BPhO Computational Challenge

Gravity & astrophysics

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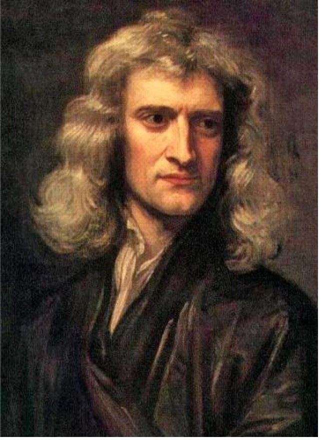
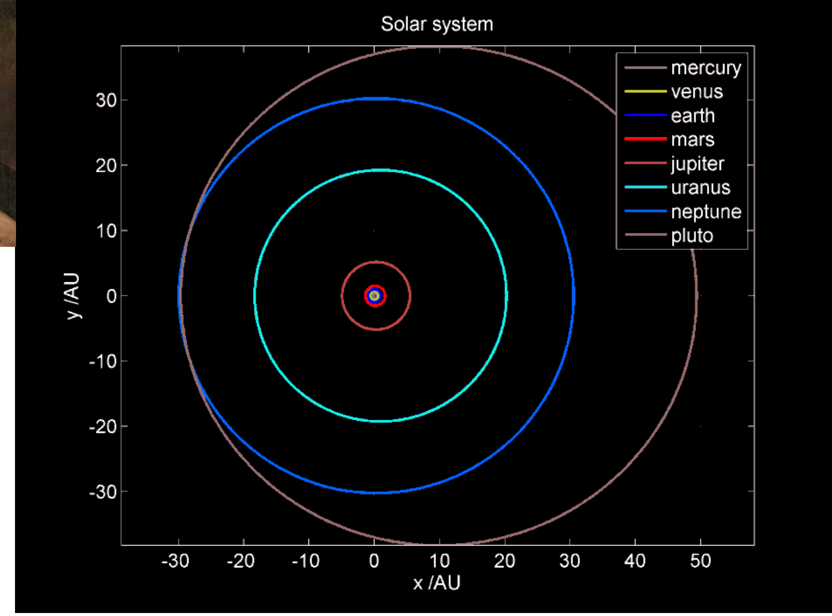
BPhO British Physics Olympiad



Isaac Newton (1642-1727) developed a mathematical model of Gravity which predicted the ***elliptical*** orbits proposed by Johannes Kepler (1571-1630)



Planet and star masses

Force of gravity

$$F = \frac{GmM}{r^2}$$

Universal gravitational constant

$$G = 6.67 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$$

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

Polar equation of ellipse

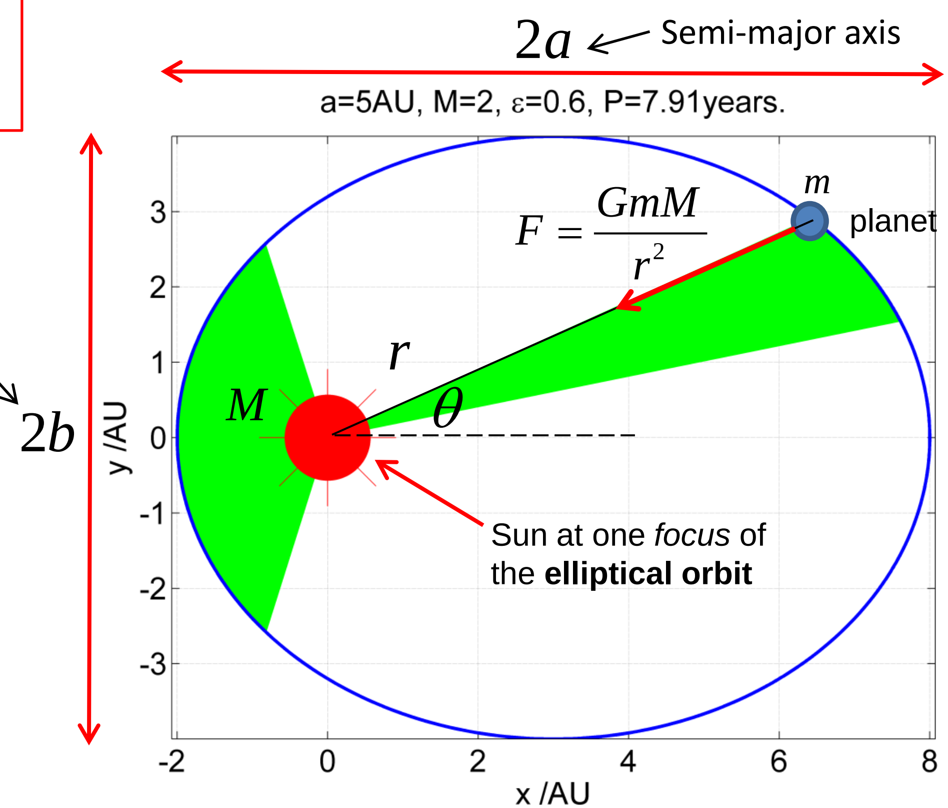
$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

Eccentricity of ellipse

$$P^2 = \frac{4\pi^2}{G(m+M)}a^3$$

Orbital period P

Semi-minor axis



Kepler's three laws are: $G = 6.67\times10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$ $\quad M_\odot = 1.9891\times10^{30}\,\mathrm{kg}$

1. *The orbit of every planet in the solar system is an ellipse with the Sun at one of the two foci.*
2. *A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.*
3. *The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.*

The wording of Kepler's laws implies a specific application to the solar system. However, the laws are more generally applicable to any system of two masses whose mutual attraction is an inverse-square law.

