Defining sequences

term in sequence

$$u_8 = 42$$

term number or index

Iterative form $u_{n+1} = f(u_n, n)$

$$u_{n+1} = u_n^2 + 1$$

$$n = 1,2,3,4,5,6,7,8$$

$$u_n = 1,2,5,26,677,458330$$

Term form $u_n = g(n)$

$$u_n = n^3$$

$$n = 1,2,3,4,5,6,7,8$$

$$u_n = 1,8,27,64,125,216,343,512$$

Summing sequences

$$S_n = u_1 + u_2 + ... + u_n$$

$$\sum_{m=1}^{n} u_m = u_1 + u_2 + ... + u_n$$

A useful shorthand for describing a sum of *n* terms is to use *sigma notation*

Arithmetic Progression (AP)

A sequence where the **difference** between subsequent terms is a **constant**

$$n = 1,2,3,4,5,6,7,8$$

$$u_n = 1,2,3,4,5,6,7,8$$

$$u_n = n$$

$$S_{100} = 1 + 2 + ... + 99 + 100$$

$$S_{100} = 100 + 99 + ... + 2 + 1$$

$$2S_{100} = 101 + 101 + ... + 101 + 101$$

$$\therefore S_{100} = \tfrac{1}{2}101 \times 100 = 5050$$

Reverse order of an arithmetic sequence and sum corresponding terms. The sum will always be the *same number*. Hence this number times the number of terms equals twice the sum of the sequence

$$u_n = a + (n-1)d$$

a is the first term
d is the *common difference*

$$u_1 = a$$

$$u_2 = a + d$$

$$u_3 = a + 2d$$

$$u_4 = a + 3d$$

......

$$S_n = a + a + d + ... + a + (n-1)d$$

$$S_n = a + (n-1)d + a + (n-2)d + ... + a$$

$$2S_n = 2a + (n-1)d + 2a + (n-1)d + ... + 2a + (n-1)d$$

$$\therefore S_n = \tfrac{1}{2}n\left(2a + (n-1)d\right)$$

$$S_n = \tfrac{1}{2}n\left(u_1 + u_n\right)$$

Sum of an Arithmetic Progression is the **mean average** of the first and last terms **multiplied by the number of terms**.

$$n = 1,2,3,4,5,6,7,8$$

$$u_n = 10,8,6,4,2,0,-2,-4$$

$$u_n = 10 - 2(n-1)$$

$$u_n = 12 - 2n$$

$$S_n = \tfrac{1}{2}n\left(u_1 + u_n\right)$$

$$S_n = \tfrac{1}{2}n\left(10 + 12 - 2n\right)$$

$$S_n = n\left(11 - n\right)$$

In this case:

$u_n < 0$	$S_n < 0 \; ; \; n > 0$
$12 - 2n < 0$	$n(11-n) < 0$
$6 < n$	$11 < n$

So find *n* when S_n is > a constant you will need to solve a *quadratic* in *n* and then round to the nearest *n* as appropriate

Geometric Progression (GP)

$$u_n = ar^{n-1}$$

r is the *common ratio*

$$u_1 = a$$
$$u_2 = ar$$
$$u_3 = ar^2$$
$$u_4 = ar^3$$
$$\ldots\ldots$$

$$S_n = \sum_{m=1}^{n} u_m = a\frac{1-r^n}{1-r}$$

Proof of formula for the sum of a geometric sequence

$$S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-1}$$
$$rS_n = \quad ar + ar^2 + \ldots\ldots + ar^{n-1} + ar^n$$
$$rS_n - S_n = ar^n - a$$
$$S_n = a\frac{r^n - 1}{r-1} = a\frac{1-r^n}{1-r}$$

Special case:

$$\lim_{n\to\infty}\left(r^n\right) = 0 \quad \text{if} \quad |r| < 1$$
$$\therefore \text{if} \quad |r| < 1$$
$$\sum_{m=1}^{\infty} ar^{m-1} = \frac{a}{1-r}$$

