Approximately regular N-gon construction scheme

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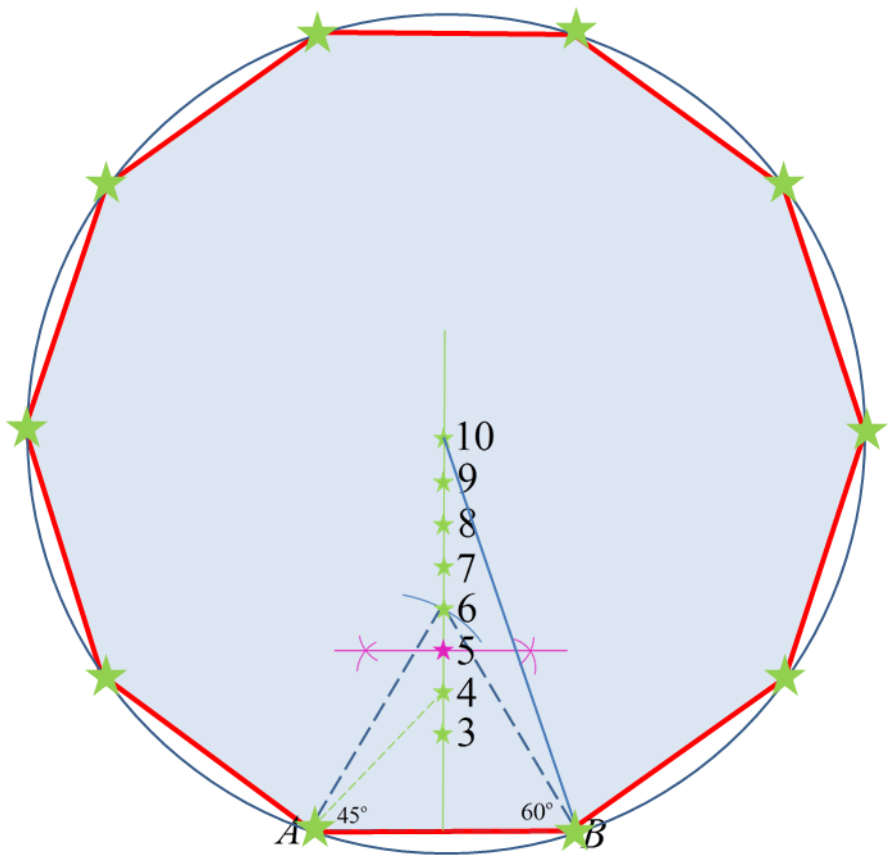
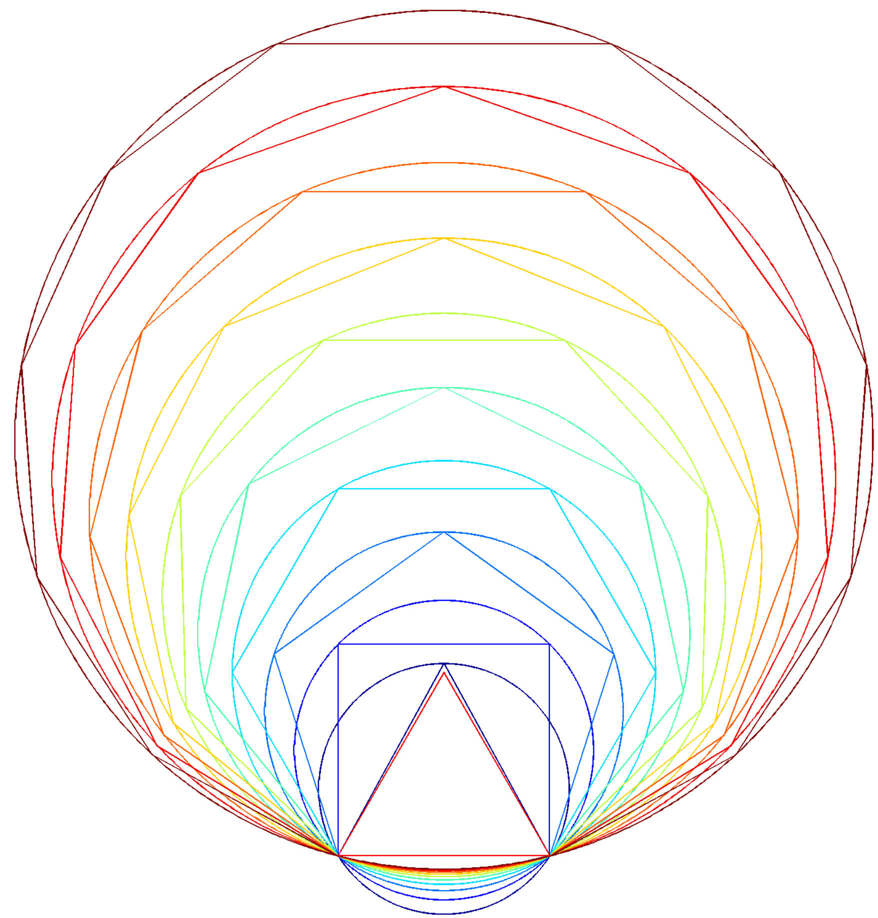
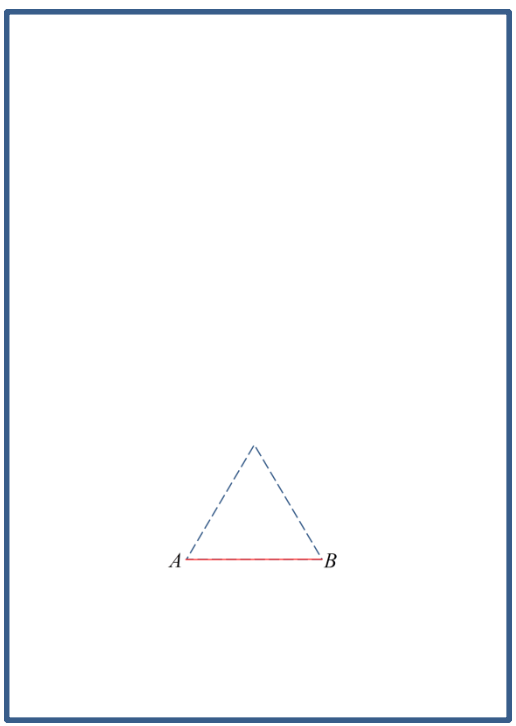
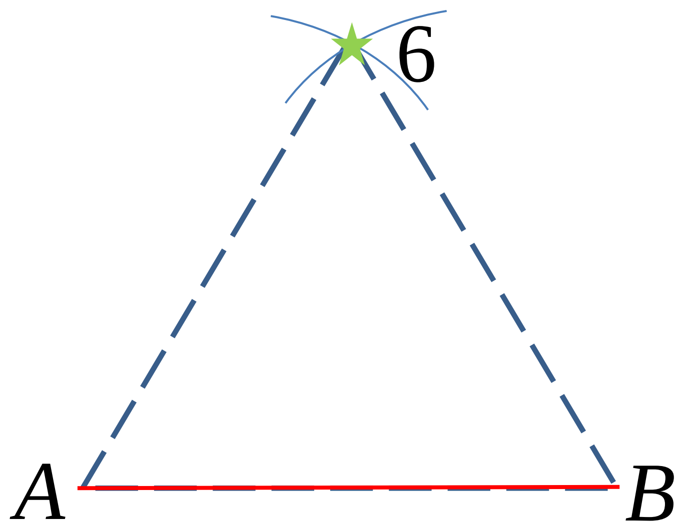


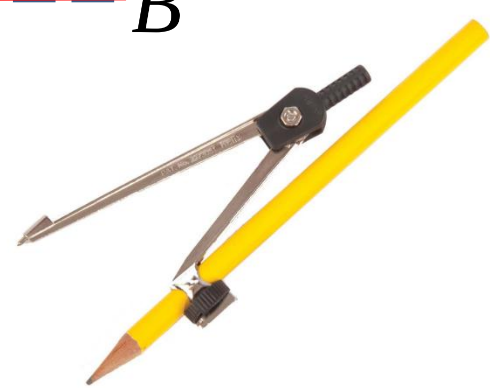


1. Draw a straight line AB, which will be the side length of the N-gon. If you plan to construct an N-gon with a large number of sides, don't make this too large otherwise you will run out of paper!

