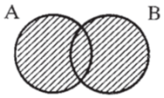
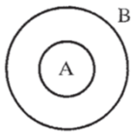
# Rules and notation of mathematical logic

**1.** ∩ 'intersection'
A ∩ B is shaded.

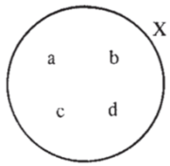

**2.** ∪ 'union'
A ∪ B is shaded.



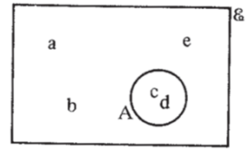

**3.** ⊂ 'is a subset of'
A ⊂ B
[B ⊄ A means 'B is *not* a subset of A']



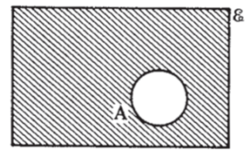

**4.** ∈ 'is a member of'
'belongs to'
b ∈ X
[e ∉ X means 'e is not a member of set X']



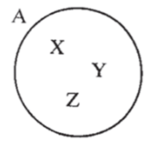

**5.** ℰ 'universal set'
ℰ = {a, b, c, d, e}




**6.** A′ 'complement of'
'not in A'
A′ is shaded
(A ∪ A′ = ℰ)



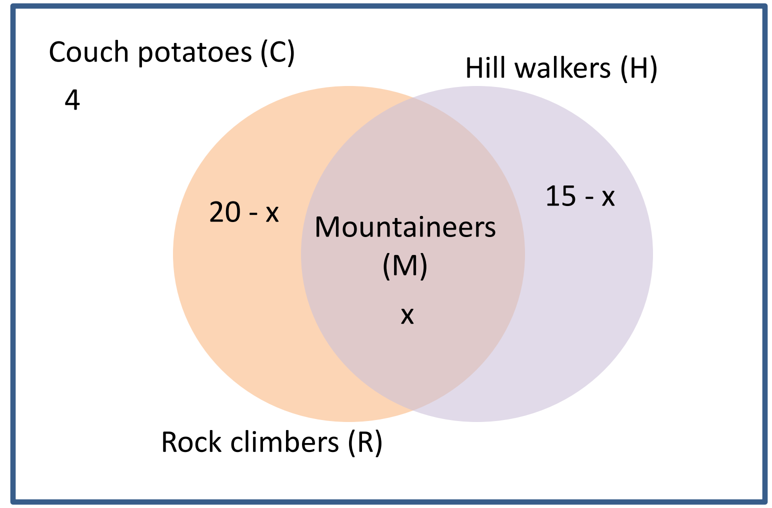

**7.** $n$(A) 'the number of elements in set A'
$n$(A) = 3



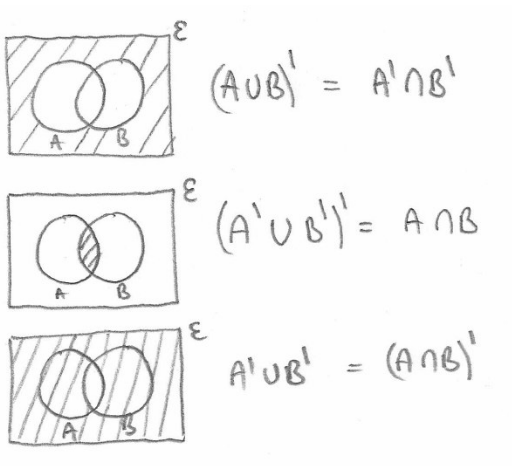

**8.** A = {$x$ : $x$ is an integer, 2 ⩽ $x$ ⩽ 9}

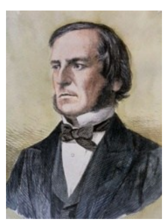
The [set of] elements $x$ [such that] $x$ is an integer and 2 ⩽ $x$ ⩽ 9.
The set A is {2, 3, 4, 5, 6, 7, 8, 9}.

