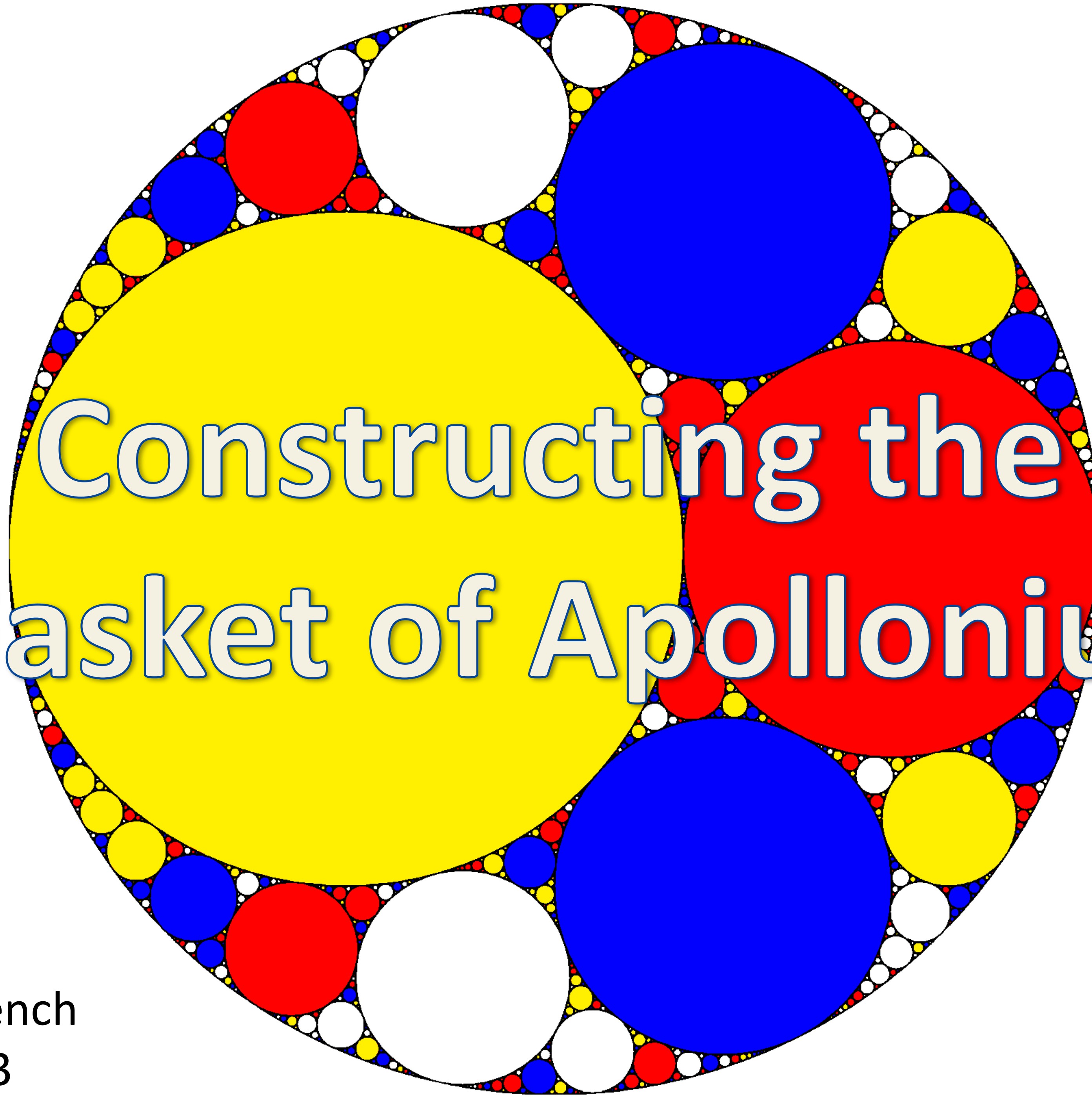
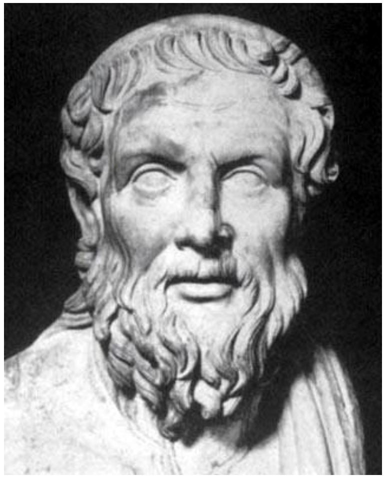
Constructing the Gasket of Apollonius



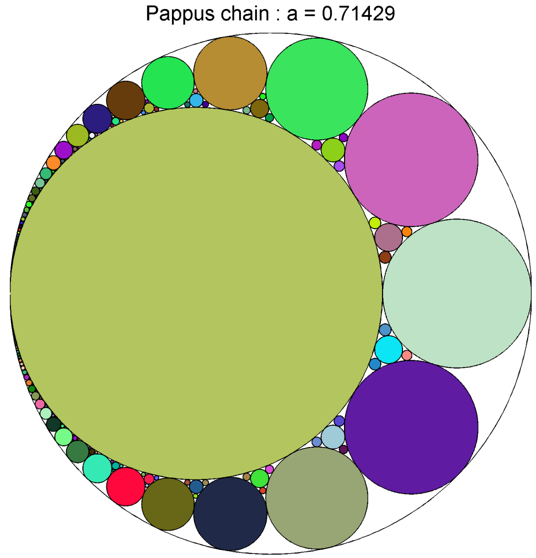
Andy French
July 2023

Mathematicians in this story



Apollonius of Perga
240BC to 190BC

Henri Poincaré
(1854-1912)



PAPPI
ALEXANDRINI
MATHEMATICAE
COLLECTIONES
A FEDERICO
COMMANDINO
VRBINATE
IN LATINVM CONVERSAE,
ET COMMENTARIIS
ILLVSTRATAE.

PISAVRI,
Apud Hieronymum Concordiam,
M. D. LXXXVIII.
Superiorum Concessu.

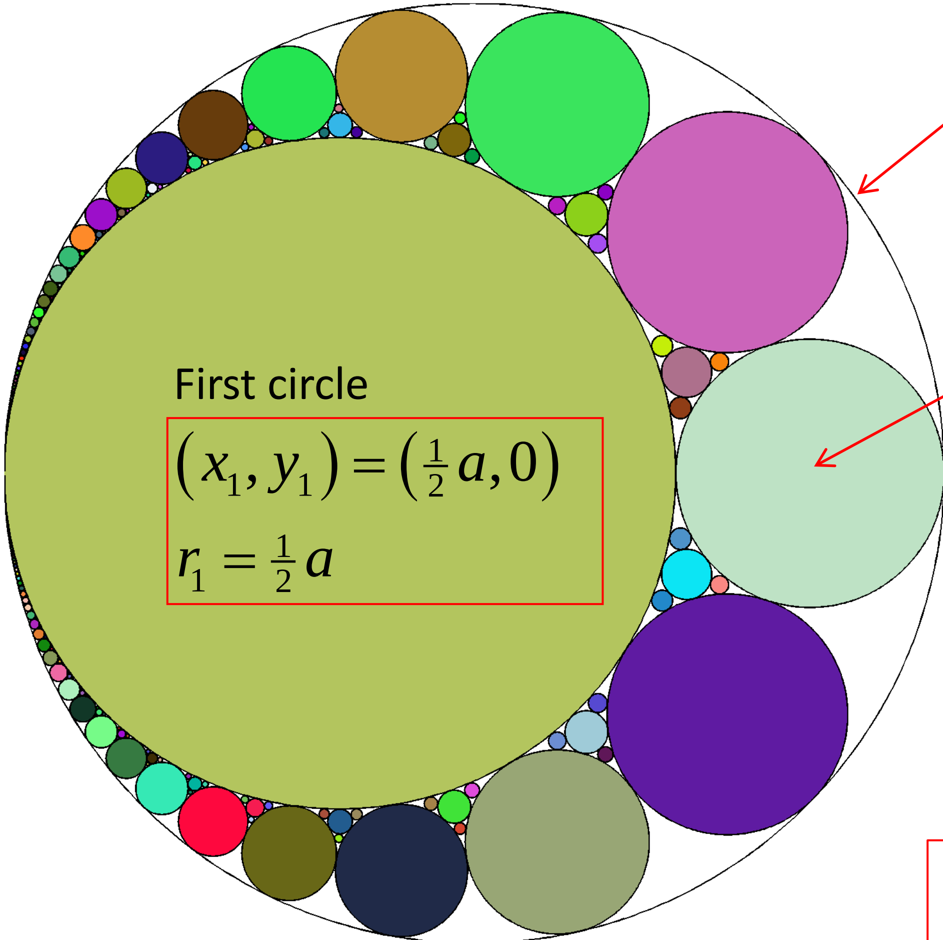
Pappus of Alexandria
290-350AD

René Descartes
1596-1650

August Ferdinand Möbius
1790-1868

Frederick Soddy
1877-1956

Pappus chain : a = 0.71429



Outer circle

$$(X, Y) = \left(\tfrac{1}{2}, 0\right), \quad R = \tfrac{1}{2}$$

Second circle

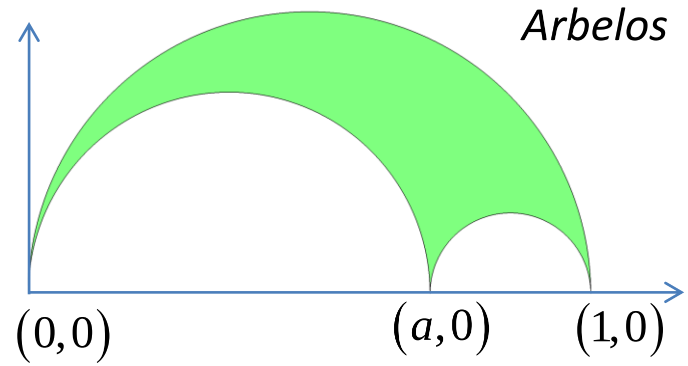
$$(x_2, y_2) = \left(\tfrac{1}{2}a + \tfrac{1}{2}, 0\right)$$