2. Span a compass between A and B and draw arcs centred on A and B respectively. Where these arcs intersect is the apex of an equilateral triangle. Label this '6'

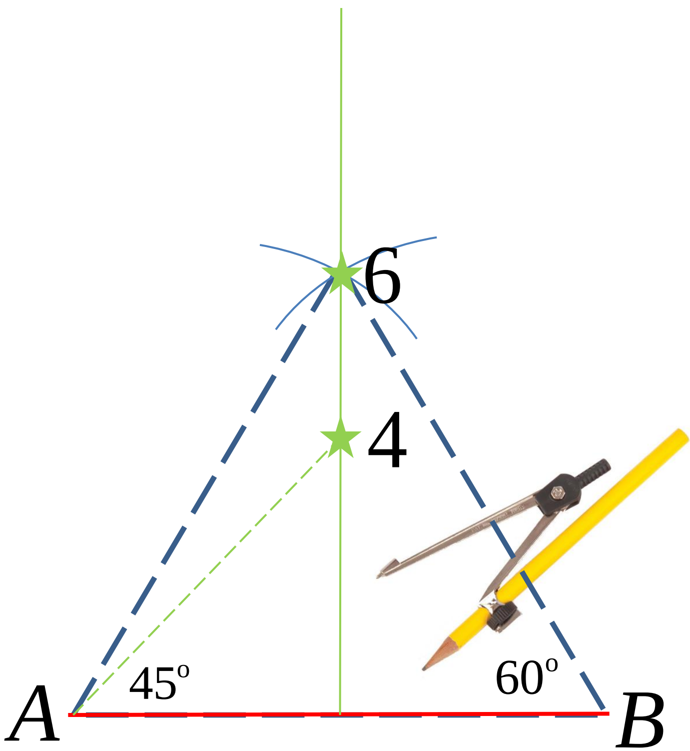




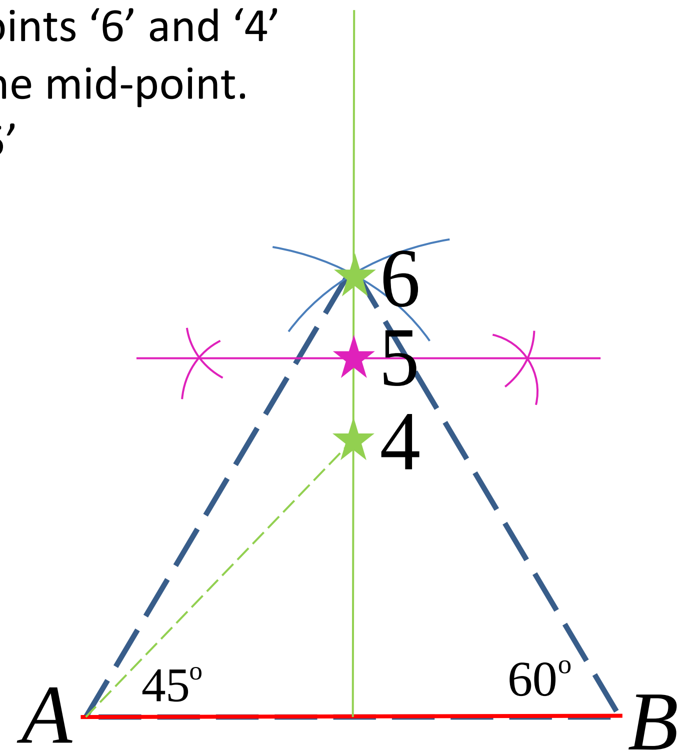


3. Construct a perpendicular bisector of line AB and hence connect the triangle apex with the base. Extend this line vertically upwards.

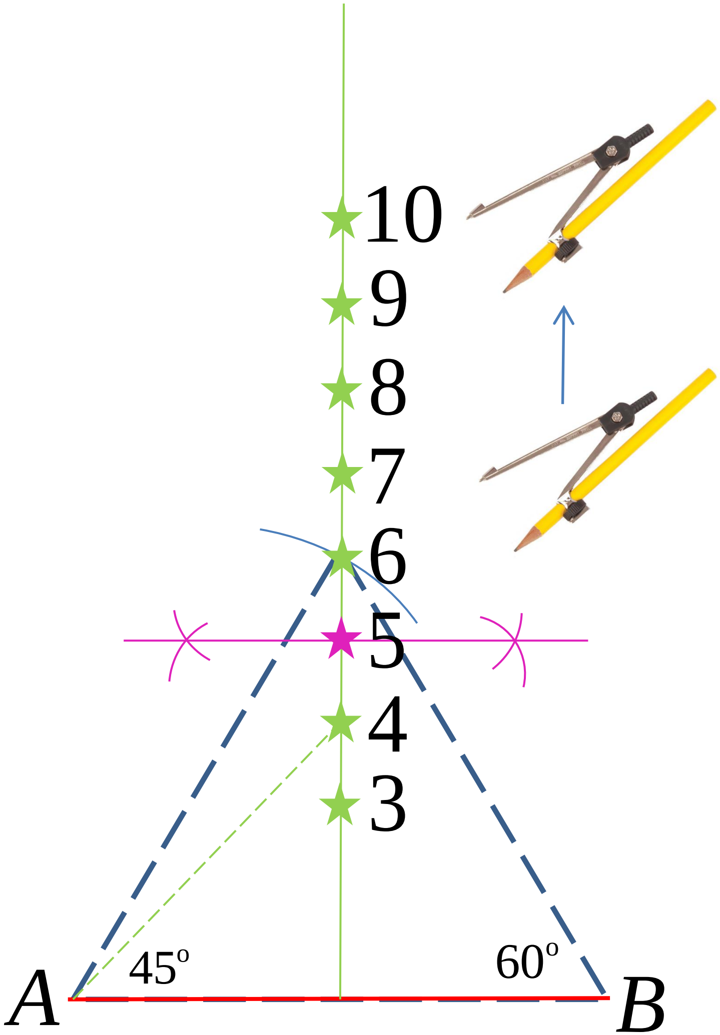
4. Use a compass to mark half of AB upwards along the perpendicular, thereby constructing a 45º angled line, to go with the 60º angled line of the equilateral triangle. Label the point along the perpendicular ‘4’.