$$\frac{dA}{dt} = \tfrac{1}{2}\sqrt{G(m+M)\left(1-\varepsilon^2\right)a}$$

This is a *constant* of the orbit

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

Polar equation of ellipse

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

Eccentricity of ellipse

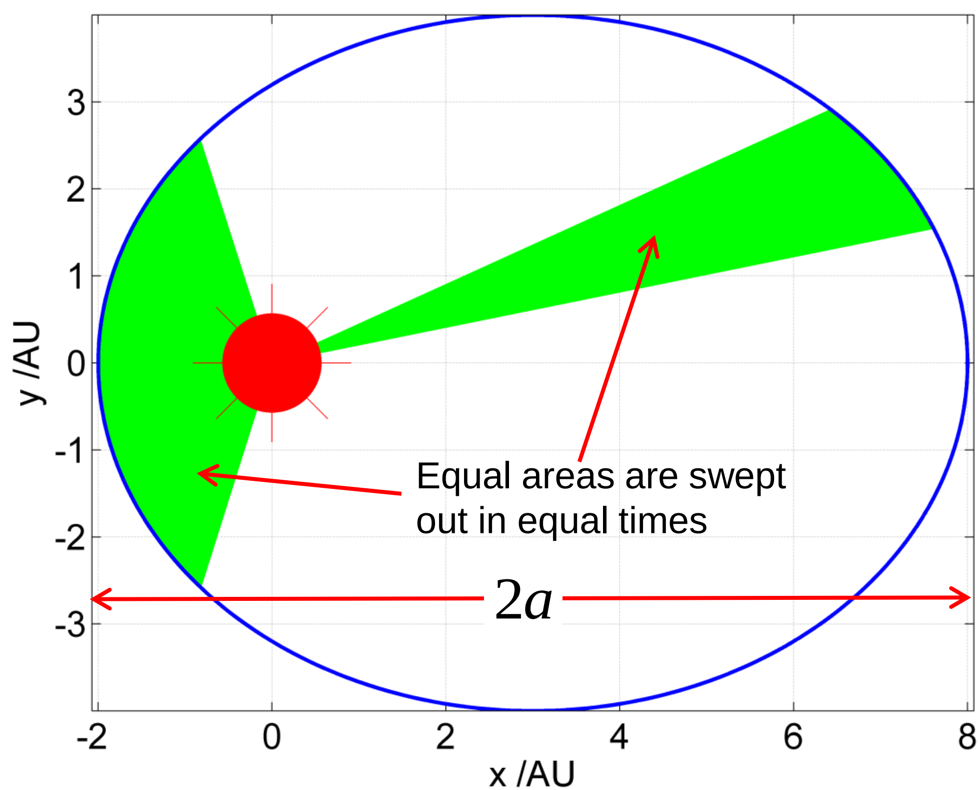
$$P^2 = \frac{4\pi^2}{G(m+M)}a^3$$

Orbital period *P*

Planet mass

Sun mass

a=5AU, M=2, ε=0.6, P=7.91years.