$$r_2 = \tfrac{1}{2}(1 - a)$$

First circle

$$(x_1, y_1) = \left(\tfrac{1}{2}a, 0\right)$$

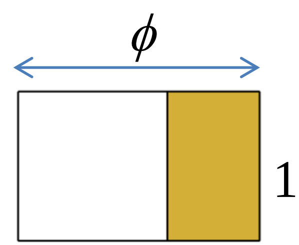
$$r_1 = \tfrac{1}{2}a$$



Pappus chain starting conditions

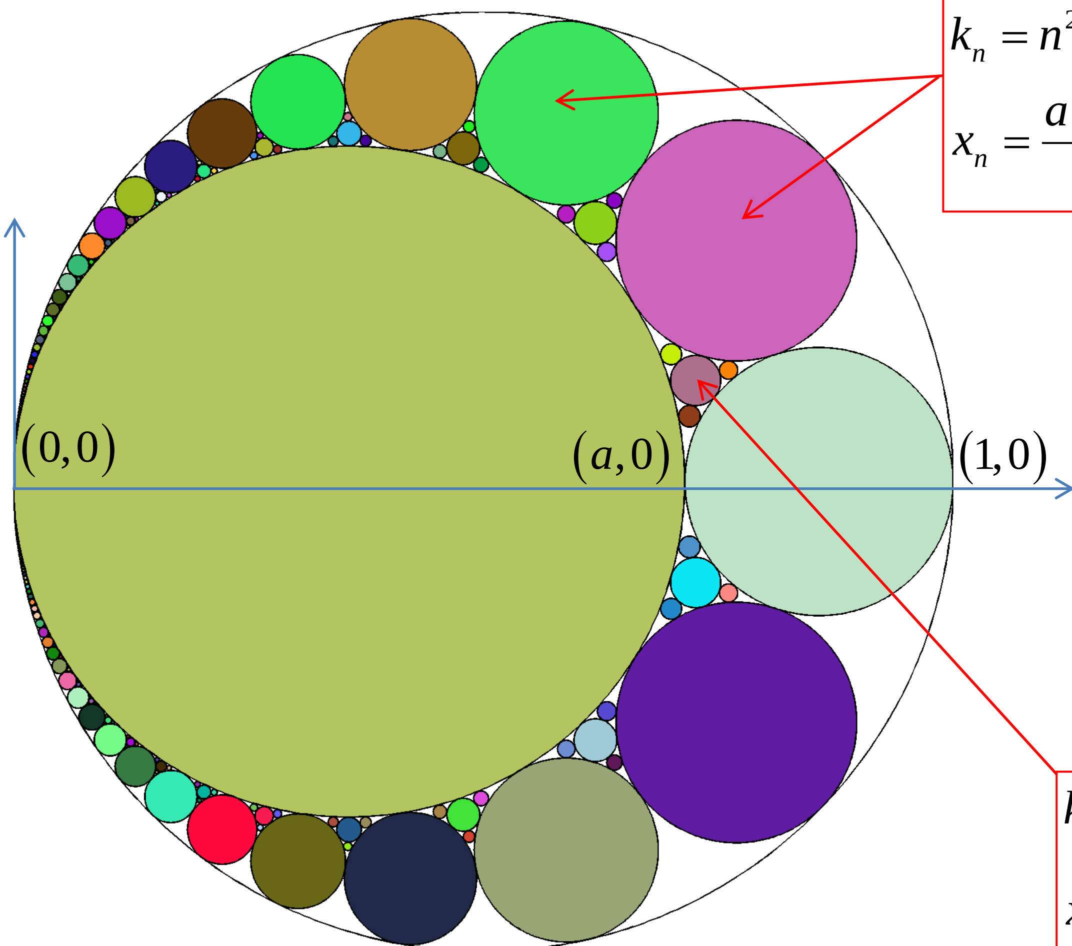
In this example a is defined via the *Golden Ratio*

$$a = \tfrac{1}{\phi} \qquad \phi = \tfrac{1}{2}\left(1 + \sqrt{5}\right)$$



Arbelos in Greek means 'shoemaker's knife'

Pappus chain : a = 0.71429



Principal circles of Pappus chain

$$k_n = n^2 (1-a)^2 + a$$

$$x_n = \frac{a(1+a)}{2k_n}, \quad y_n = \frac{na(1-a)}{k_n}, \quad r_n = \frac{a(1-a)}{2k_n}$$

Pappus chain starting conditions

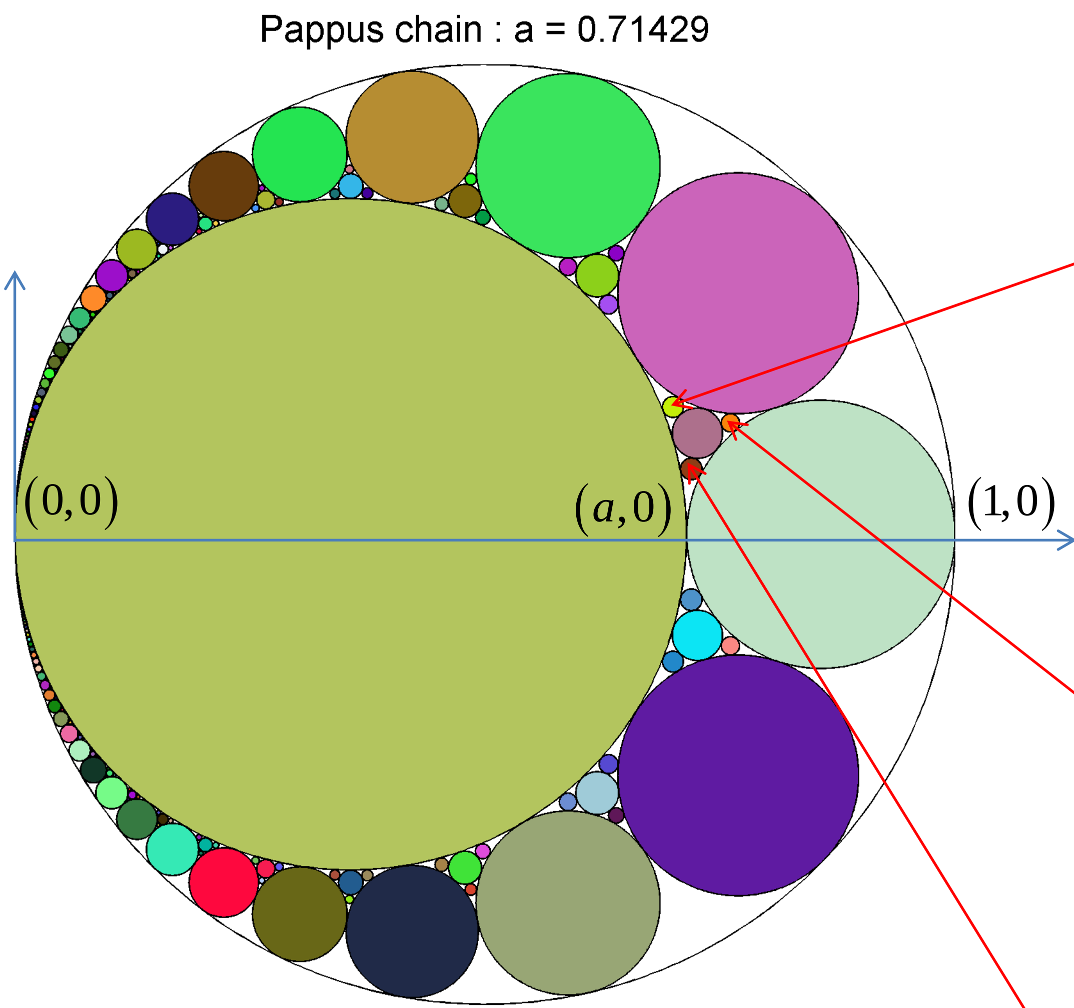
Inner tangent circles

$$k_n = 4 + 4n(n-1)(1-a)^2 + a(a-1)$$

$$x_n = \frac{a(7+a)}{2k_n}, \quad y_n = \frac{2a(1-a)(2n-1)}{k_n}$$

$$r_n = \frac{a(1-a)}{2k_n}$$

https://mathworld.wolfram.com/PappusChain.html



More tangent circles!