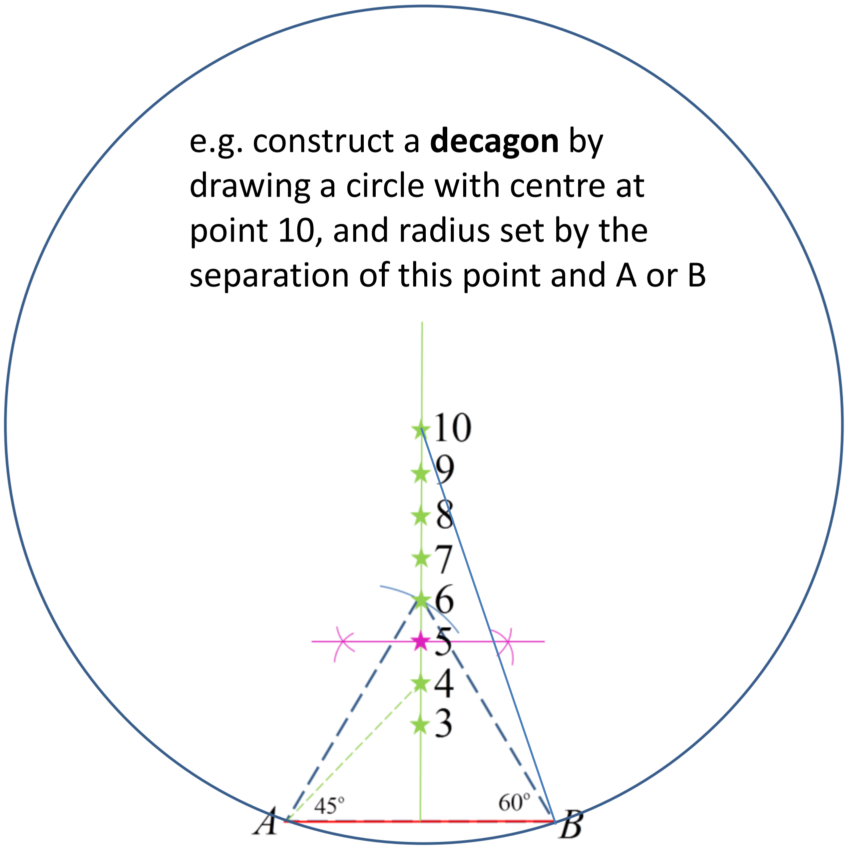
5. Now construct the perpendicular bisector between points ‘6’ and ‘4’ and mark the mid-point. Label this ‘5’



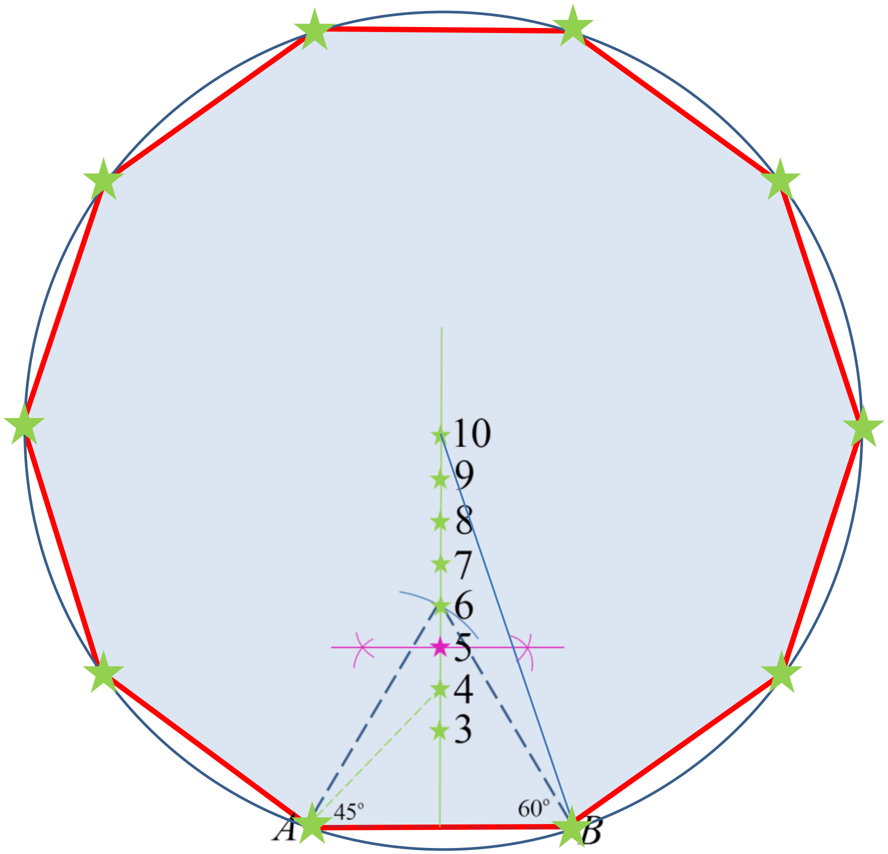
6. Use the compass to measure the separation between '5' and '6', and then 'walk' this separation in a ladder-like fashion up the vertical perpendicular line. Where the compass intersects with the line, mark a point and label these 7,8,9.... etc. For completeness, also mark point '3' below '4'.



7. Now choose the number of sides of your desired polygon (or construct them all!)
   Using a compass, draw a circle of radius from point N to either A or B.

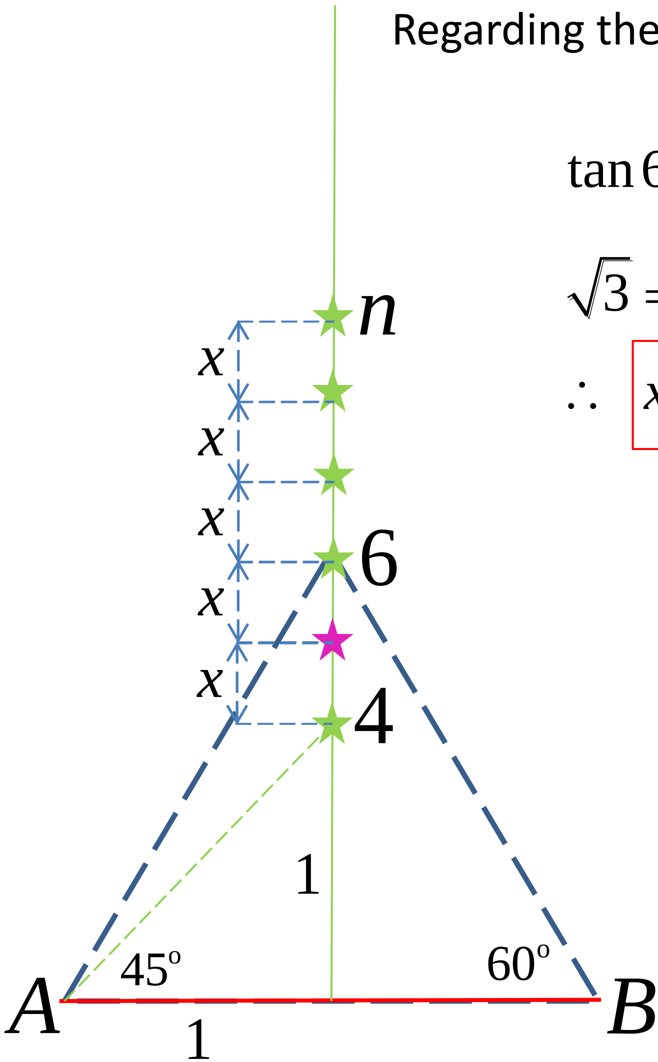


8. Set the compass to the distance AB and ‘walk’ the compass round the circle you have just drawn. The intersections will be the vertices of the N-gon.