Johannes Kepler
1571-1630

$$(60R_\oplus + \delta)^2 = d^2 + (60R_\oplus)^2$$

$$(60R_\oplus)^2 + 120R_\oplus\delta + \delta^2 = d^2 + (60R_\oplus)^2$$

$$120R_\oplus\delta + \delta^2 = d^2 \Rightarrow 120R_\oplus\delta \approx d^2$$

$$\therefore \delta \approx \frac{d^2}{120R_\oplus}$$

In **one second**, the Moon will travel

$$d = \frac{2\pi \times 60R_\oplus}{28 \times 24 \times 3600}$$

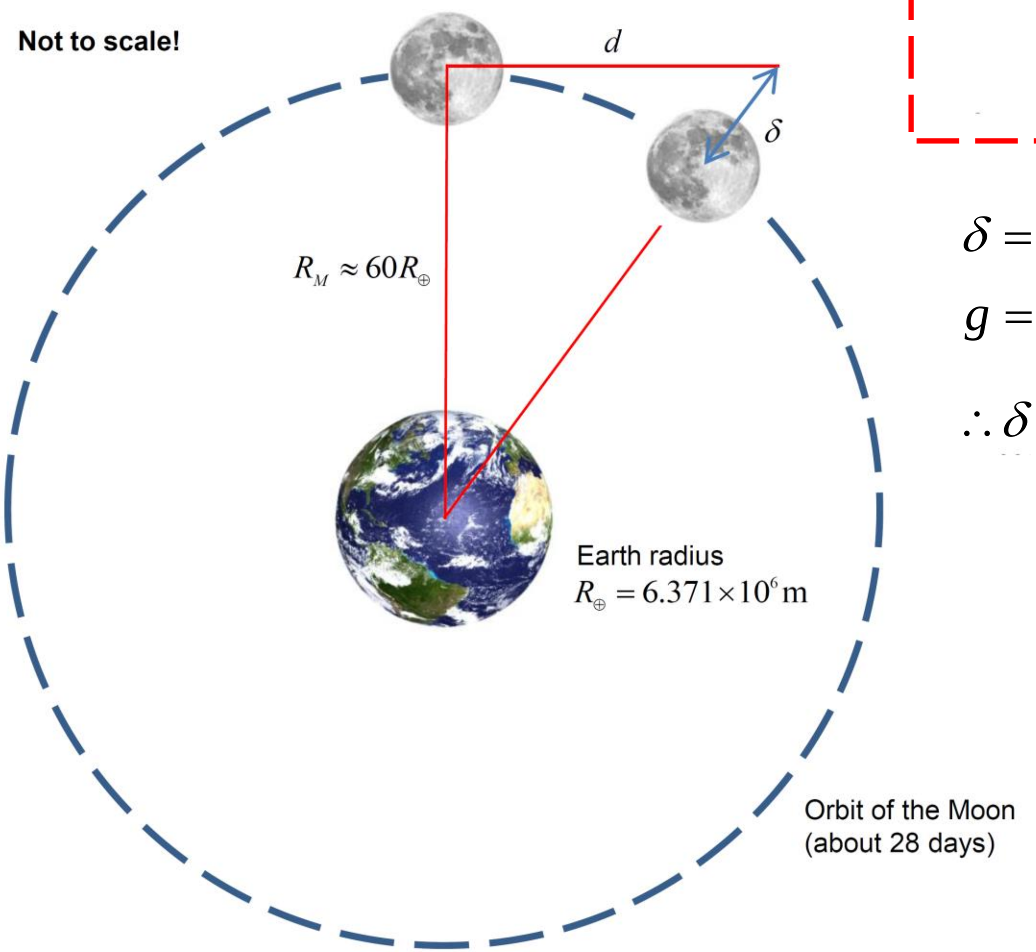
$$d \approx 993\text{m}$$

Hence in **one second the Moon will fall**

$$\delta \approx \frac{d^2}{120R_\oplus}$$

$$\delta \approx \frac{992.81^2}{120 \times 6.371 \times 10^6}$$

$$\delta \approx 1.3\text{mm}$$



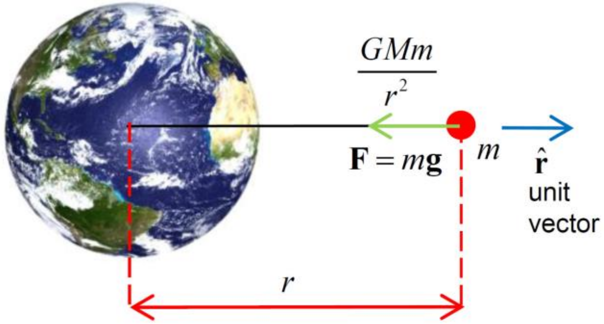
Moonfall

$$\delta = \tfrac{1}{2} g_M t^2$$

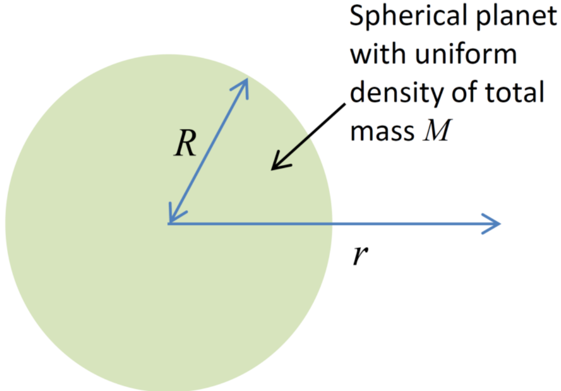
$$g = 9.81\text{ms}^{-2} \qquad t = 1\text{s}$$

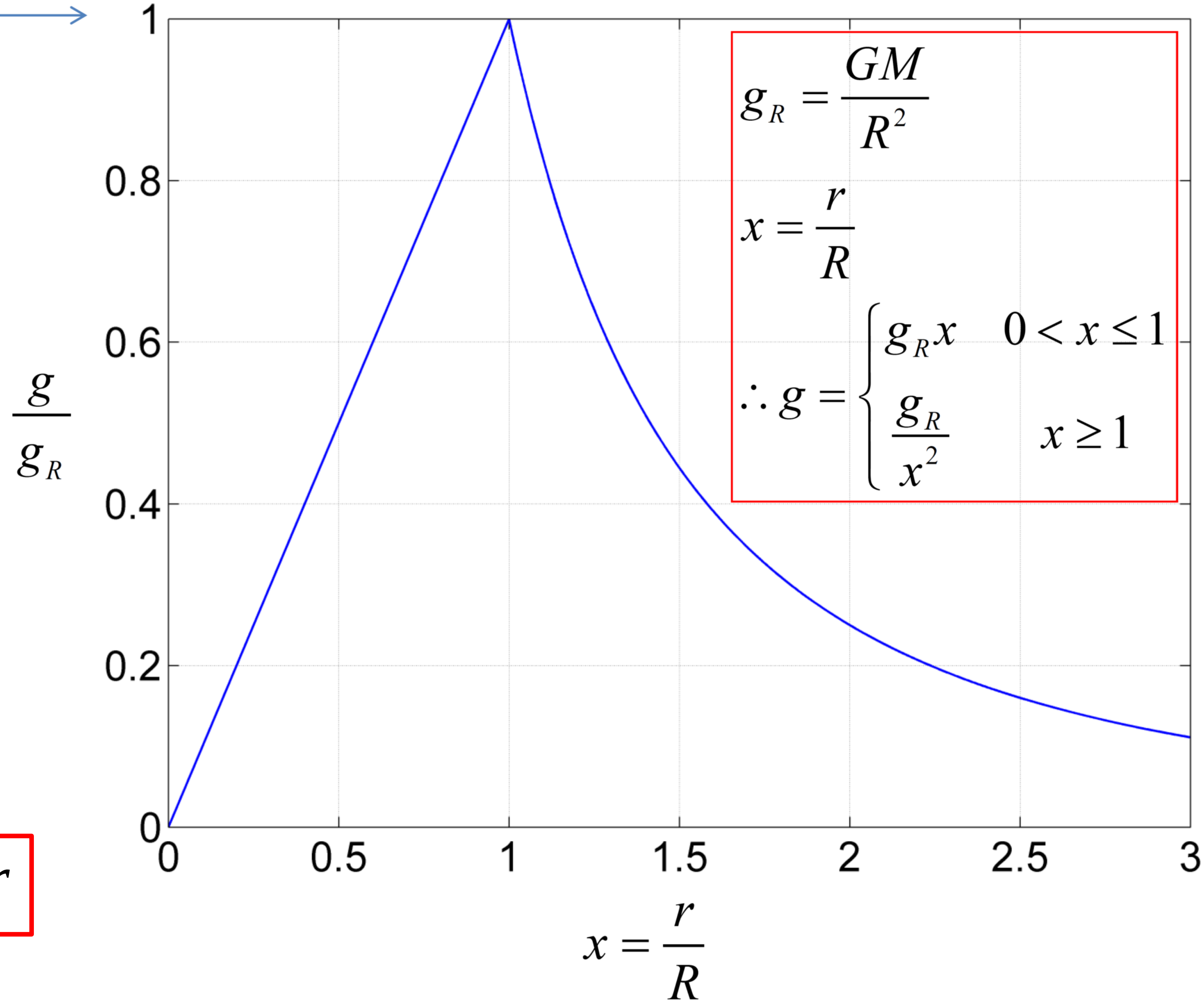
$$\therefore \delta = \tfrac{1}{2} \times \frac{9.81}{60^2} \approx 1.3\text{mm}$$

$$g_M = \frac{g_\oplus}{60^2}$$



Gravitational field strength inside and outside a uniformly dense spherical planet