$$k_n = 9 + 3n(3n-2)(1-a)^2 - a(a-1)$$

$$x_n = \frac{a(17+a)}{2k_n}, \qquad y_n = \frac{3a(1-a)(3n-1)}{k_n}$$

$$r_n = \frac{a(1-a)}{2k_n}$$

$$k_n = 9 + 12n(n-1)(1-a)^2 + a(4a-1)$$

$$x_n = \frac{a(17+7a)}{2k_n}, \qquad y_n = \frac{6a(1-a)(2n-1)}{k_n}$$

$$r_n = \frac{a(1-a)}{2k_n}$$

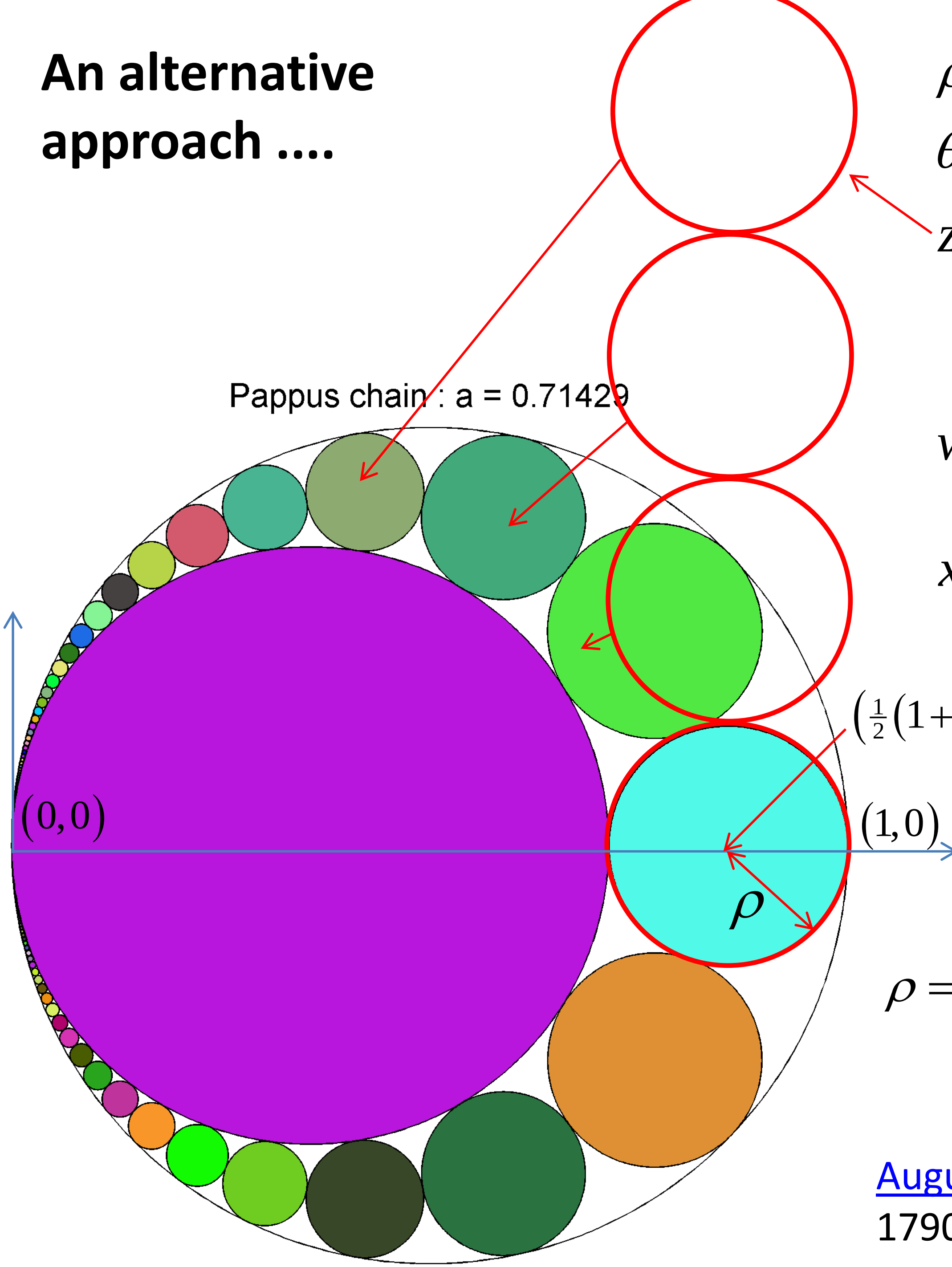
$$k_n = 12 + 3n(3n-4)(1-a)^2 + a(4a-7)$$

$$x_n = \frac{a(17+a)}{2k_n}, \qquad y_n = \frac{3a(1-a)(3n-2)}{k_n}$$

$$r_n = \frac{a(1-a)}{2k_n}$$

https://mathworld.wolfram.com/PappusChain.html

An alternative approach ....



$$\rho = \tfrac{1}{2}(1-a)$$

$$\theta = 0 \ldots 2\pi$$

Define vertical circle stack as complex numbers

$$z_n = \tfrac{1}{2}\left(1+a\right) + \rho\cos\theta + i\left(\rho\sin\theta + (n-1)(1-a)\right)$$

$$w_n = \frac{a}{z_n}$$

$$x_n = \mathrm{Re}\left(w_n\right), \quad y_n = \mathrm{Im}\left(w_n\right)$$

Note the *circles* transform, *not* the circle centres!

Determine Pappus chain via a **Möbius complex transformation** (an *inversion*) of a vertical stack of circles