This is a *convergent* sequence

$$S_n = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \ldots\ldots + \frac{1}{2^{n-1}}$$
$$S_n = \frac{1-2^{-n}}{1-\frac{1}{2}} = 2\left(1-2^{-n}\right)$$
$$S_\infty = 2$$

GP Example 1

Consider the sequence: 3, 6, 12, 24, 48, ... Find S_{10}

$$u_n = 3\times 2^{n-1}$$
$$S_n = 3\frac{2^n - 1}{2-1}$$
$$S_n = 3\left(2^n - 1\right)$$
$$\therefore S_{10} = 3069$$

GP Example 2

Consider the sequence: 1024, -512, 256, -128, 32, ... Find the sum to infinity

$$u_n = 1024\times\left(-\tfrac{1}{2}\right)^{n-1}$$
$$S_n = 1024\frac{1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)}$$
$$S_n = \frac{2048}{3}\left(1 - \frac{(-1)^n}{2^n}\right)$$
$$\therefore S_\infty = \frac{2048}{3} = 682\tfrac{2}{3}$$

GP Example 3

$$u_3 = 63$$
$$u_5 = 567$$
$$S_{10} = ?$$

$$u_n = ar^{n-1}$$
$$63 = ar^2$$
$$567 = ar^4$$
$$\therefore \frac{567}{63} = 9 = r^2 \Rightarrow r = 3$$

$$63 = ar^2$$
$$63 = 9a$$
$$7 = a$$
$$\therefore u_n = 7\times 3^{n-1}$$
$$\therefore S_n = \tfrac{7}{2}\left(3^n - 1\right)$$
$$\therefore S_{10} = 206{,}668$$

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$$u_{n+1} = u_n^2 + 1$$
$$n = 1, 2, 3, 4, 5, 6, 7, 8$$
$$u_n = 1, 2, 5, 26, 677, 458330$$

Term form $u_n = g(n)$

$$u_n = n^3$$
$$n = 1, 2, 3, 4, 5, 6, 7, 8$$
$$u_n = 1, 8, 27, 64, 125, 216, 343, 512$$

These are *divergent* sequences

Summing sequences: using sigma notation

$$\sum_{n=1}^{4} g(n) = g(1) + g(2) + g(3) + g(4)$$

$$\sum_{n=1}^{5} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

$$\sum_{m=1}^{n} (m+1)^3 = (1+1)^3 + (2+2)^3 + (3+2)^3 + ... + (n+1)^3$$

Some useful properties of sigma notation

$$\sum_n A g(n) = A \sum_n g(n)$$

$$\sum_n \{ f(n) + g(n) \} = \sum_n f(n) + \sum_n g(n)$$

$$\sum_n (n+1)^2 = \sum_n (n^2 + 2n + 1) = \sum_n n^2 + 2\sum_n n + \sum_n 1$$
$$= \tfrac{1}{6} n(n+1)(2n+1) + \tfrac{6}{6} n(n+1) + \tfrac{6}{6} n$$
$$= \tfrac{1}{6} n(n+1)(2n+7) + \tfrac{6}{6} n$$
$$= \tfrac{1}{6} n \{ 2n^2 + 9n + 8 \}$$

Difference formula $g(n) = f(n+1) - f(n)$

$$g(1) = f(2) - f(1)$$
$$g(2) = f(3) - f(2)$$
$$g(3) = f(4) - f(3)$$
.

.

$$g(m-1) = f(m) - f(m-1)$$
$$g(m) = f(m+1) - f(m)$$

$$\sum_{m=1}^{n} g(m) = f(n+1) - f(1)$$

$$\sum_{m=1}^{n} \{ f(m+1) - f(m) \} = f(n+1) - f(1)$$

$$f(m+1) - f(m) = (m+1)^2 - m^2 = 2m + 1$$
$$(n+1)^2 - 1 = n^2 + 2n$$
$$\therefore \sum_{m=1}^{n} (2m+1) = n^2 + 2n$$
$$2\sum_{m=1}^{n} m + n = n^2 + 2n$$
$$\therefore \sum_{m=1}^{n} m = \tfrac{1}{2} n(n+1)$$

