2x + 3$$