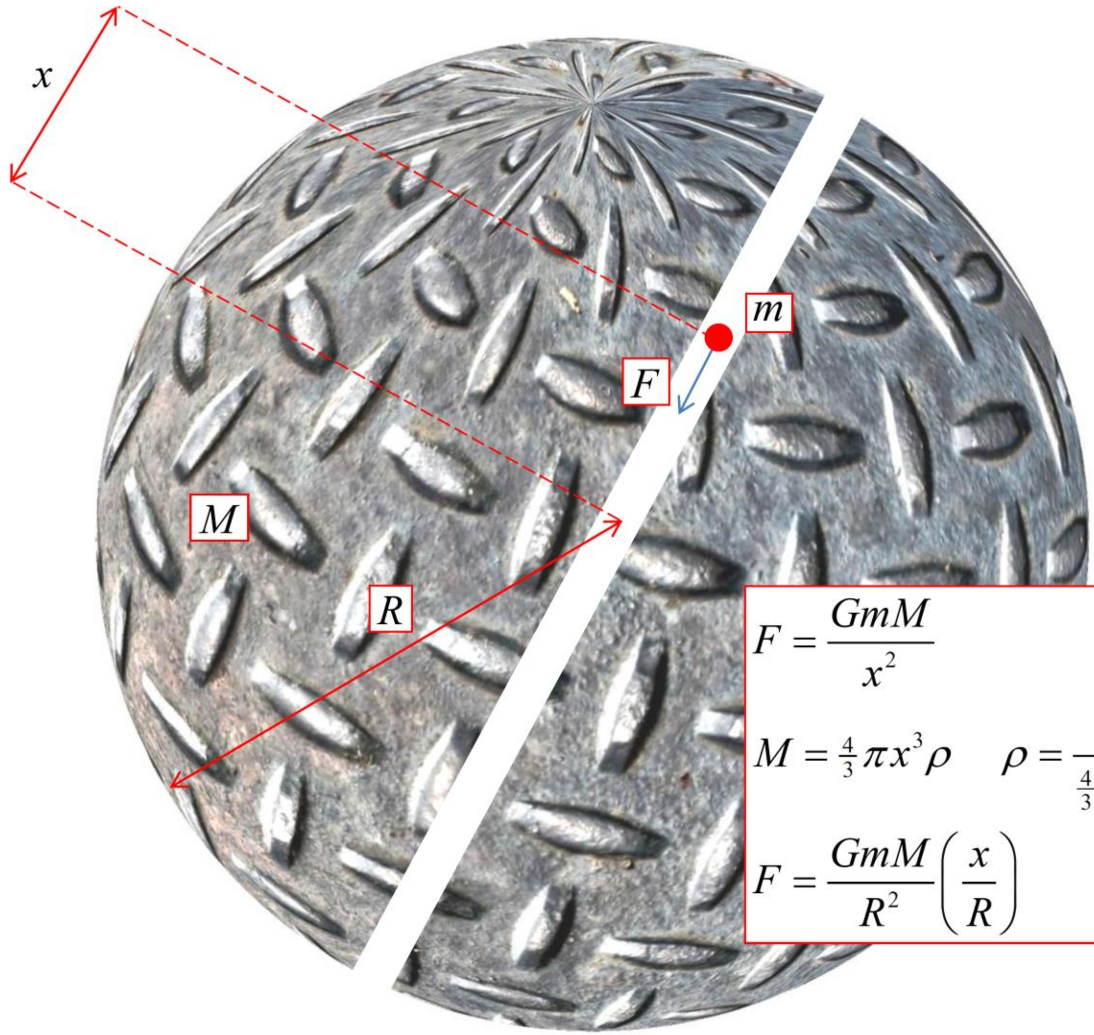
Inside planet

$$g = \frac{GM}{r^2}$$

$$M = \tfrac{4}{3}\pi r^3 \rho$$

$$\therefore g = \tfrac{4}{3}\pi G \rho r$$

Cybertron diametric transport



$$F = \frac{GmM}{x^2}$$

$$M = \tfrac{4}{3}\pi x^3 \rho \qquad \rho = \frac{M}{\tfrac{4}{3}\pi R^3}$$

$$F = \frac{GmM}{R^2}\left(\frac{x}{R}\right)$$

Simple
Harmonic
Motion

$$\ddot{x} = -\left(\frac{2\pi}{P}\right)^2 x$$

$$x = R\cos\left(\frac{2\pi t}{P}\right)$$

$$P = \sqrt{\frac{3\pi}{G\rho}}$$

$$m\ddot{x} = -\frac{G\left(\tfrac{4}{3}\pi x^3 \rho\right)m}{x^2} \Rightarrow \ddot{x} = -\tfrac{4}{3}\pi G\rho x$$

$$\rho_{\text{titanium}} = 4{,}510\,\text{kgm}^{-3}$$

$$\rho_{\text{earth}} = 5{,}513\,\text{kgm}^{-3}$$

$$P_{\text{cybertron}} = 93\,\text{min}$$

$$P_{\text{earth}} = 84\,\text{min}$$

Object	Mass in Earth masses	Distance from Sun in AU	Radius in Earth radii	Rotational period /days	Orbital period /years	Gravitational field (in terms of g = 9.81 ms^-2)
Saturn	95.16	9.58	9.45	0.44	29.63	1.07
Uranus	14.50	19.29	4.01	0.72	84.75	0.90
Jupiter	317.85	5.20	11.21	0.41	11.86	2.53
Sun	332,837	-	109.12	-	-	27.95
Neptune	17.20	30.25	3.88	0.67	166.34	1.14
Pluto	0.00	39.51	0.19	6.39	248.35	0.09
Mars	0.107	1.523	0.53	1.03	1.88	0.38
Venus	0.815	0.723	0.95	243.02	0.62	0.90
Mercury	0.055	0.387	0.38	58.65	0.24	0.37
Earth	1.000	1.000	1.00	1.00	1.00	1.00