$$\rho = \tfrac{1}{2}(1-a)$$

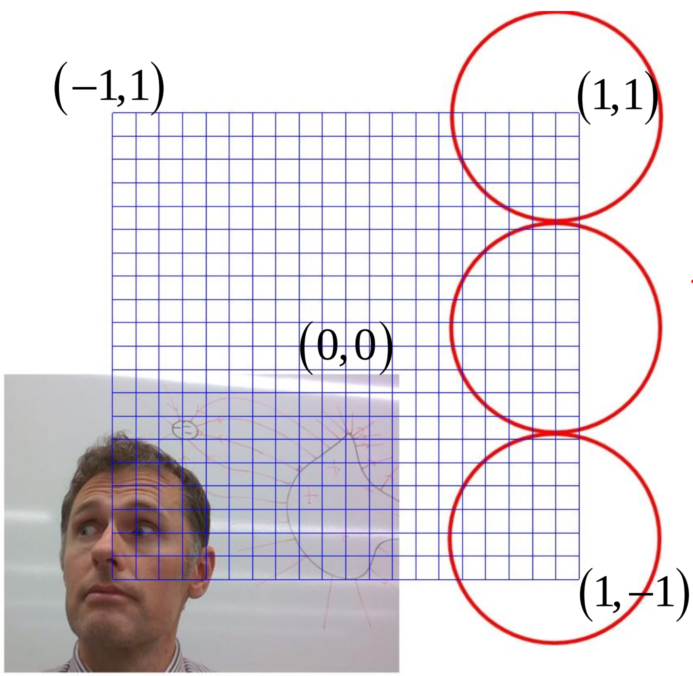
August Ferdinand Möbius
1790-1868

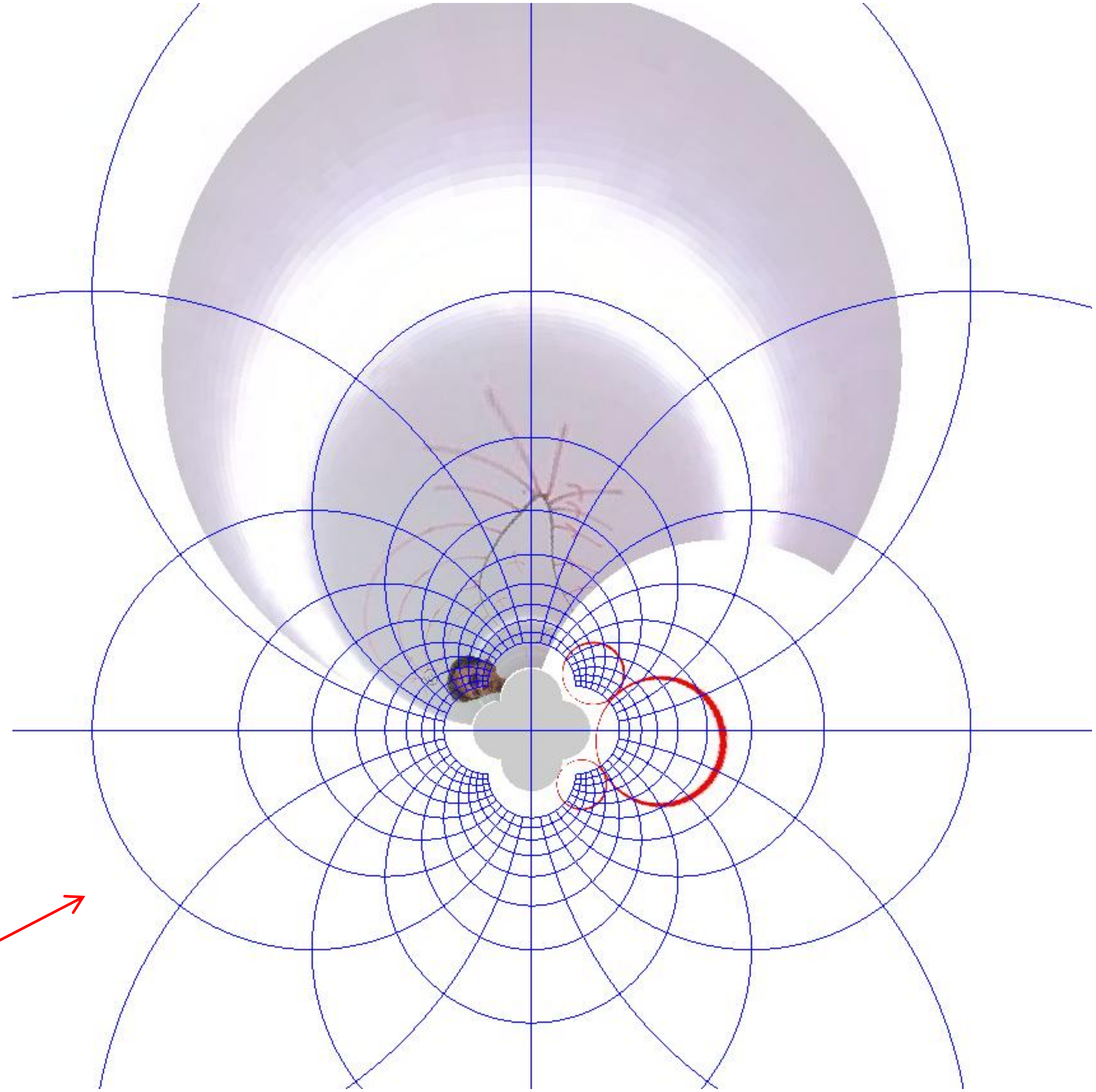
$$z = x + iy$$

$$z \rightarrow \frac{1}{z}$$

$$x \rightarrow \mathrm{Re}(z)$$

$$y \rightarrow \mathrm{Im}(z)$$





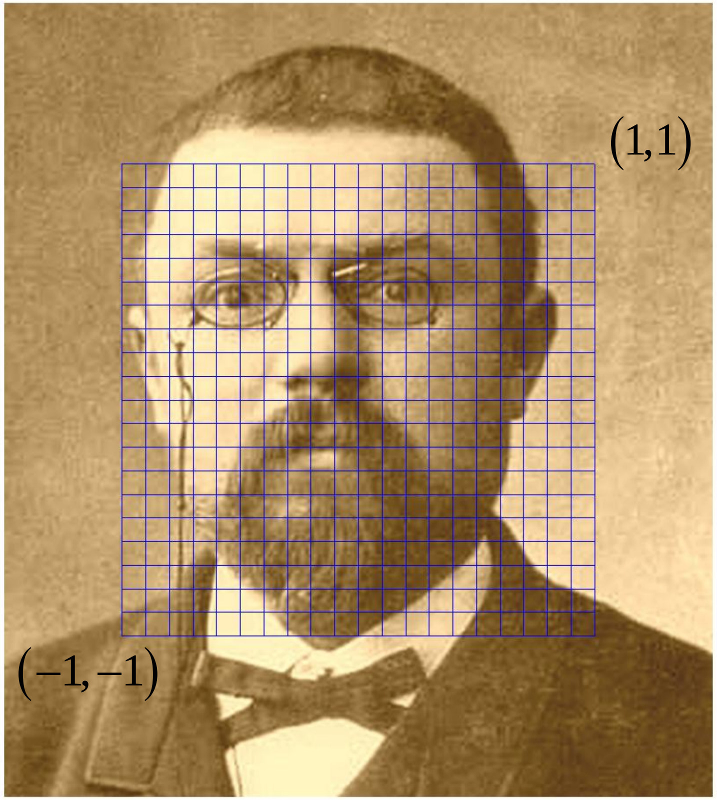
Inversion complex transform

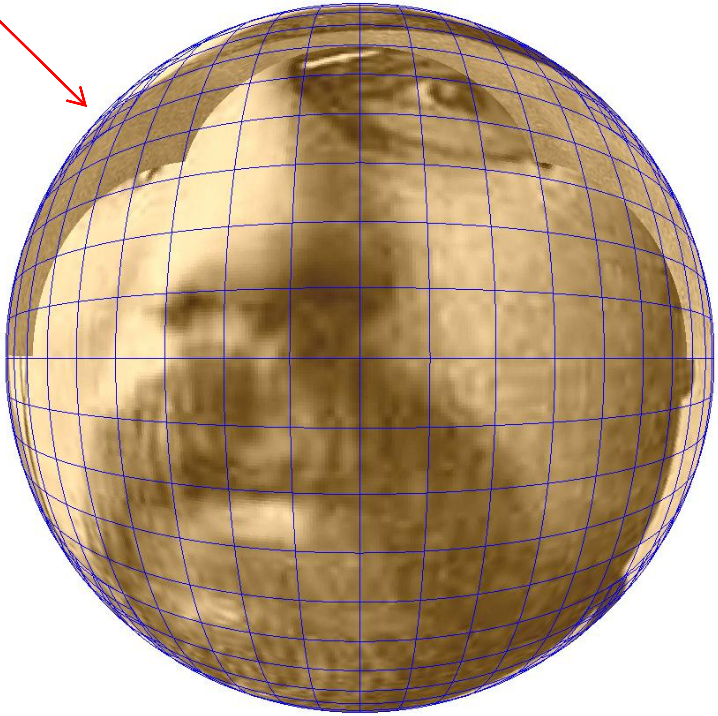
Aside: this is a **Poincaré disc transform**.
Everything is mapped to the unit circle.

$$x \to \frac{2x}{x^2 + y^2 + 1}$$

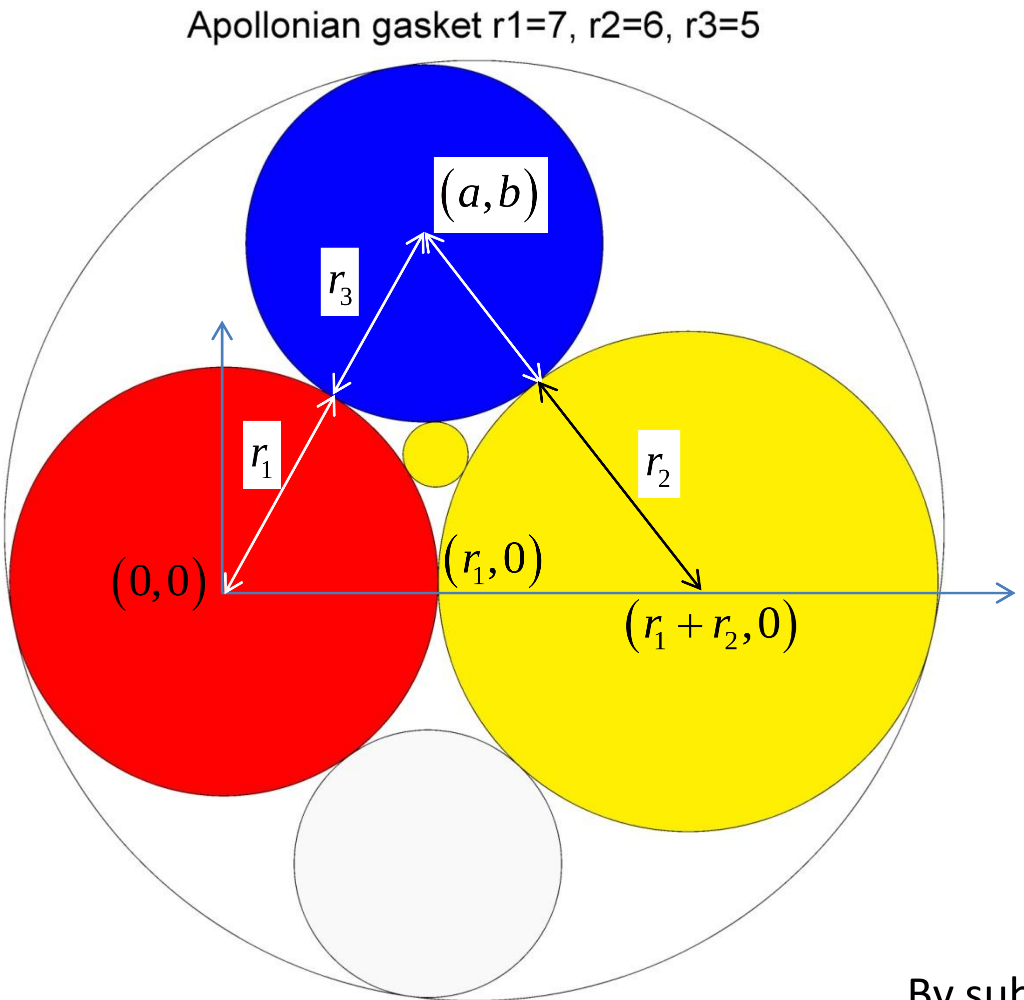
$$y \to \frac{2y}{x^2 + y^2 + 1}$$

This is visually equivalent to a reflection in a spherical mirror





[Henri Poincaré](#) (1854-1912)