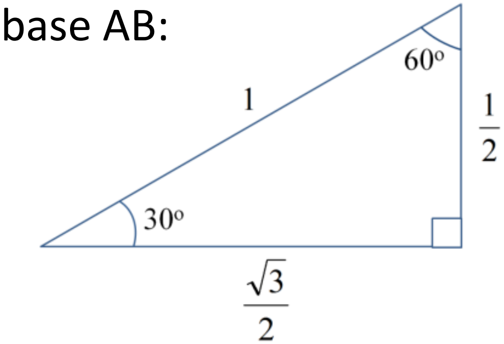


9. Connect up the vertices using a ruler and you have an (approximately) regular N-gon!

So how approximate is it? ...



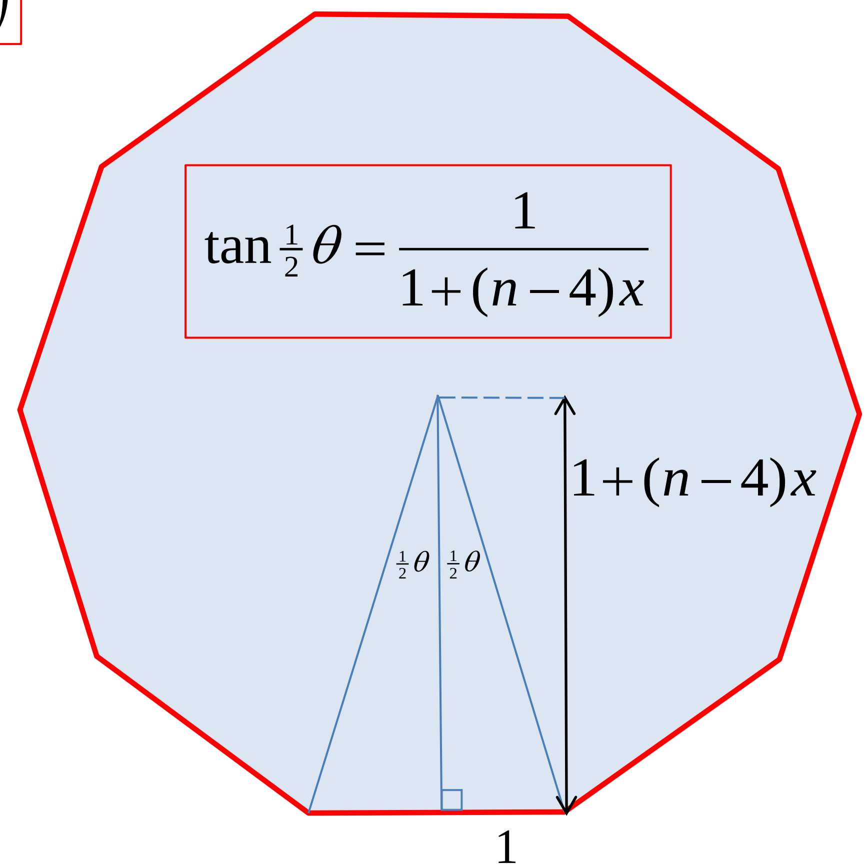
Regarding the original equilateral triangle with base AB:



$$\tan 60^{\circ} = \frac{1+2x}{1}$$

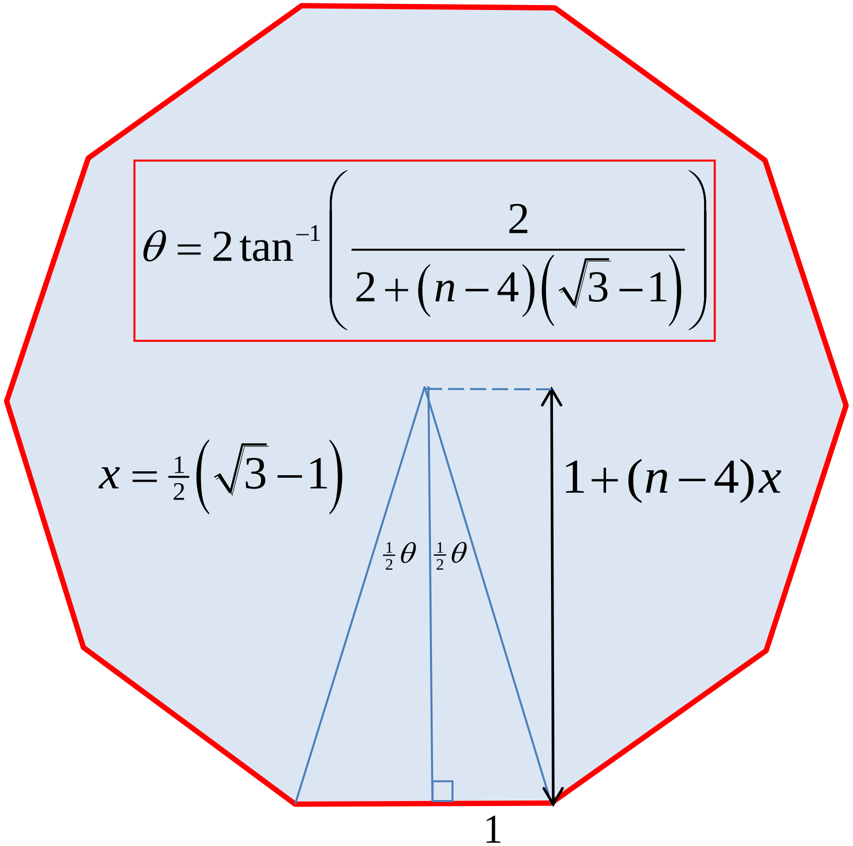
$$\sqrt{3} = 1+2x$$

$$\therefore \quad \boxed{x = \tfrac{1}{2}\left(\sqrt{3}-1\right)}$$



$$\boxed{\tan \tfrac{1}{2}\theta = \frac{1}{1+(n-4)x}}$$

If the N-gon were regular $\boxed{\theta = \dfrac{360^{\circ}}{n}}$



$$\tan\tfrac{1}{2}\theta = \frac{1}{1+(n-4)x}$$

$$x = \tfrac{1}{2}\left(\sqrt{3}-1\right)$$

$$\theta = 2\tan^{-1}\left(\frac{1}{1+(n-4)\frac{1}{2}\left(\sqrt{3}-1\right)}\right)$$

