The quadratic formula

The roots of a quadratic equation $y = ax^2 + bx + c$ are the values of *x* such that *y* = 0. Graphically, this is where the quadratic curve cuts the *x* axis. We can derive a formula for the roots of a quadratic, if one firstly *completes the square*.

$$y = ax^2 + bx + c$$
$$y = a\left\{x^2 + \frac{b}{a}x\right\} + c$$
$$y = a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right\} + c$$
$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

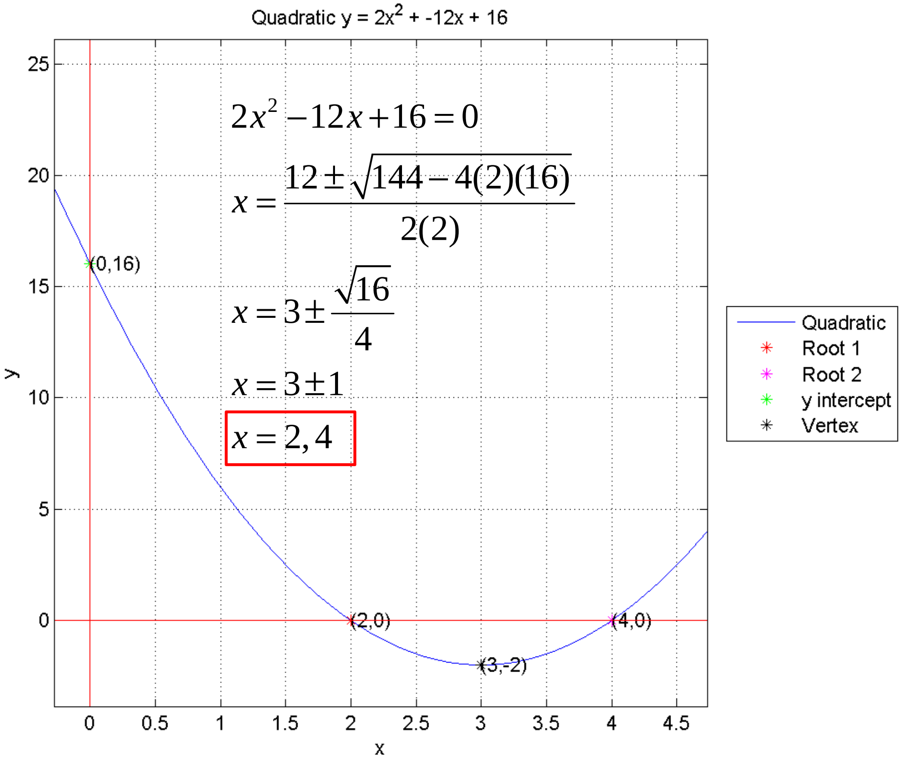
$$y = 0$$
$$0 = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$
$$\frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2$$
$$\frac{b^2}{4a^2} - \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2$$

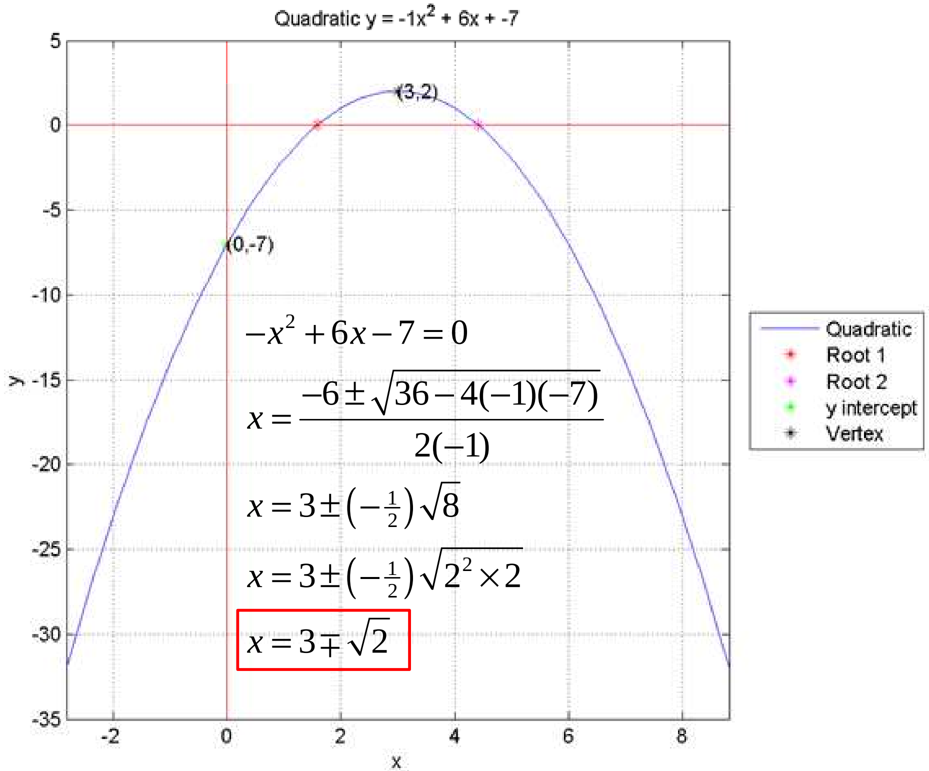
$$\frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$
$$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$
$$\pm\frac{\sqrt{b^2 - 4ac}}{2a} = x + \frac{b}{2a}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The quadratic formula

$$\boxed{\begin{aligned} &ax^2 + bx + c = 0 \\ &x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}}$$





The discriminant $\Delta = b^2 - 4ac$

All quadratic equations have solutions to $y = 0$, but not all of these roots are *real*. i.e. some quadratic equations do not cross the x axis. For those that do, there can also be a *single* crossing point rather than two if the vertex is also the root.