For *our* Solar System:

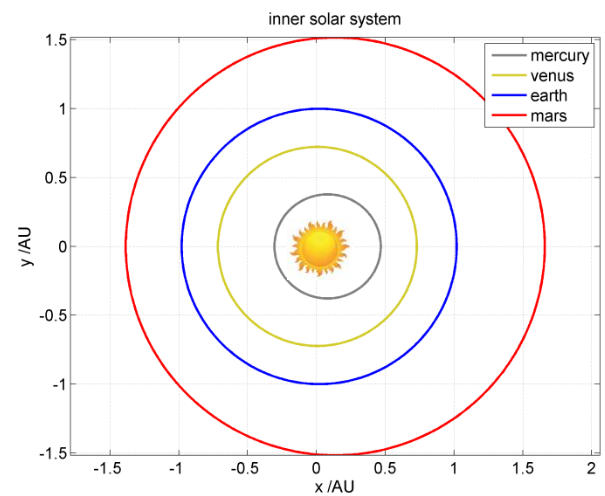
$$m \ll M_\odot$$

$$P^2 = \frac{4\pi^2}{G\left(m + M_\odot\right)} a^3$$

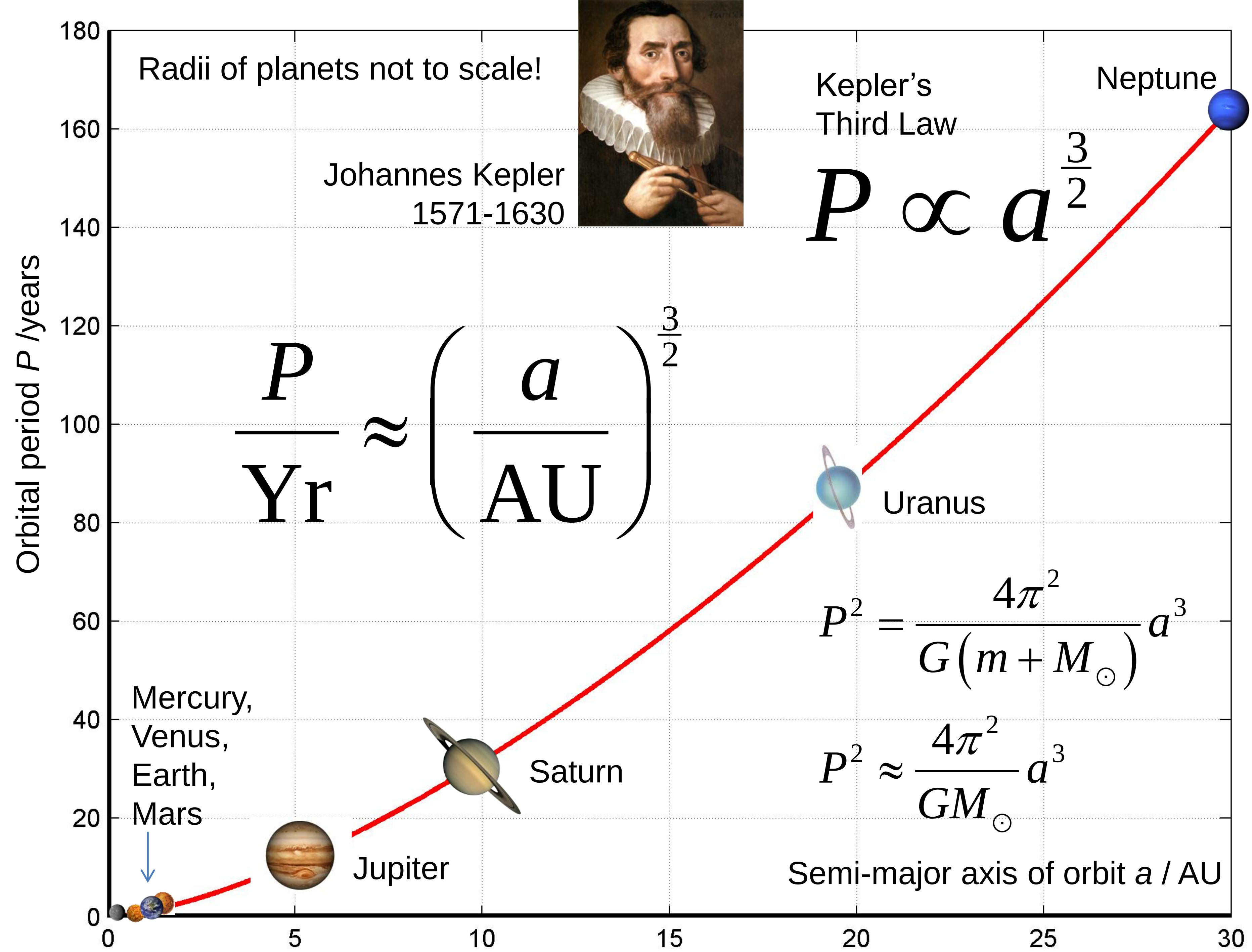
$$P^2 \approx \frac{4\pi^2}{GM_\odot} a^3$$

$$\mathrm{Yr}^2 = \frac{4\pi^2}{GM_\odot} \mathrm{AU}^3$$

$$\therefore \frac{P}{\mathrm{Yr}} \approx \left(\frac{a}{\mathrm{AU}}\right)^{\frac{3}{2}}$$