Alternative three-circle initial conditions

Inputs: $\boxed{r_1\ ,\ r_2\ ,\ r_3}$

Pythagoras' Theorem:

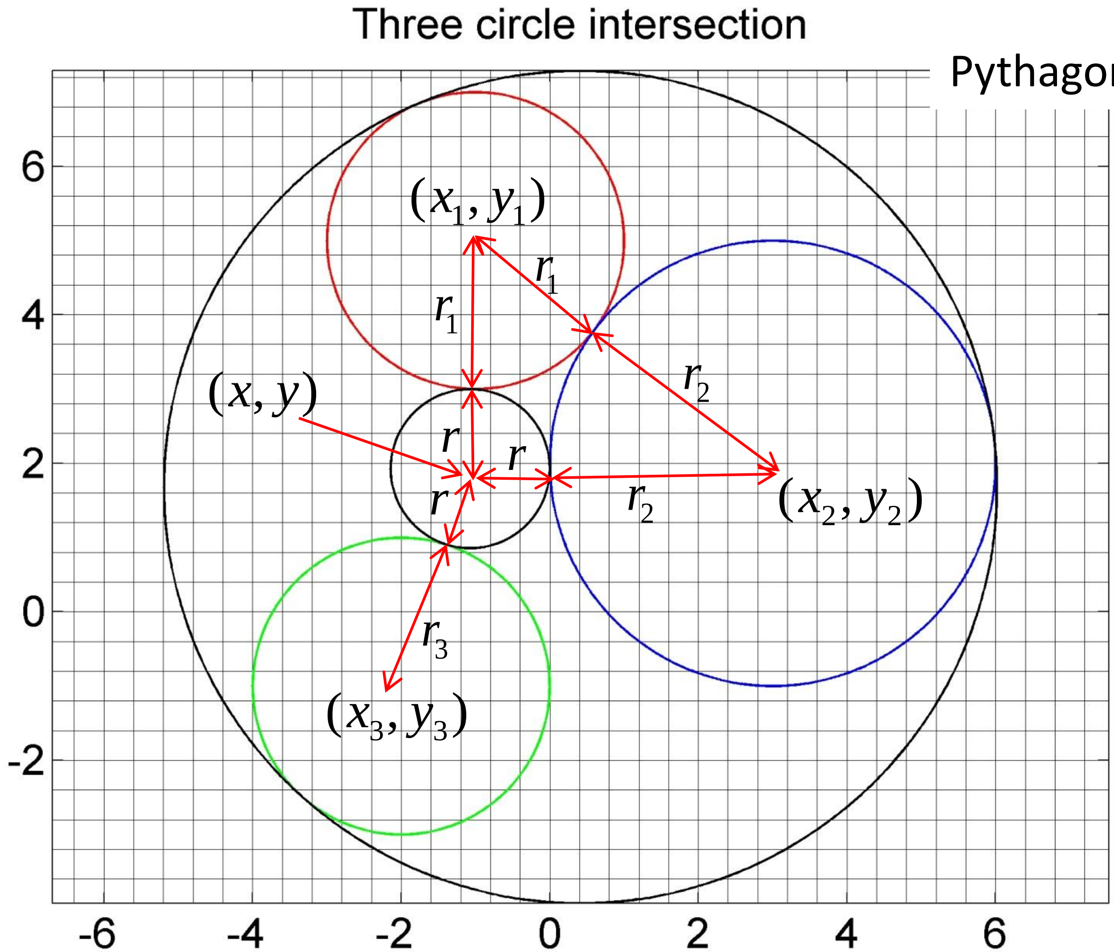
$$a^2 + b^2 = \left(r_1 + r_3\right)^2$$

$$\left(-a + r_1 + r_2\right)^2 + b^2 = \left(r_2 + r_3\right)^2$$

By subtracting these equations:

$$\boxed{\begin{aligned} a &= \frac{(r_1+r_2)^2 - (r_3+r_2)^2 + (r_1+r_3)^2}{2\left(r_1+r_2\right)} \\ b &= \sqrt{(r_1+r_3)^2 - a^2} \end{aligned}}$$

i.e. determine centre of **third** mutually tangent circle



Pythagoras:

$$(r_1+r)^2=(x-x_1)^2+(y-y_1)^2$$
$$(r_2+r)^2=(x-x_2)^2+(y-y_2)^2$$
$$(r_3+r)^2=(x-x_3)^2+(y-y_3)^2$$

Subtract pairs of equations, expand, and reduce to a quadratic equation in *r*. The pair of solutions yields the inner circle and the outer *circumcircle*.

$$r=\frac{-C_1\pm\sqrt{C_1^2-C_0C_2}}{C_2}$$
$$x=A_0+A_1r$$
$$y=B_0+B_1r$$

$$A_0=\frac{K_a\left(y_1-y_3\right)+K_b\left(y_2-y_1\right)}{2D}$$

$$B_0=-\frac{K_a\left(x_1-x_3\right)+K_b\left(x_2-x_1\right)}{2D}$$

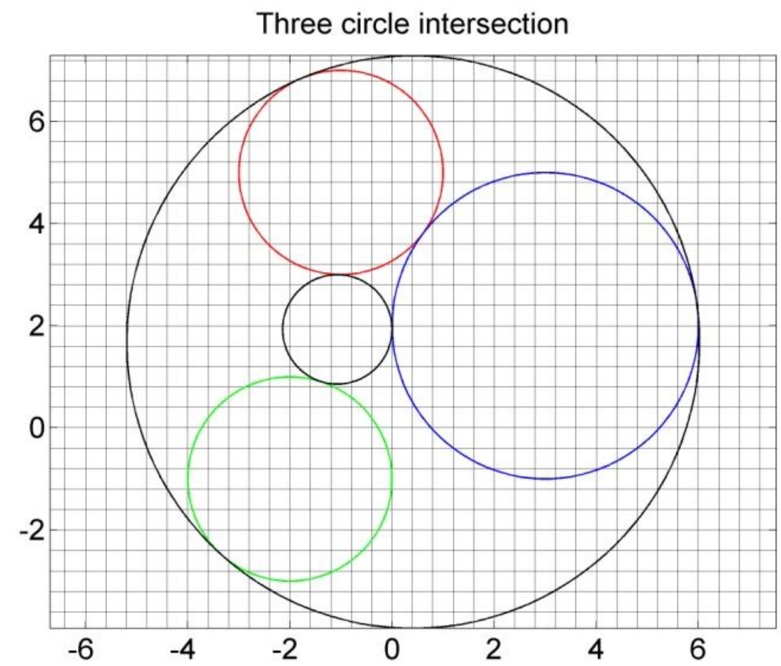
$$A_1=-\frac{r_1(y_2-y_3)+r_2(y_3-y_1)+r_3(y_1-y_2)}{D}$$

$$B_1=\frac{r_1(x_2-x_3)+r_2(x_3-x_1)+r_3(x_1-x_2)}{D}$$

$$K_a=r_1^2-r_2^2-x_1^2+x_2^2-y_1^2+y_2^2$$
$$K_b=r_1^2-r_3^2-x_1^2+x_3^2-y_1^2+y_3^2$$
$$D=x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)$$