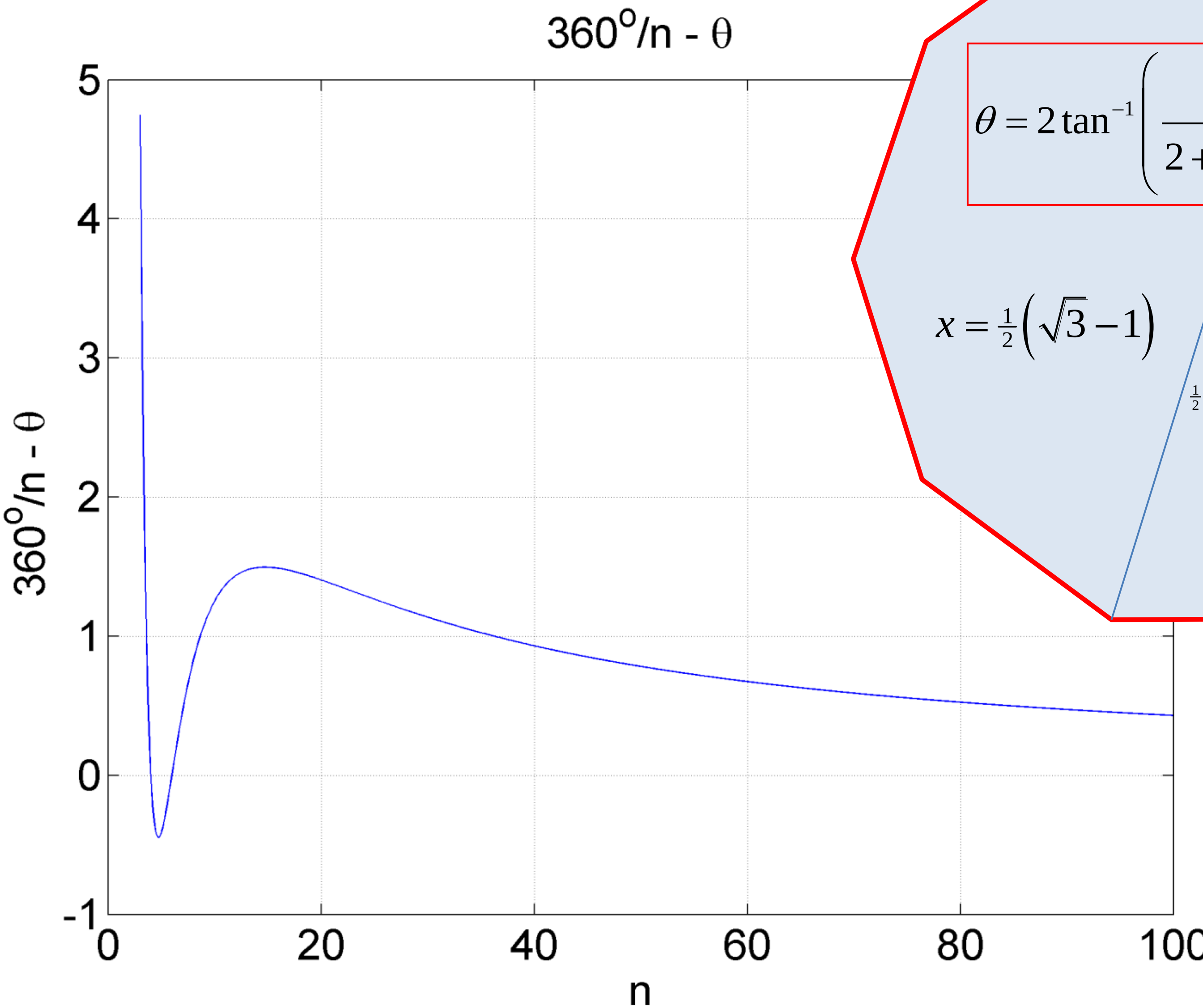
$$\boxed{\theta = 2\tan^{-1}\left(\frac{2}{2+(n-4)\left(\sqrt{3}-1\right)}\right)}$$

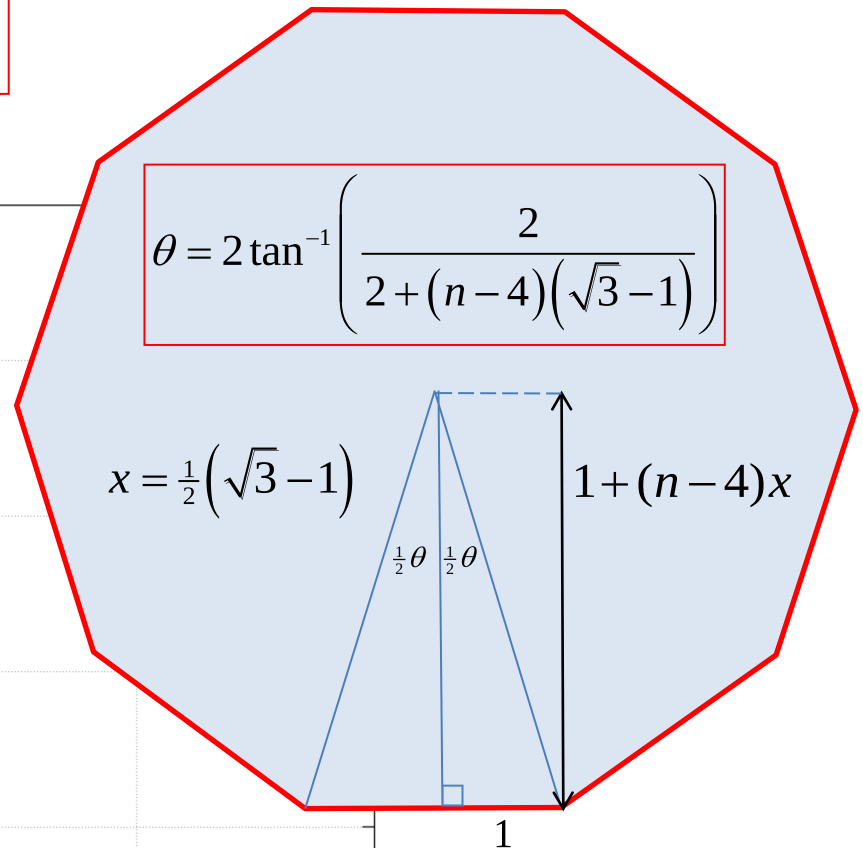
With the exception of $n = 4$ (square) and $n = 6$ (hexagon) the construction is only *approximate*, although a discrepancy of **only about a degree** means any difference would be hard to detect without direct measurement.

As n increases, the approximation *improves*. This is fairly intuitive given the radius of the construction circle increases with n while the polygon side length AB remains the same. Relative to the circumference, the step side becomes increasingly insignificant until, in the infinite limit, a circle is constructed.

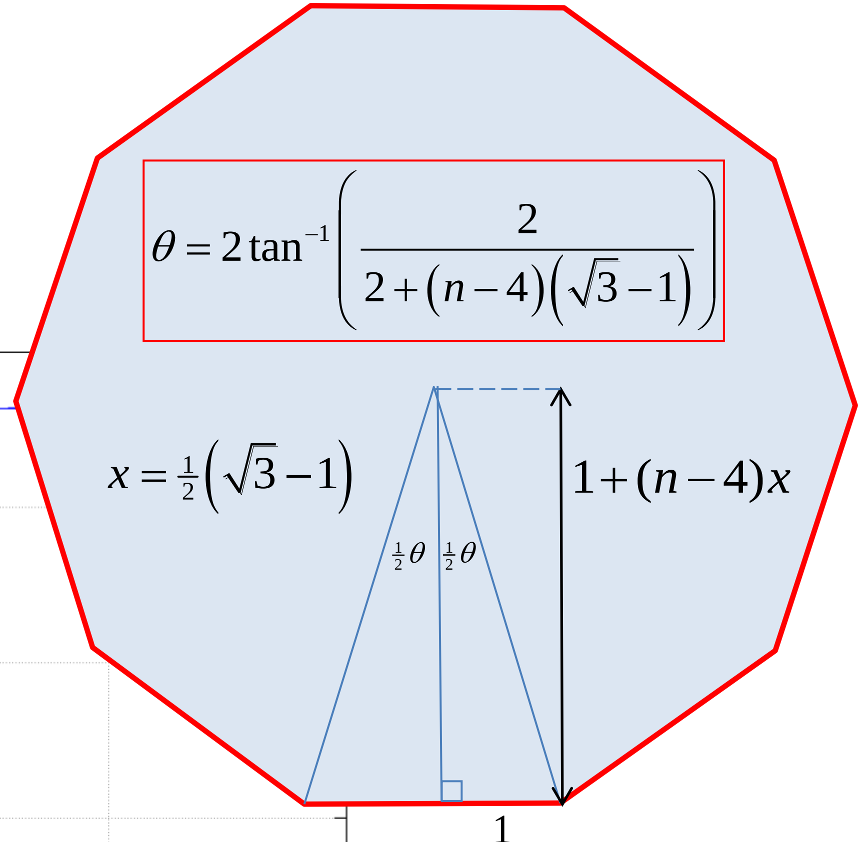
n	θ / º	360º/n	360º/n - θ
3	115.25	120	4.75
4	90	90	0
5	72.41	72	-0.41
6	60	60	0
7	50.97	51.43	0.46
9	38.92	40	1.08
40	8.07	9.00	0.93