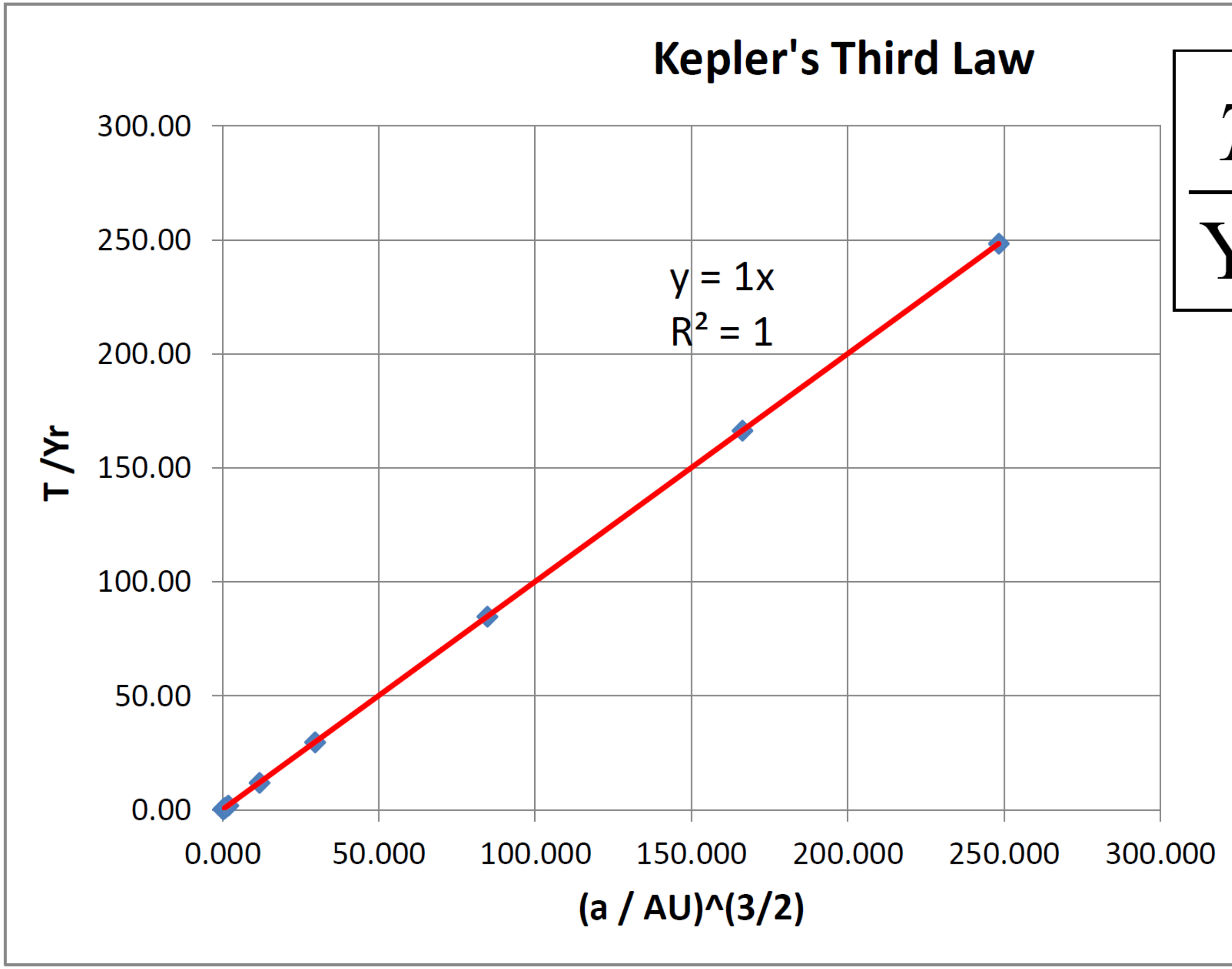


$$1\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m}$$



Radii of planets not to scale!
Johannes Kepler
1571-1630
Kepler's
Third Law
$P \propto a^{\frac{3}{2}}$
Neptune
$\frac{P}{\mathrm{Yr}} \approx \left(\frac{a}{\mathrm{AU}}\right)^{\frac{3}{2}}$
Uranus
$P^2 = \frac{4\pi^2}{G\left(m + M_\odot\right)} a^3$
$P^2 \approx \frac{4\pi^2}{GM_\odot} a^3$
Mercury,
Venus,
Earth,
Mars
Saturn
Jupiter
Semi-major axis of orbit *a* / AU
Orbital period *P* /years
0
20
40
60
80
100
120
140
160
180
0
5
10
15
20
25
30

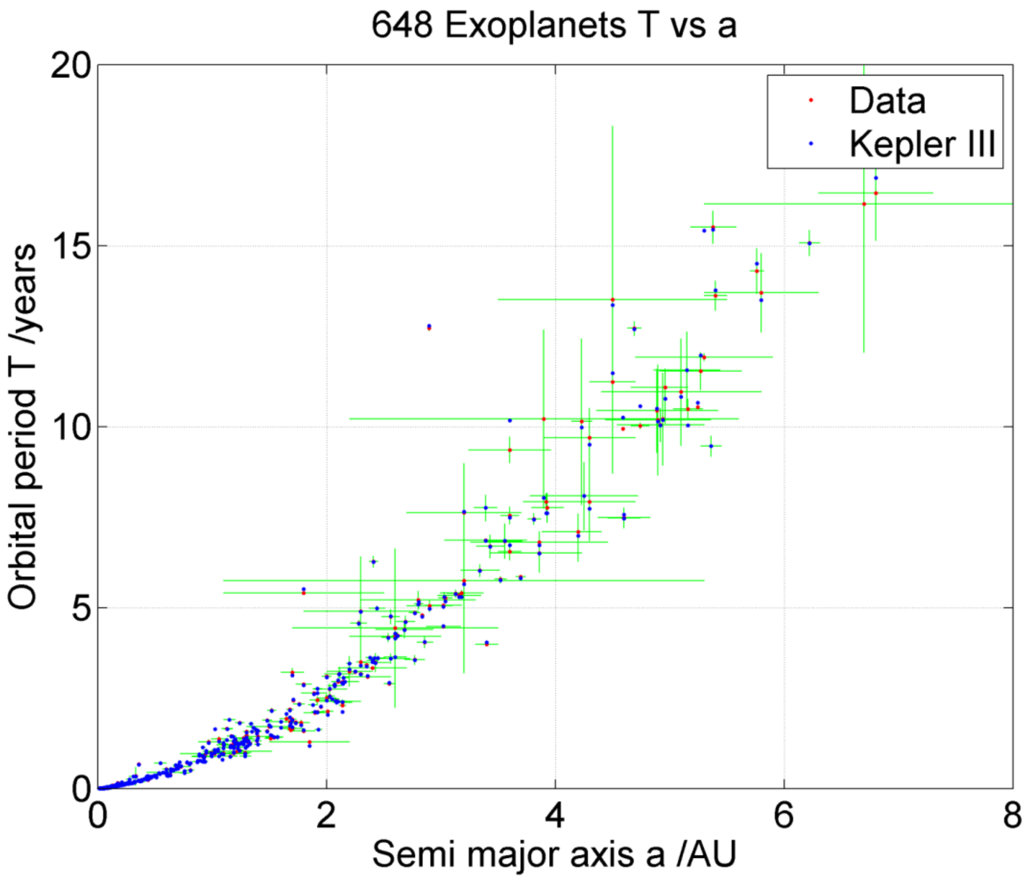
$$1\text{AU} = 1.496 \times 10^{11}\,\text{m}$$