**9.** ∅ or { } 'empty set'
(Note ∅ ⊂ A for any set A)




30 attendees at the local BMC meeting. 20 rock climbers and 15 hill walker + 4 couch potatoes. How many mountaineers?
4 + 20 – x + x + 15 – x = 30
Hence 39 – x = 30  => **x = 9  mountaineers**



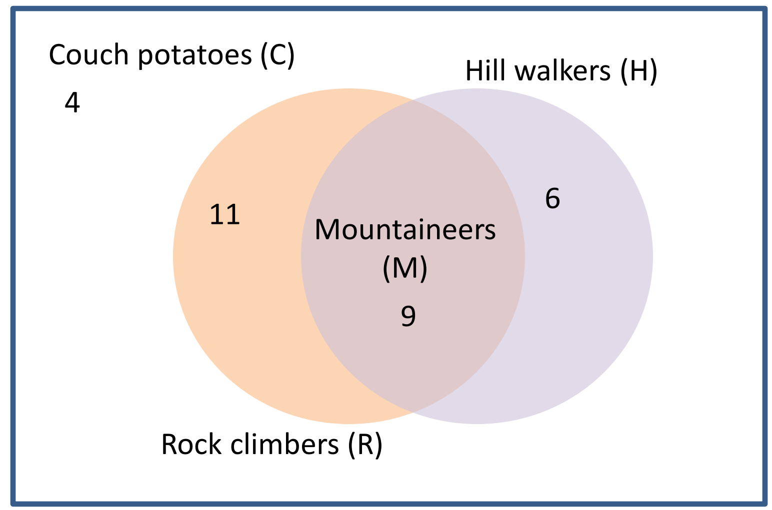



[George Boole](George Boole)
1815-1864

**Boolean algebra**

$$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$$
$$A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$$
$$A\cap(A\cup B)=A$$
$$A\cup(A\cap B)=A$$
$$A'\cap B'=(A\cup B)'$$
$$A'\cup B'=(A\cap B)'$$

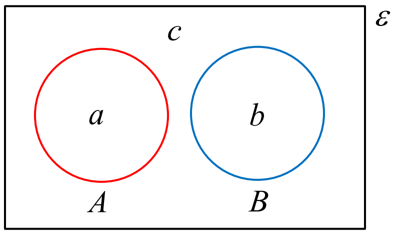
Last two are the *De-Morgan laws*




n(M) = 9;   n(R) = 20;   n( R U H ) = 26;   n( ε ) = 30

## Venn diagrams and probability

### Mutual exclusivity




Sets $A$ and $B$ are ***disjoint*** if their intersection is null. This means $A$ and $B$ are ***mutually exclusive*** i.e. *you can be a member of set* $A$, *or* $B$, *but not both*

$$A \cap B = \varnothing$$
$$\therefore n(A \cap B) = 0$$

Define set sizes:

$$n(A) = a,\ n(B) = b,\ n\left((A \cup B)'\right) = c$$

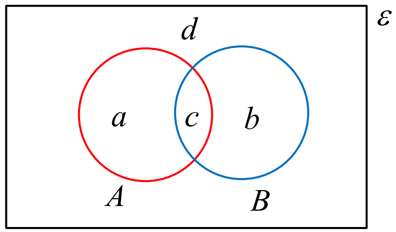
And hence define probabilities

$$P(A) = \frac{a}{a+b+c},\ P(B) = \frac{b}{a+b+c}$$
$$P(A \text{ or } B) \equiv P(A \cup B) = \frac{a+b}{a+b+c}$$

$$\therefore P(A \text{ or } B) \equiv P(A \cup B) = P(A) + P(B)$$

This is the (probability) criteria for $A$ and $B$ to be **mutually exclusive**.

### Independence




Define set sizes:

$$n(A \cap B') = a,\quad n(B \cap A') = b,$$
$$n\left((A \cap B)\right) = c,\ n\left((A \cup B)'\right) = d$$

and hence define probabilities

$$P(A) = \frac{a+c}{a+b+c+d},\ P(B) = \frac{b+c}{a+b+c+d}$$
$$P(A \,\&\, B) \equiv P(A \cap B) = \frac{c}{a+b+c+d}$$

The (probability) criteria for $A$ and $B$ to be **independent** is.