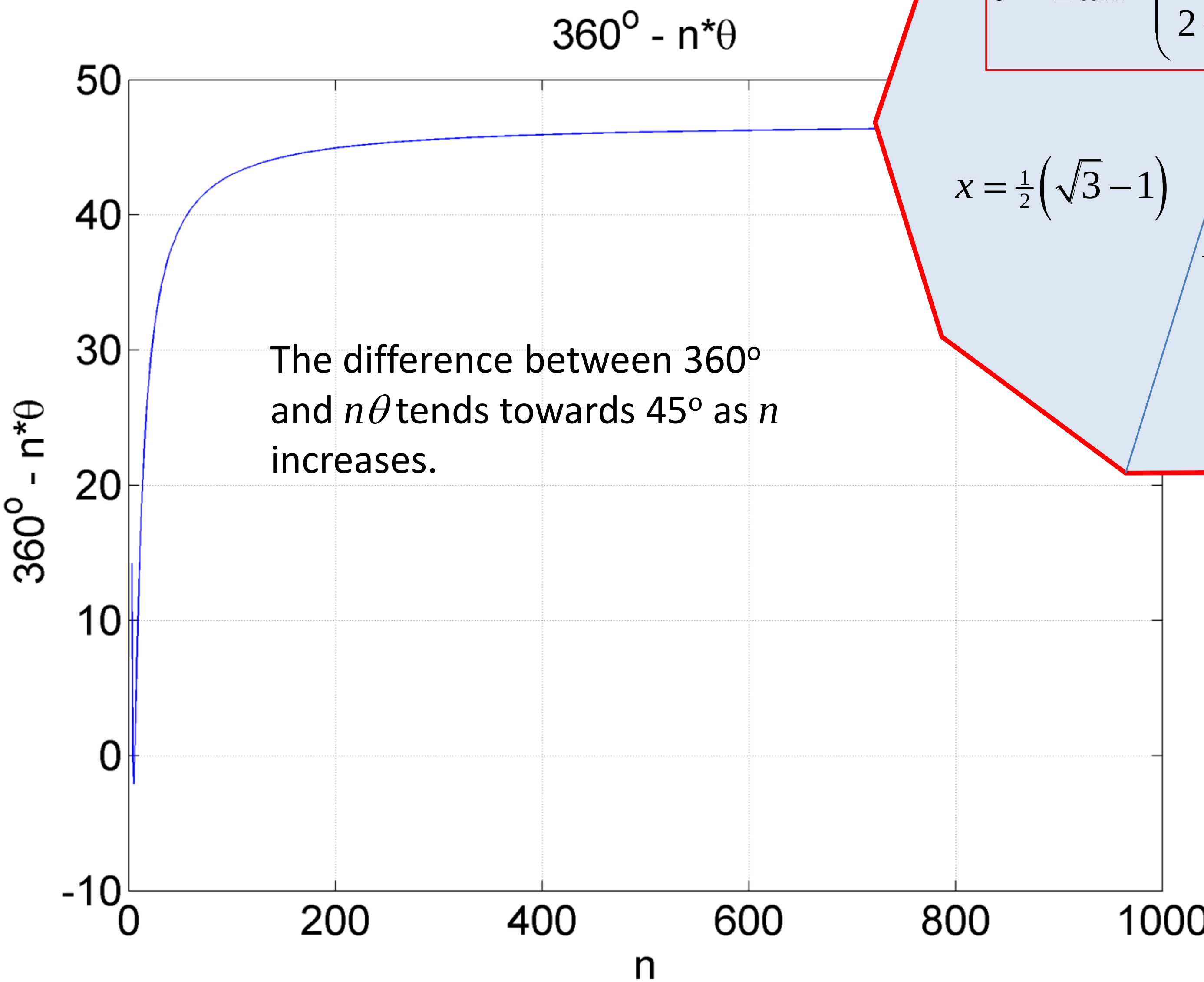
$$\theta = 2\tan^{-1}\left(\frac{2}{2+(n-4)\left(\sqrt{3}-1\right)}\right) \approx \frac{360^{\circ}}{n}$$