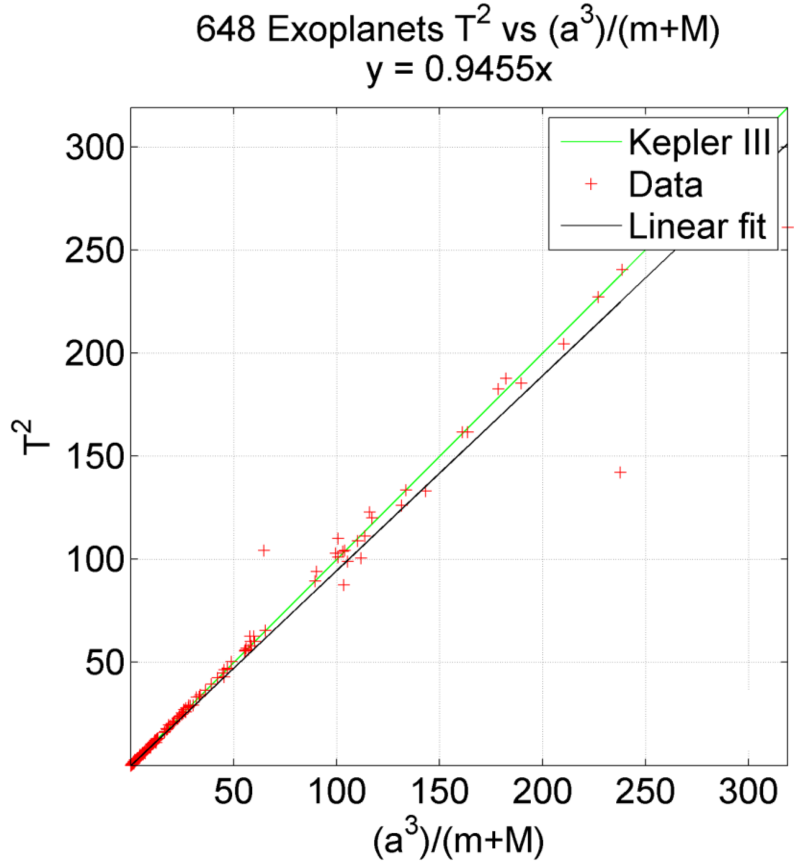


$$\frac{T}{\text{Yr}} = \left(\frac{R}{\text{AU}}\right)^{\frac{3}{2}}$$

A *very strong* correlation of Kepler III to orbital data for planets in our solar system!

Challenge #1: Replicate this Kepler III correlation in Excel or Python or MATLAB





Kepler III for exoplanets

$$P^2 = \frac{4\pi^2}{G(m+M)} a^3$$

$$\mathrm{Yr}^2 = \frac{4\pi^2}{GM_\odot} \mathrm{AU}^3$$

$$\left(\frac{P}{\mathrm{Yr}}\right)^2 = \left(\frac{m+M}{M_\odot}\right)^{-1} \left(\frac{a}{\mathrm{AU}}\right)^3$$

$$1\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m} \qquad M_\odot = 1.99 \times 10^{30}\,\mathrm{kg}$$

Challenge #2: Plot elliptical orbits of the planets using Excel, Python, MATLAB

Assume all orbits are **ellipses** with the Sun at the (left) focus. Let this sun position be the origin of a Cartesian coordinate system, and assume the sun is stationary.

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}} \quad \therefore b = a\left(1-\varepsilon^2\right)$$