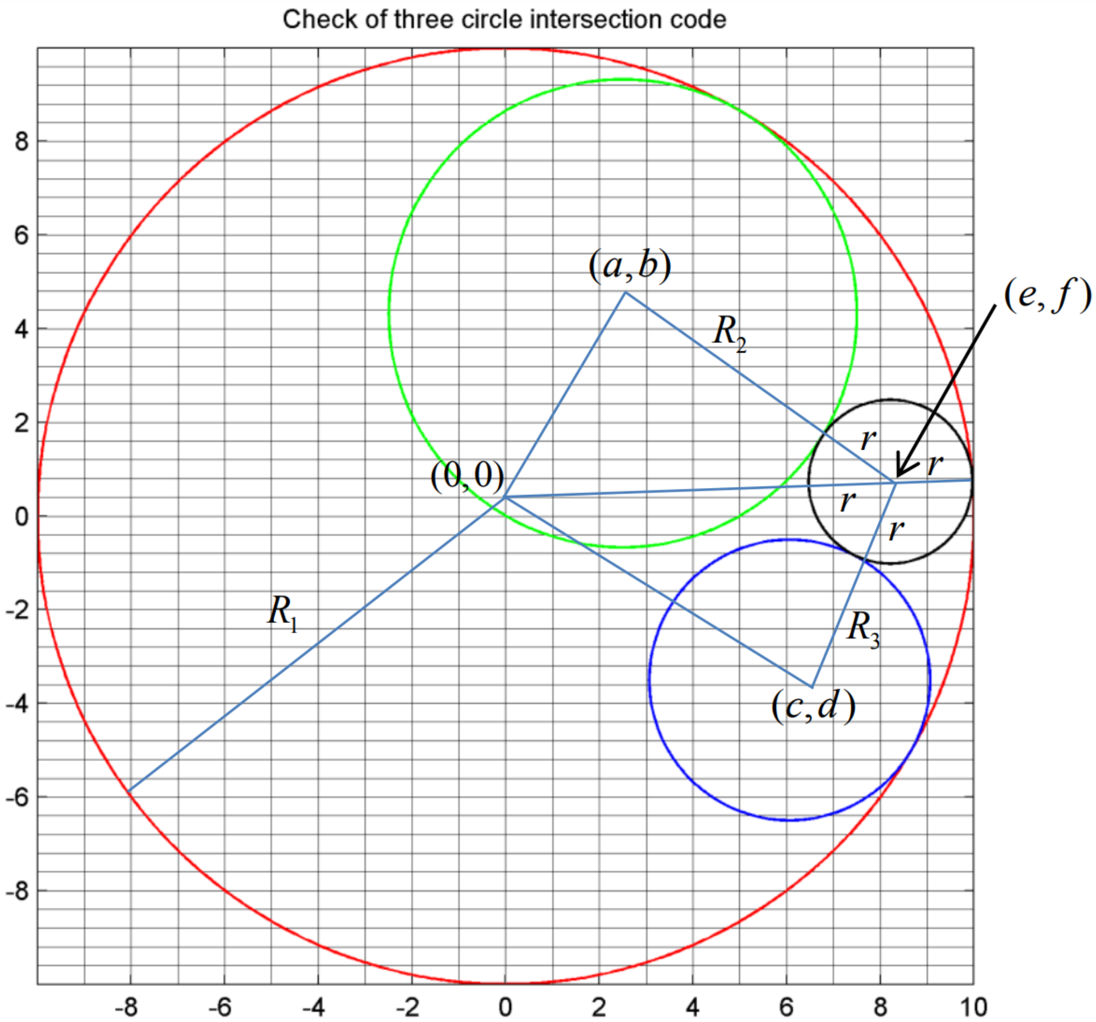
https://math.stackexchange.com/questions/3100828/calculate-the-circle-that-touches-three-other-circles

```
%Solve for centre (x,y) and radius r of circle that touches three other circles
function [x1,y1,r1,x2,y2,r2] = three_circle_touch( x_1,y_1,r_1,x_2,y_2,r_2,x_3,y_3,r_3 )

% Find constant of circle #2- circle #1
K_a = -r_1^2+r_2^2+x_1^2-x_2^2+y_1^2-y_2^2;

% Find constant of circle #3- circle #1
K_b = -r_1^2+r_3^2+x_1^2-x_3^2+y_1^2-y_3^2;

% Find constants of [x=A_0+A_1*r, y=B_0+B_1*r]
D = x_1*(y_2-y_3)+x_2*(y_3-y_1)+x_3*(y_1-y_2);
A_0 =(K_a*(y_1-y_3)+K_b*(y_2-y_1))/(2*D);
B_0 = -(K_a*(x_1-x_3)+K_b*(x_2-x_1))/(2*D);
A_1 = -(r_1*(y_2-y_3)+r_2*(y_3-y_1)+r_3*(y_1-y_2))/D;
B_1 = (r_1*(x_2-x_3)+r_2*(x_3-x_1)+r_3*(x_1-x_2))/D;
```




```
% Find constants of C_0 + 2*C_1*r + C_2^2 = 0
C_0 = A_0^2-2*A_0*x_1+B_0^2-2*B_0*y_1-r_1^2+x_1^2+y_1^2;
C_1 = A_0*A_1-A_1*x_1+B_0*B_1-B_1*y_1-r_1;
C_2 = A_1^2+B_1^2-1;

% Solve for r,x,y
r1=( -sqrt(C_1^2-C_0*C_2)-C_1)/C_2;
x1 = A_0 + A_1*r1; y1 = B_0 + B_1*r1;
r2=( sqrt(C_1^2-C_0*C_2)-C_1)/C_2;
x2 = A_0 + A_1*r2; y2 = B_0 + B_1*r2;
```

MATLAB code for (externally tangent) 'kissing circles'



$$(R_1 - r)^2 = e^2 + f^2$$

$$(R_2 + r)^2 = (e-a)^2 + (f-b)^2$$

$$(R_3 + r)^2 = (e-c)^2 + (f-d)^2$$

$$r = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

$$e = A + Br$$

$$f = C + Dr$$

$$\alpha = 1 - B^2 - D^2$$

$$\beta = -2R_1 - 2AB - 2CD$$

$$\gamma = R_1^2 - A^2 - C^2$$

$$A = \frac{\left(R_2^2 - R_1^2\right)d - \left(R_3^2 - R_1^2\right)b + c^2 b + d^2 b - a^2 d - b^2 d}{2\left(cb - ad\right)}$$

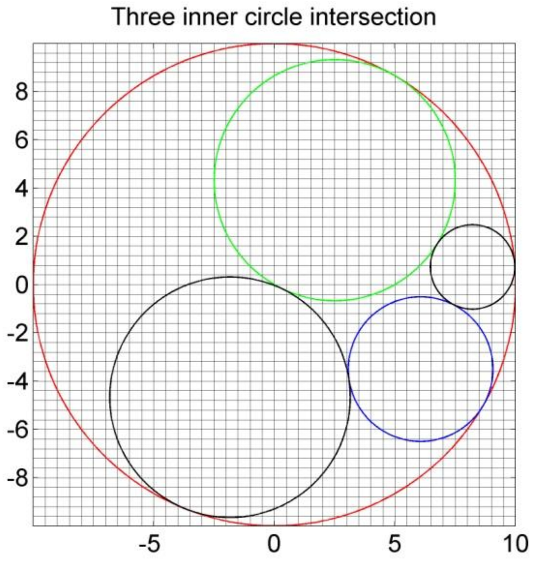
$$C = \frac{\left(R_2^2 - R_1^2\right)c - \left(R_3^2 - R_1^2\right)a + c^2 a + d^2 a - a^2 c - b^2 c}{2\left(da - bc\right)}$$

$$B = \frac{\left(R_2 + R_1\right)d - \left(R_3 + R_1\right)b}{cb - ad}$$

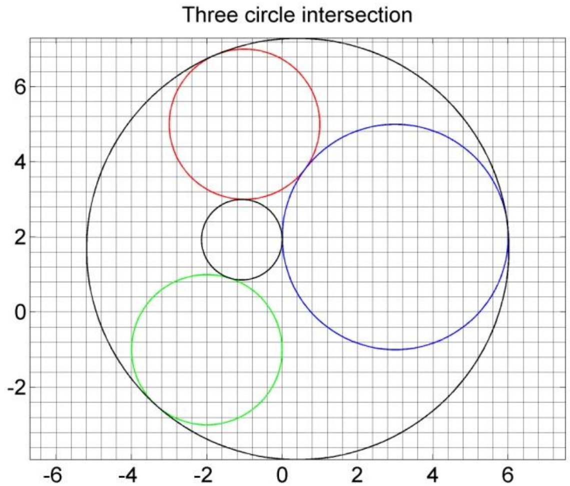
$$D = \frac{\left(R_2 + R_1\right)c - \left(R_3 + R_1\right)a}{da - bc}$$

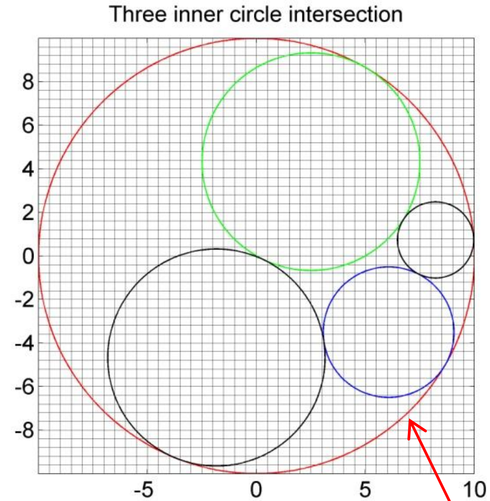
MATLAB code

```
%Solve for centre (x,y) and radius r of circles that touch two circles
%within the larger outer circle with centre (x1,y1) and radius R1
function [xx1,yy1,rr1,xx2,yy2,rr2] = three_circle_touch( x1,y1,R1,x2,y2,R2,x3,y3,R3 )

%Solve assuming larger circle centre is (0,0)
a = x2 - x1; b = y2 - y1; c = x3 - x1; d = y3 - y1;

%Compute coefficients of e = A+B*r and f = C+D*r  where x = e + x1 and y = f + y1

A = ( R2^2 - R1^2 )*d - (R3^2 - R1^2 )*b + (c^2) * b + (d^2) * b - (a^2) *d - (b^2) *d;
A = A/( 2*c*b - 2*a*d );

B = (R2 + R1)*d - (R3+R1)*b;  B = B/( c*b - a*d );

C = ( R2^2 - R1^2 )*c - (R3^2-R1^2)*a +(c^2)*a +(d^2)*a -(a^2)*c -(b^2)*c;
C = C/(2*d*a - 2*b*c);

D = (R2 + R1)*c - (R3 + R1)*a; D = D/(d*a - b*c);

%Now solve quadratic for r
aa = 1 - B^2 -D^2; bb = -2*R1 -2*A*B - 2*C*D; cc = R1^2 - A^2 - C^2;
rr1 = ( - bb + sqrt( bb^2 -4*aa*cc ) )/(2*aa); rr2 = ( - bb - sqrt( bb^2 -4*aa*cc ) )/(2*aa);

%Determine circle centre
xx1 = A + B*rr1 + x1; yy1 = C + D*rr1 + y1; xx2 = A + B*rr2 + x1; yy2 = C + D*rr2 + y1;
```