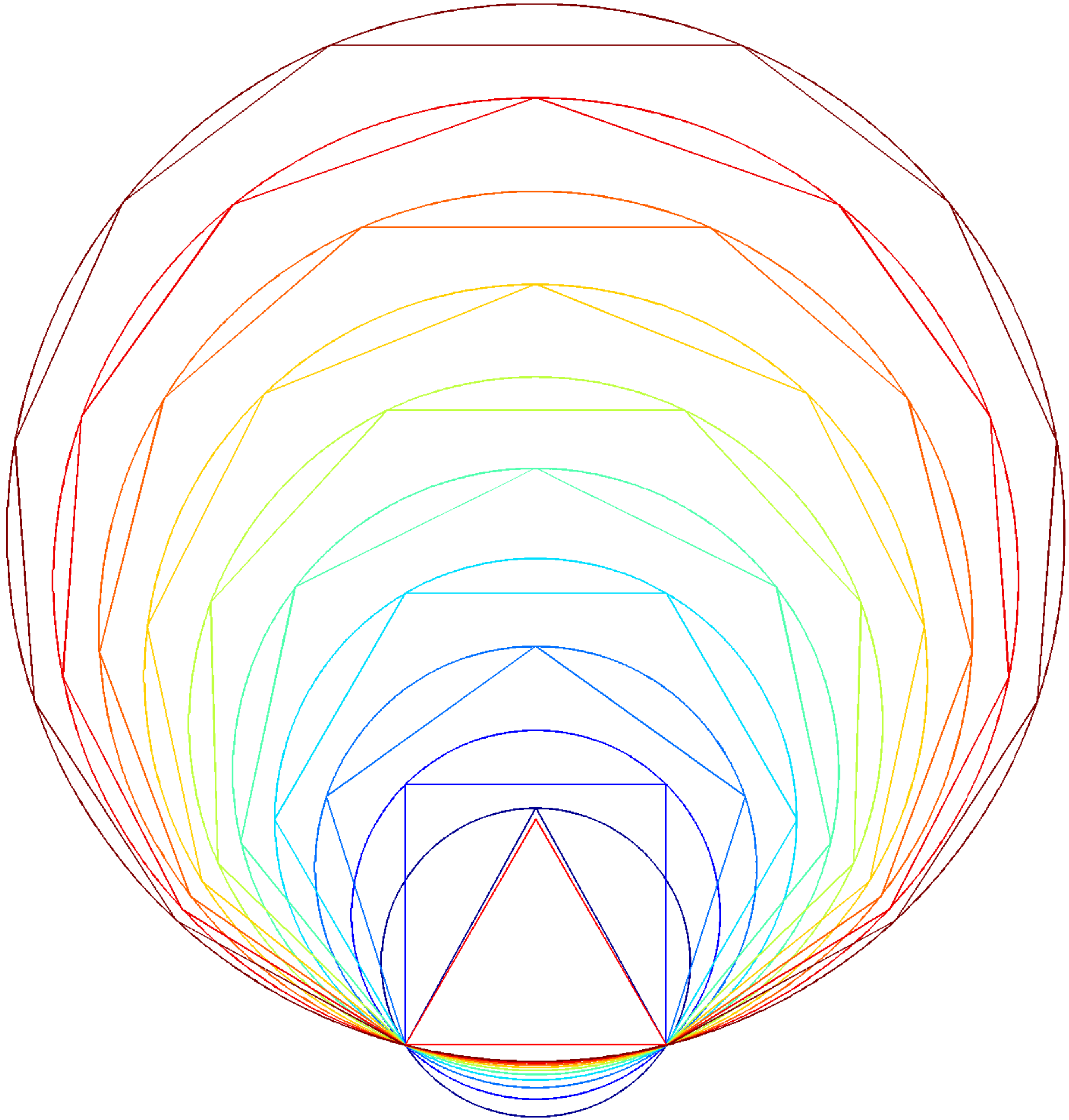


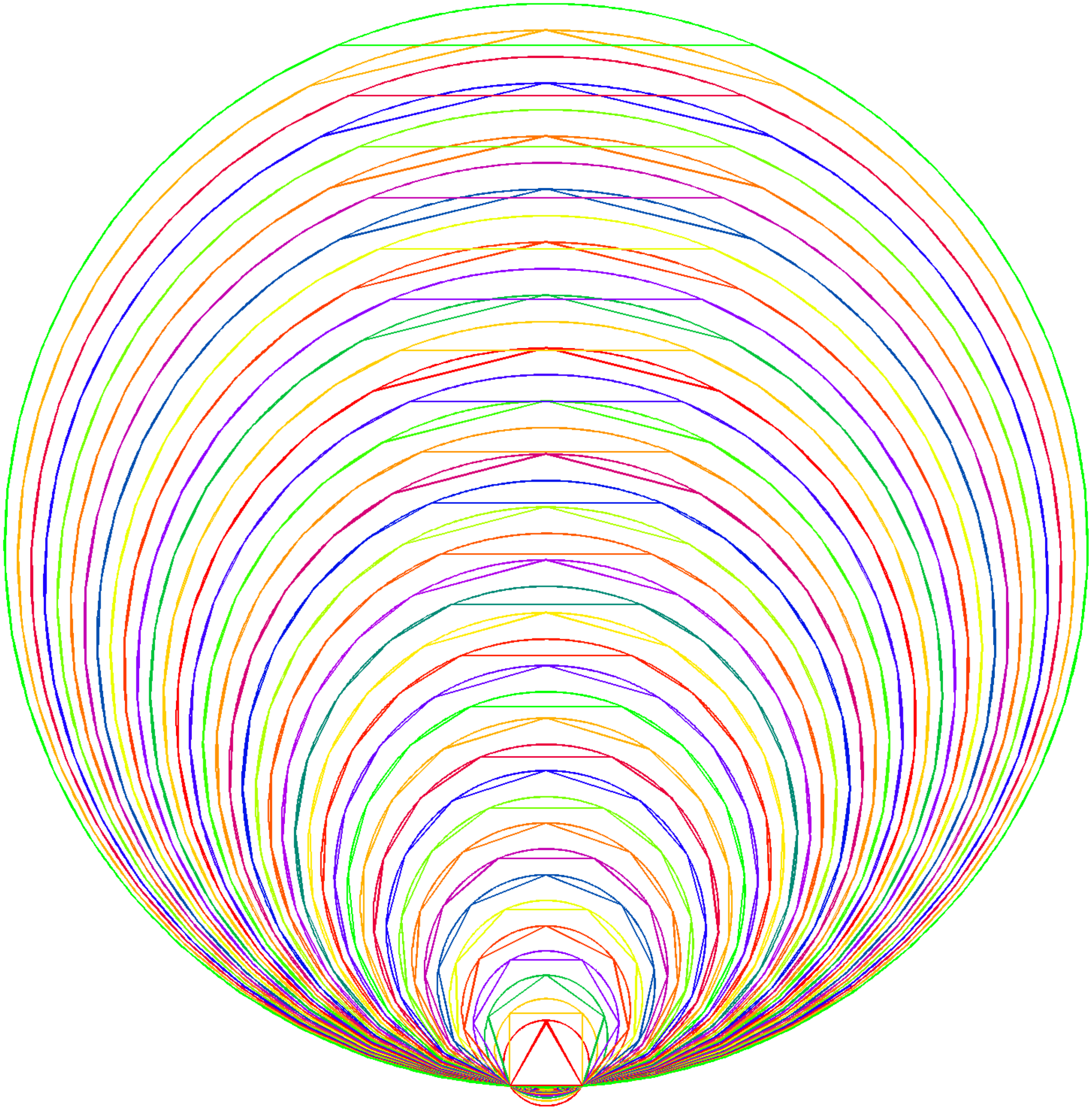


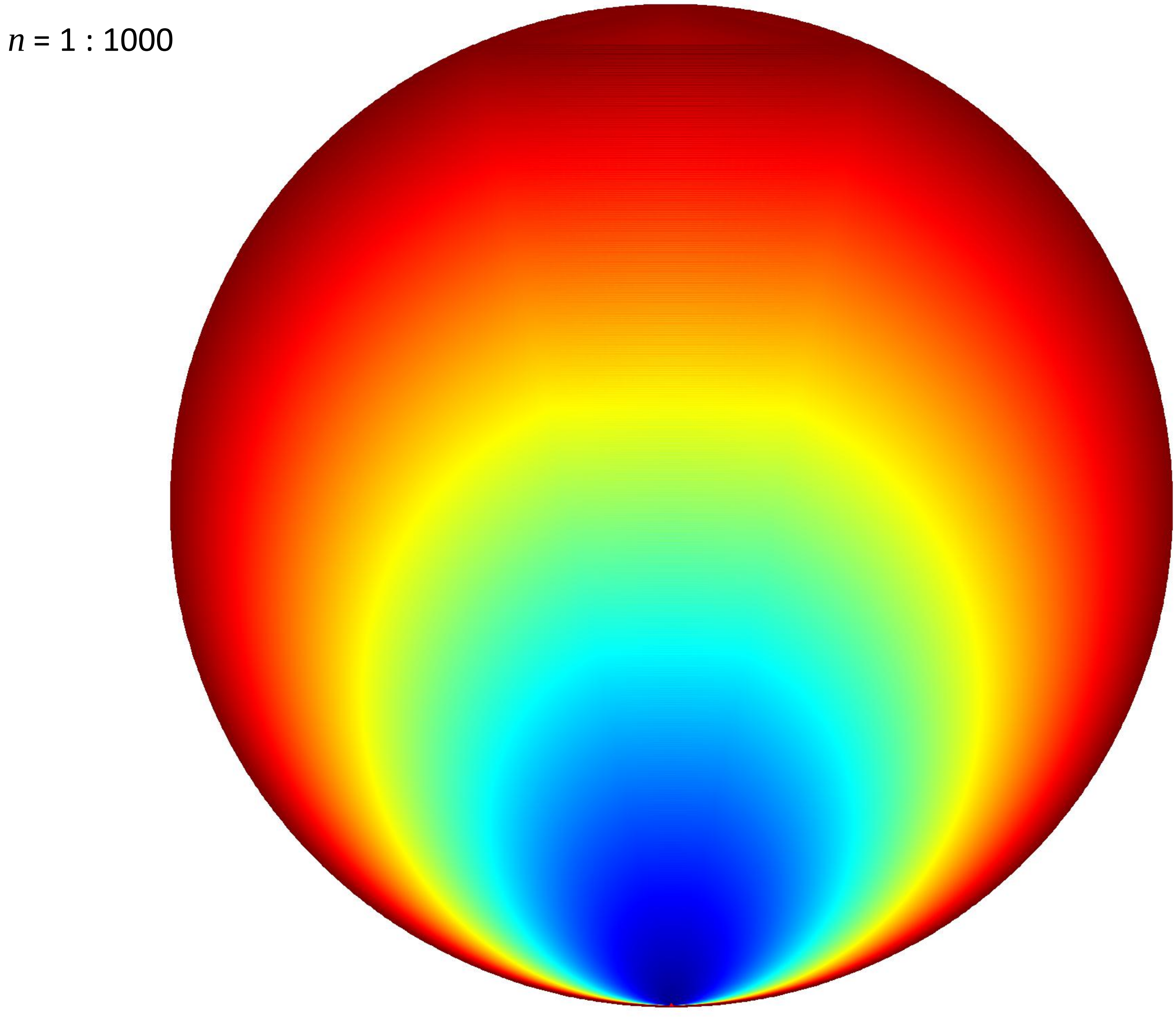
$$\theta = 2\tan^{-1}\left(\frac{2}{2+(n-4)\left(\sqrt{3}-1\right)}\right) \approx \frac{360^\circ}{n}$$