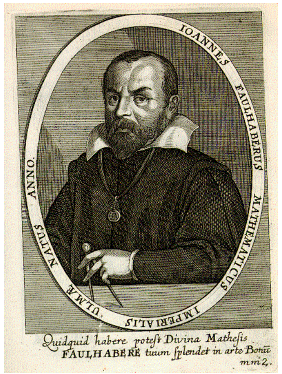
Sums of integer powers

These can all be derived by application of an appropriate difference formula

$$\sum_{m=1}^{n} 1 = n$$
$$\sum_{m=1}^{n} m = 1 + 2 + 3 + ... + n = \tfrac{1}{2} n(n+1)$$
$$\sum_{m=1}^{n} m^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \tfrac{1}{6} n(n+1)(2n+1)$$
$$\sum_{m=1}^{n} m^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = \tfrac{1}{4} n^2 (n+1)^2$$

Also by the difference formula

$$\sum_{m=1}^{n} \frac{1}{m(m+1)} = \sum_{m=1}^{n} \left\{ \frac{1}{m} - \frac{1}{m+1} \right\} = \frac{n}{n+1}$$

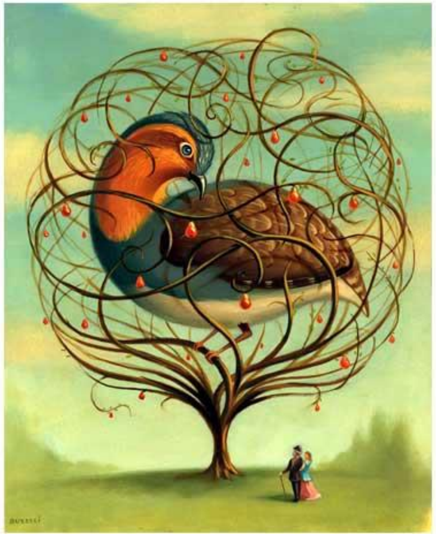


Faulhaber's formula can be used to generate a closed form expression for the sum of any integer powers of integers. This involves *Bernoulli numbers.*

The total number of presents in the carol ***The Twelve Days of Christmas*** is:

1 +
1 + 2 +
1 + 2 + 3 +
1 + 2 + 3 + 4 +
1 + 2 + 3 + 4 + 5 +
1 + 2 + 3 + 4 + 5 + 6 +
1 + 2 + 3 + 4 + 5 + 6 + 7 +
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 +
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 +
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 +
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 +
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12

= **364**



There is actually a formula for this type of sum. If *n* is the 'number of days of Christmas' and *p* is the total number of presents:

$$p = \tfrac{1}{6} n(n+1)(n+2)$$

e.g. (1/6) x 12 x 13 x 14 = 364

Note if Christmas *lasted all year*

p = (1/6) x 365 x 366 x 367 = **8,171,255** presents!

$$\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \tfrac{1}{2}n(n+1)$$

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\ldots+n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

Proof of the Christmas present formula

On the nth day of Christmas, my true love gave to me a total of:

$$1+2+\ldots+n = \tfrac{1}{2}n(n+1)$$ presents.

Therefore the cumulative sum after *n* days is:

$$S_n = \sum_{k=1}^{n} \tfrac{1}{2}k(k+1)$$

$$S_n = \tfrac{1}{2}\sum_{k=1}^{n} k^2 + \tfrac{1}{2}\sum_{k=1}^{n} k$$

$$S_n = \tfrac{1}{2}\tfrac{1}{6}n(n+1)(2n+1) + \tfrac{1}{2}\tfrac{1}{2}n(n+1)$$

$$S_n = \tfrac{1}{12}n(n+1)\left(2n+1+3\right)$$

$$S_n = \tfrac{1}{12}n(n+1)\left(2n+4\right)$$

$$S_n = \tfrac{1}{6}n(n+1)\left(n+2\right)$$