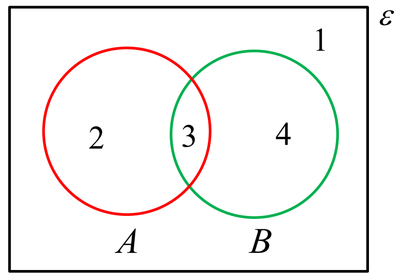
$$P(A \,\&\, B) \equiv P(A \cap B) = P(A)P(B)$$

For this to be true:

$$\underbrace{\frac{c}{a+b+c+d}}_{P(A\&B)} = \underbrace{\frac{a+c}{a+b+c+d}}_{P(A)}\ \underbrace{\frac{b+c}{a+b+c+d}}_{P(B)}$$
$$\therefore c(a+b+c+d) = (a+c)(b+c)$$
$$\therefore ca + cb + c^2 + cd = ab + cb + ac + c^2$$
$$\therefore cd = ab$$

Alas, there isn't a nice visual picture-only Venn-diagram representation of independence.

### Bayes' Theorem and conditional probability




$$P(A) = \tfrac{5}{10} = \tfrac{1}{2}$$
$$P(B) = \tfrac{7}{10}$$
$$P(B \mid A) = \tfrac{3}{5}$$
$$P(A \mid B) = \tfrac{3}{7}$$
$$P(A \,\&\, B) = P(A \cap B) = \tfrac{3}{10}$$
$$P(A \text{ or } B) = P(A \cup B) = \tfrac{9}{10}$$

$$P(A)P(B \mid A) = \tfrac{1}{2} \times \tfrac{3}{5} = \tfrac{3}{10}$$
$$P(B)P(A \mid B) = \tfrac{7}{10} \times \tfrac{3}{7} = \tfrac{3}{10}$$

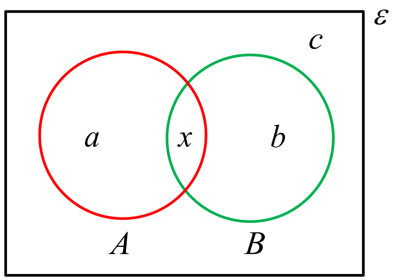
i.e. **Bayes's Theorem** holds:

$$P(A)P(B \mid A) = P(B)P(A \mid B)$$

Are A and B independent?

$$P(A \,\&\, B) = \tfrac{3}{10}$$
$$P(A)P(B) = \left(\tfrac{1}{2}\right)\left(\tfrac{7}{10}\right) = \tfrac{7}{20}$$
$$\therefore P(A \,\&\, B) \neq P(A)P(B)$$

A and B are *not* independent.




**Proof of Bayes' Theorem using Venn diagrams**

$$P(A) = \tfrac{x+a}{a+x+b+c}$$
$$P(B) = \tfrac{x+b}{a+x+b+c}$$
$$P(B \mid A) = \tfrac{x}{x+a}$$
$$P(A \mid B) = \tfrac{x}{x+b}$$
$$P(A \,\&\, B) = P(A \cap B) = \tfrac{x}{a+x+b+c}$$
$$P(A \text{ or } B) = P(A \cup B) = \tfrac{a+x+b}{a+x+b+c}$$

$$P(A)P(B \mid A) = \tfrac{x+a}{a+x+b+c} \times \tfrac{x}{x+a} = \tfrac{x}{a+x+b+c}$$
$$P(B)P(A \mid B) = \tfrac{x+b}{a+x+b+c} \times \tfrac{x}{x+b} = \tfrac{x}{a+x+b+c}$$

$$\therefore\ P(A)P(B \mid A) = P(B)P(A \mid B)$$

Note in this example:

$$n(\varepsilon) = a + x + b + c$$
$$n(A) = a + x$$
$$n(B) = b + x$$
$$n(A \cap B') = a$$
$$n(B \cap A') = b$$