The discriminant $\Delta = b^2 - 4ac$ of a quadratic $y = ax^2 + bx + c$ determines which of these three possible scenarios applies.

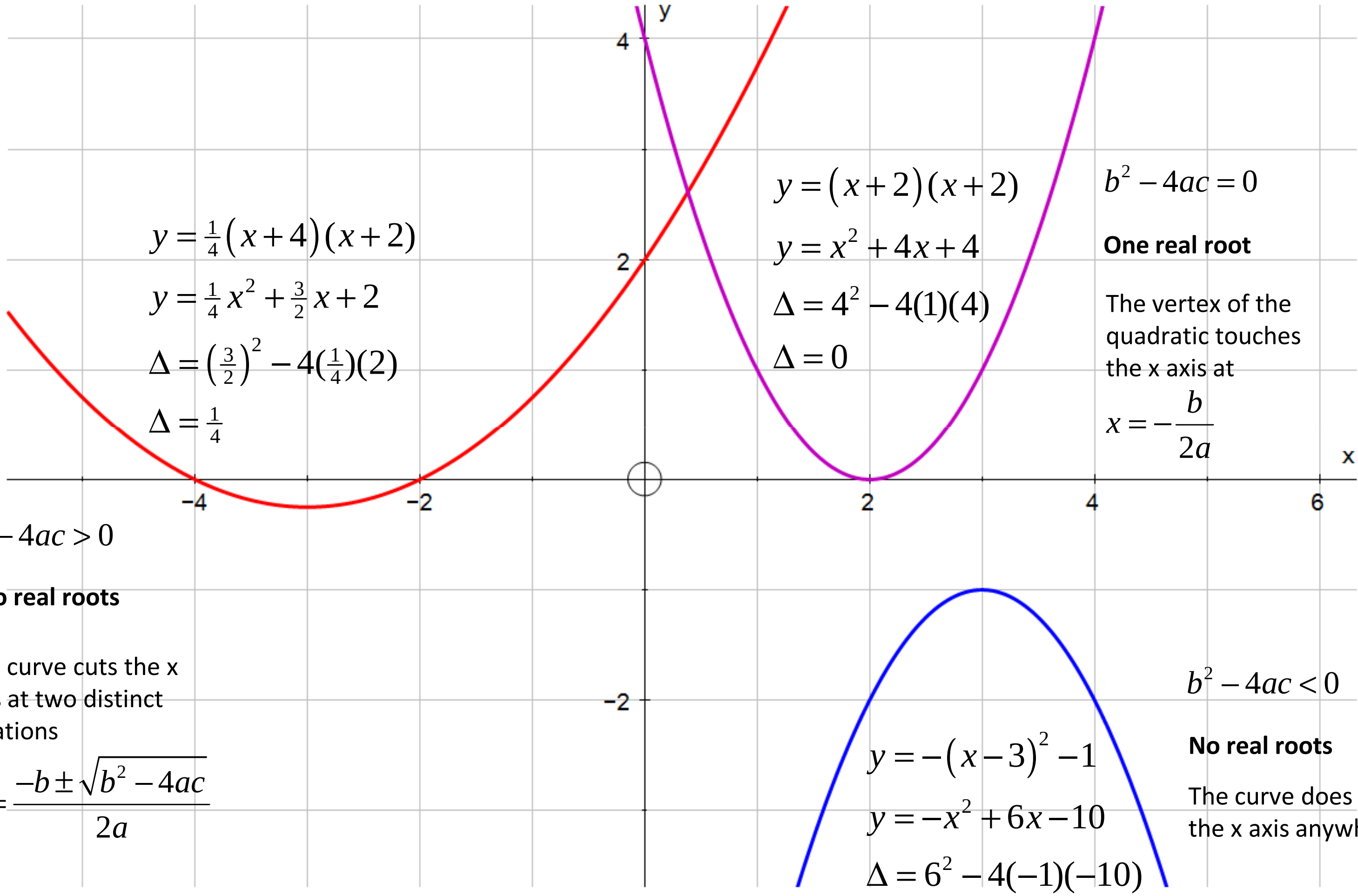
The quadratic formula

$$\boxed{\begin{aligned} &ax^2 + bx + c = 0 \\ &x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}}$$

We can express the *complex roots* of a quadratic with a negative discriminant using the imaginary number

$i = \sqrt{-1}$

$$y = -(x-3)^2 - 1$$
$$y = 0$$
$$-1 = (x-3)^2$$
$$\pm\sqrt{-1} = x - 3$$
$$x = 3 \pm i$$



$$y = \tfrac{1}{4}(x+4)(x+2)$$
$$y = \tfrac{1}{4}x^2 + \tfrac{3}{2}x + 2$$
$$\Delta = \left(\tfrac{3}{2}\right)^2 - 4\left(\tfrac{1}{4}\right)(2)$$
$$\Delta = \tfrac{1}{4}$$

$b^2 - 4ac > 0$

Two real roots

The curve cuts the x axis at two distinct locations

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = (x+2)(x+2)$$
$$y = x^2 + 4x + 4$$
$$\Delta = 4^2 - 4(1)(4)$$
$$\Delta = 0$$

$b^2 - 4ac = 0$

One real root

The vertex of the quadratic touches the x axis at $x = -\dfrac{b}{2a}$

$$y = -(x-3)^2 - 1$$
$$y = -x^2 + 6x - 10$$
$$\Delta = 6^2 - 4(-1)(-10)$$
$$\Delta = -4$$

$b^2 - 4ac < 0$

No real roots

The curve does not cut the x axis anywhere