René Descartes
1596-1650





Frederick Soddy
1877-1956

A more 'complex' method due to Descartes (and rediscovered by Soddy)

$$z_{1,2,3} = x_{1,2,3} + iy_{1,2,3}$$

Fourth circle centre: $x_4 = \mathrm{Re}(z_4), \quad y_4 = \mathrm{Im}(z_4),$

$$k_{1,2,3} = \frac{1}{r_{1,2,3}}$$

'***Curvature***' (or 'bend'). Note this is **negative** for internally tangent circles

$$k_4 = k_1 + k_2 + k_3 \pm 2\sqrt{k_1k_2 + k_2k_3 + k_3k_1}$$

$$r_4 = \left|1/k_4\right|$$

Alas these +/- don't match up!

$$z_4 = z_1k_1 + z_2k_2 + z_3k_3 \pm 2\sqrt{k_1k_2z_1z_2 + k_2k_3z_2z_3 + k_3k_1z_1z_3}$$

Slight snag: which pair of the *four* possible solutions to you use? Code solution: run a check using the original Pythagorean equations

Recipe for generating the Apollonian Gasket

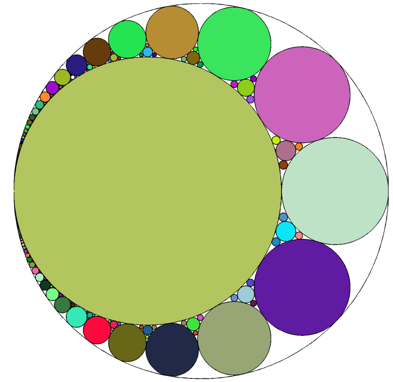
1. **Run initial condition.** Pappus chain, or three circles. If the latter, determine the circumcircle and another two tangent circles.

2. **Determine a N x 3 array of ‘new kisses’.** For each remaining ‘space’ there will be three circles that can be mutually tangent to a fourth circle.

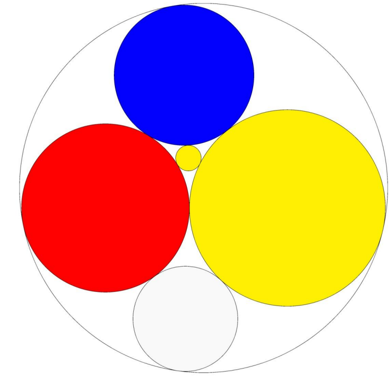
3. Step through ‘new kisses’ array and **compute (via Descartes/Soddy or the circle intersection recipes) the new circles.** Make a fresh ‘new kisses array’ to avoid creating the same circles.

4. **Repeat** until circle radii drop below a defined minimum.

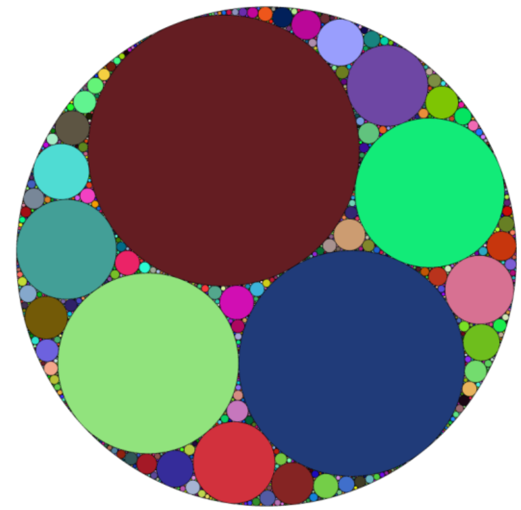
Pappus chain : a = 0.71429

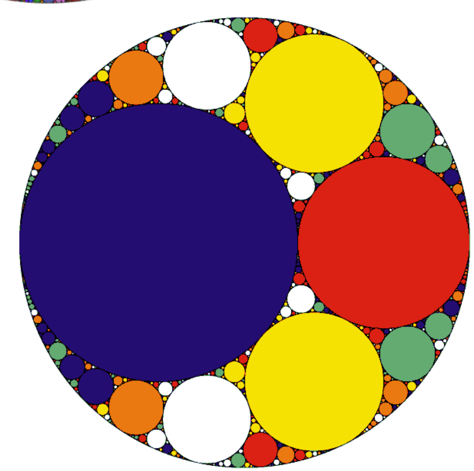


Apollonian gasket r1=7, r2=6, r3=5



Apollonian gasket r1=6, r2=5, r3=4

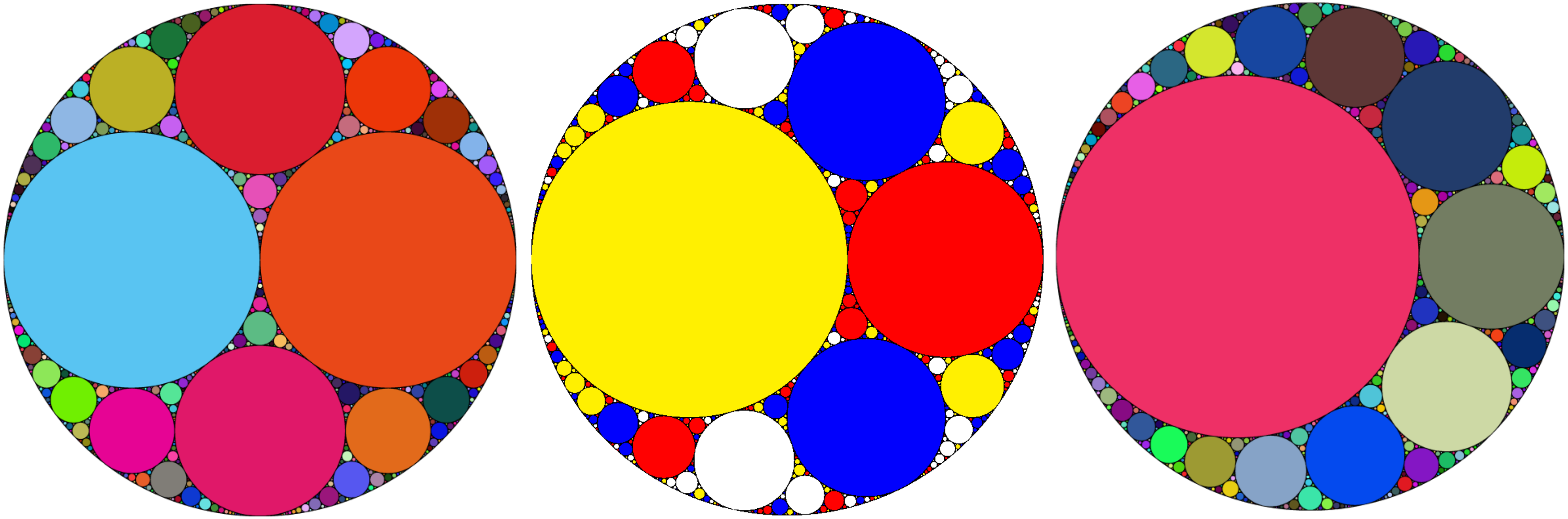




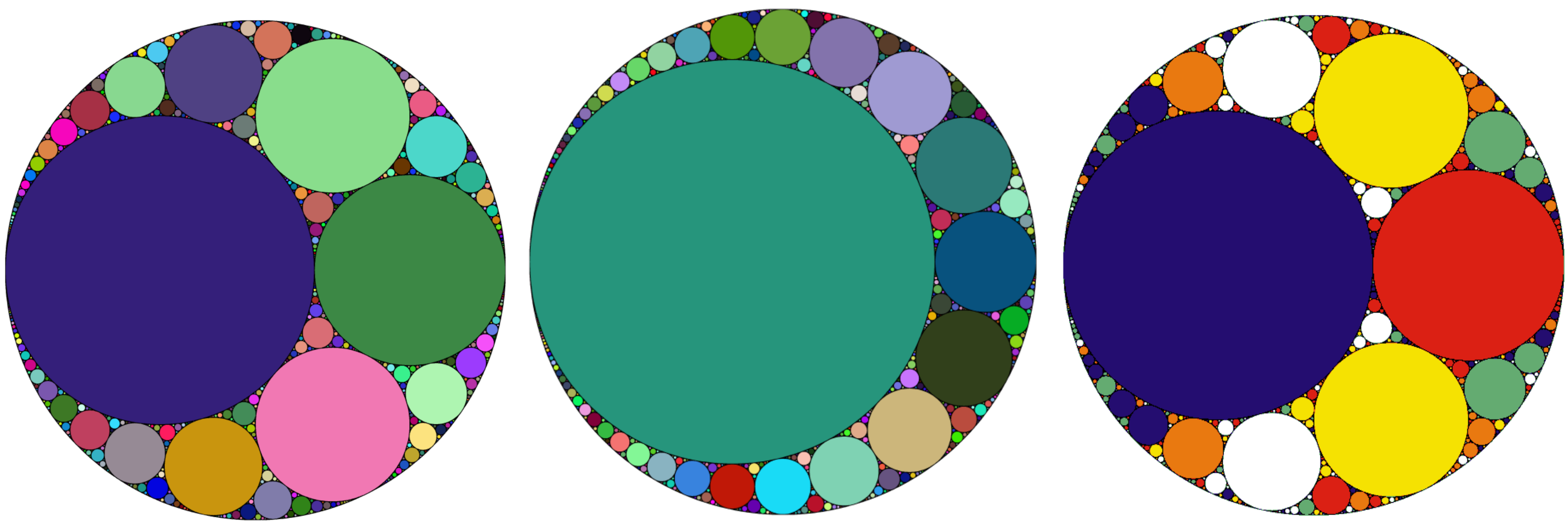
My MATLAB inputs:

Pappus: Chain of 100 then 20 rounds of finding new kisses.
min r of 0.0001. Note circumcircle radius is 0.5.