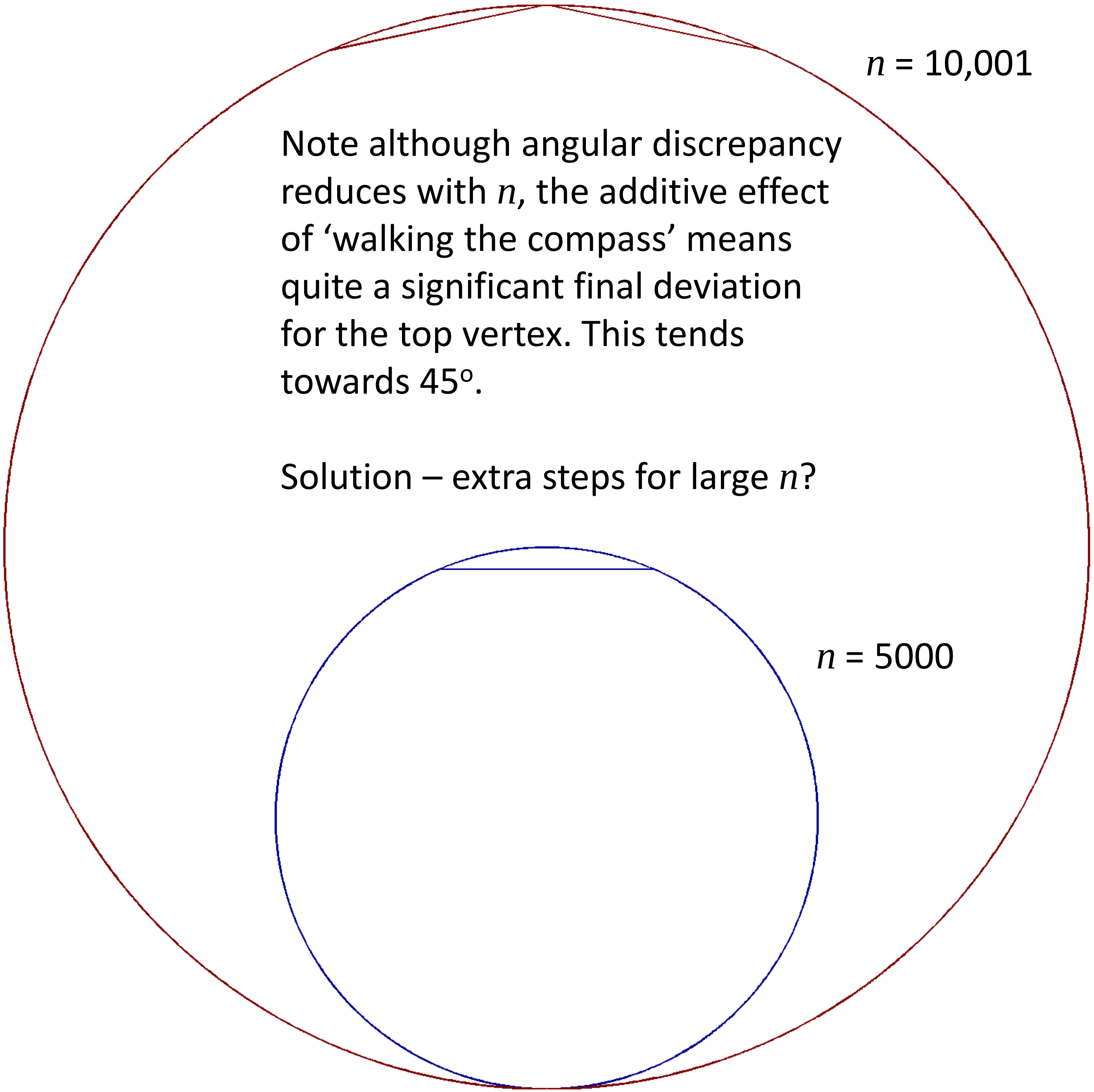









n = 1 : 1000



Note although angular discrepancy reduces with n, the additive effect of ‘walking the compass’ means quite a significant final deviation for the top vertex. This tends towards 45^{o}.

Solution – extra steps for large n?

The approximate N-gon angle formula will therefore yield a series of approximations of π

$$\boxed{\begin{aligned} &2\tan^{-1}\left(\frac{2}{2+(n-4)\left(\sqrt{3}-1\right)}\right)\approx\frac{2\pi}{n} \\ &\therefore \pi \approx n\tan^{-1}\left(\frac{2}{2+(n-4)\left(\sqrt{3}-1\right)}\right)\end{aligned}}$$

$$\boxed{\tan^{-1}x = x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\tfrac{1}{9}x^9-\ldots.. \quad |x|<1}$$

Maclaurin Expansion

$$\frac{2}{2+(3-4)\left(\sqrt{3}-1\right)}=\frac{2}{3-\sqrt{3}}=\frac{6+2\sqrt{3}}{9-3}=\boxed{\tfrac{1}{3}\left(3+\sqrt{3}\right)}$$

$$\frac{2}{2+(4-4)\left(\sqrt{3}-1\right)}=\boxed{1}$$

$$\frac{2}{2+(5-4)\left(\sqrt{3}-1\right)}=\frac{2}{1+\sqrt{3}}=\frac{2-2\sqrt{3}}{1-3}=\boxed{\sqrt{3}-1}$$

$$\frac{2}{2+(12-4)\left(\sqrt{3}-1\right)}=\frac{2}{8\sqrt{3}-6}=\frac{16\sqrt{3}+12}{192-36}=\boxed{\frac{4\sqrt{3}+3}{39}}$$

Using just the first term of the Maclaurin expansion

$$\pi \approx \frac{48\sqrt{3}+36}{39}=3.0548\ldots$$

$$\pi = 3.14159\ldots$$