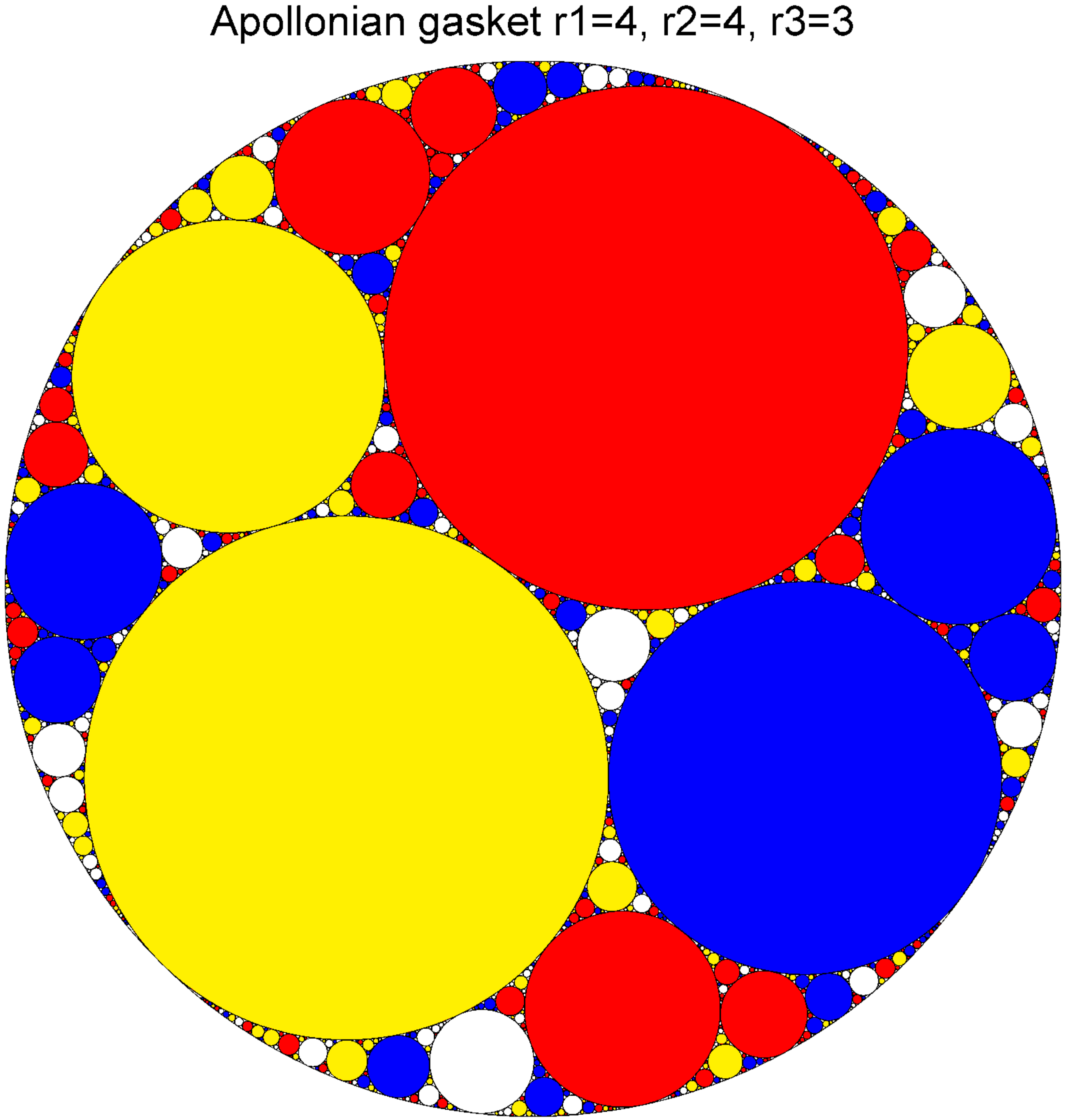
Three circles: Max circle radius 10.
Then 50 rounds of finding new kisses.
min r of 0.02.



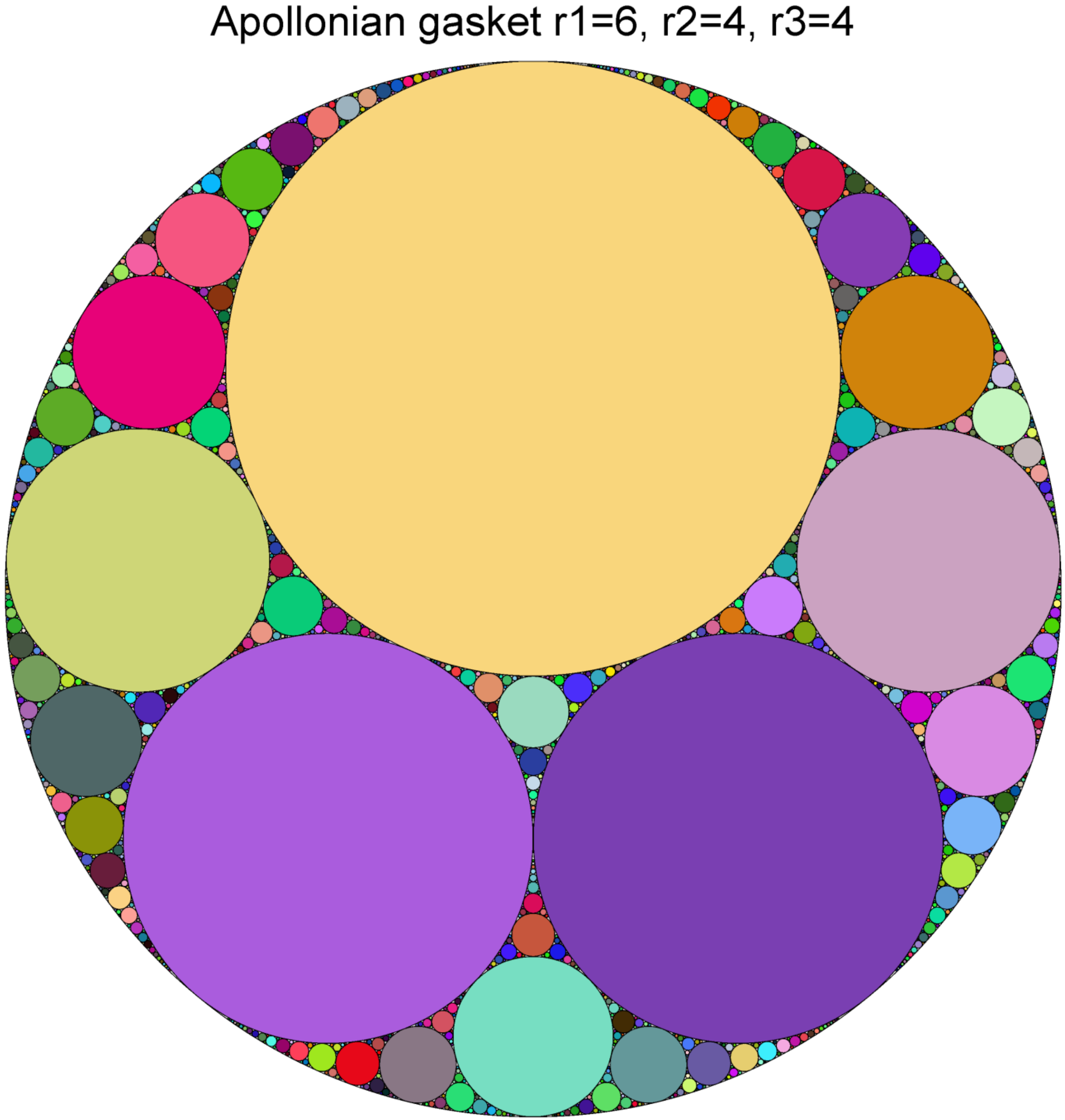
Pappus chain inputs



Three circle inputs



Three circle inputs



Malin Christersson's VERY fast gasket generator!

Not secure | malinc.se/math/geometry/apolloniangasketen.php

about

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Pythagoras & Thales

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Circumscribed and Inscribed Circles

Apollonian Gasket

Ruler and Compass

Malin Christersson 2019

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← Circumscribed and Inscribed Circles

Ruler and Compass →

Apollonian Gasket

☑ Show Apollonian circles

☐ Show perpendicular circles.

☑ Show dots.

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