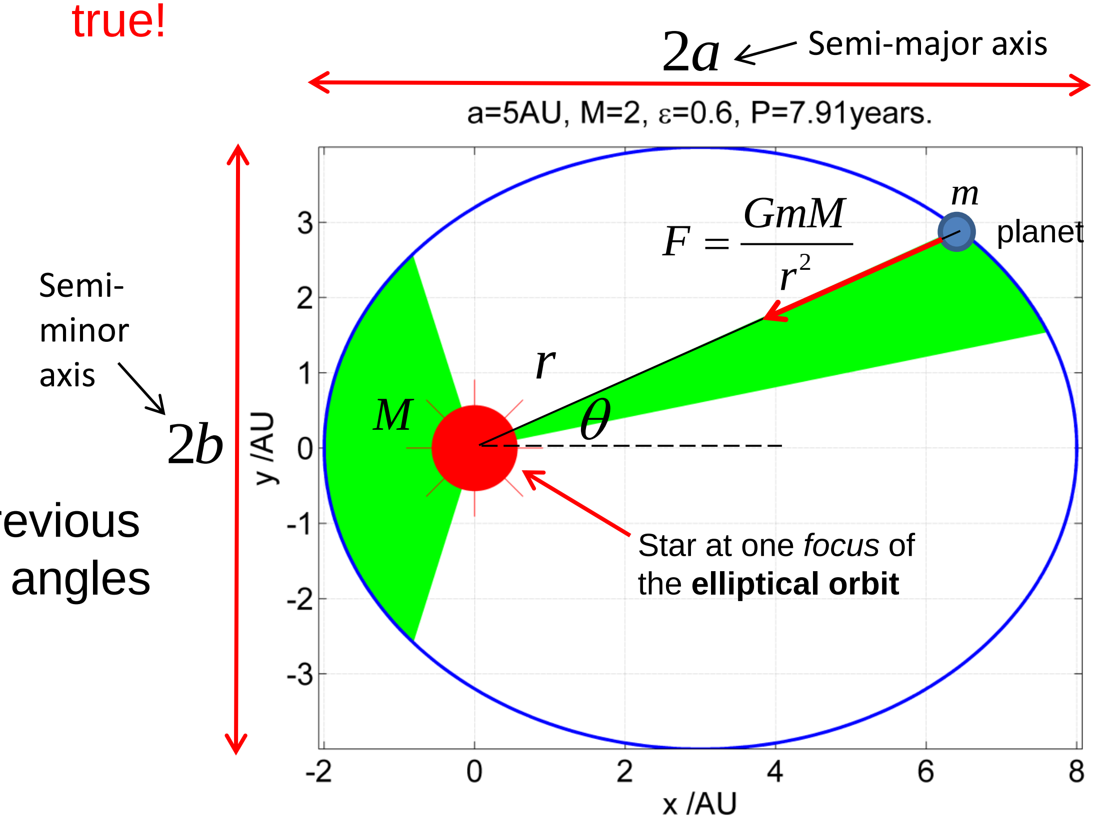
$$P^2 = \frac{4\pi^2}{G(m+M)}a^3$$

$$x = r\cos\theta,\;\; y = r\sin\theta$$

$$\theta = 0....2\pi$$

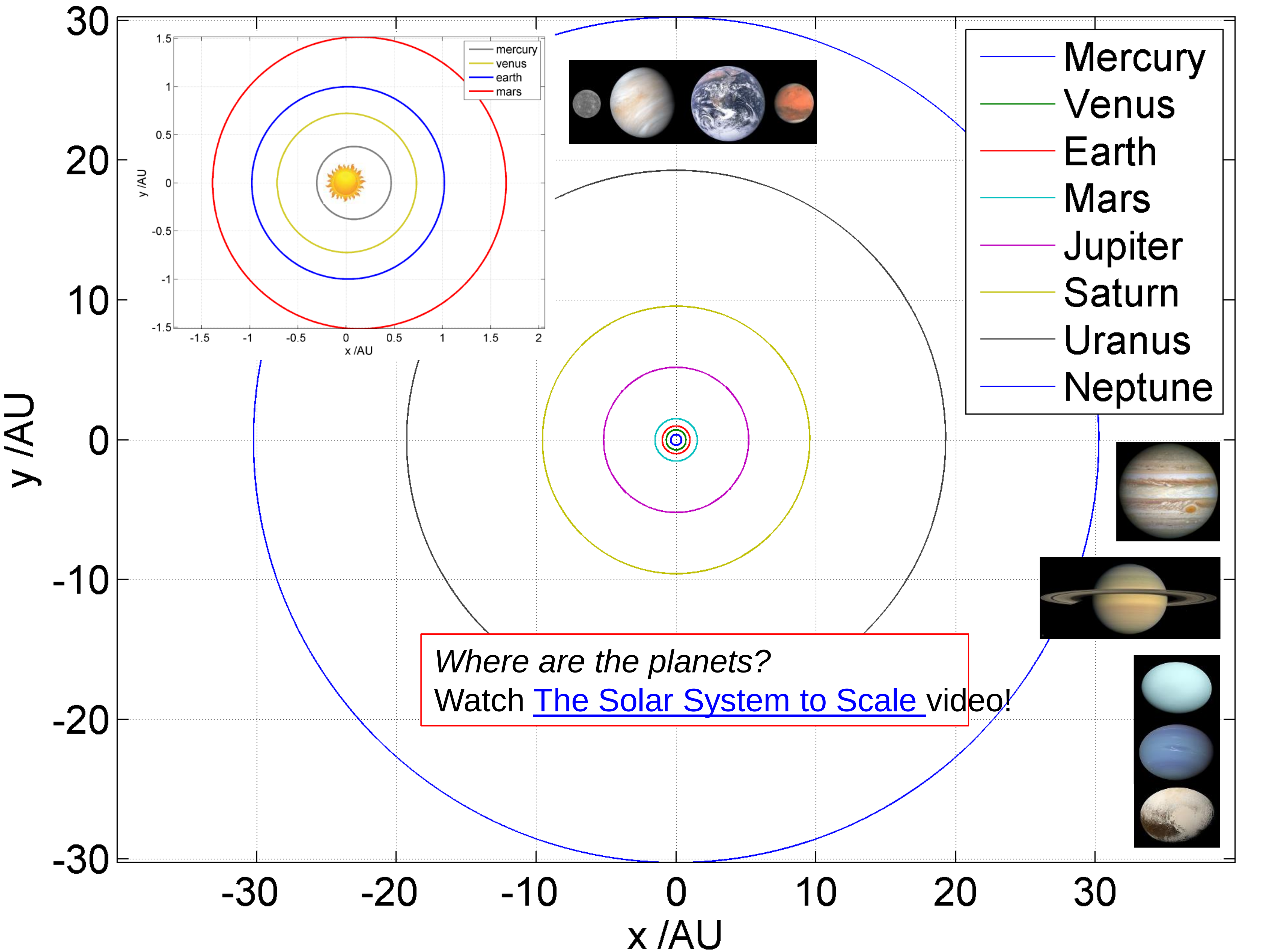
Plot the inner five planets on a separate scale to the outer planets

We will assume at this point all elliptical orbits are in the same plane ... but this is not quite true!



Use the data in the table on the previous slide. Use a 1,000 linearly spaced angles θ for each orbit.

Use an axis scale of AU


30
20
10
0
-10
-20
-30
y /AU
x /AU
Mercury
Venus
Earth
Mars
Jupiter
Saturn
Uranus
Neptune
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
2
y /AU
x /AU
mercury
venus
earth
mars
Where are the planets?
Watch The Solar System to Scale video!

Challenge #3: Create a 2D animation of the solar system orbits

Use an axis scale of AU

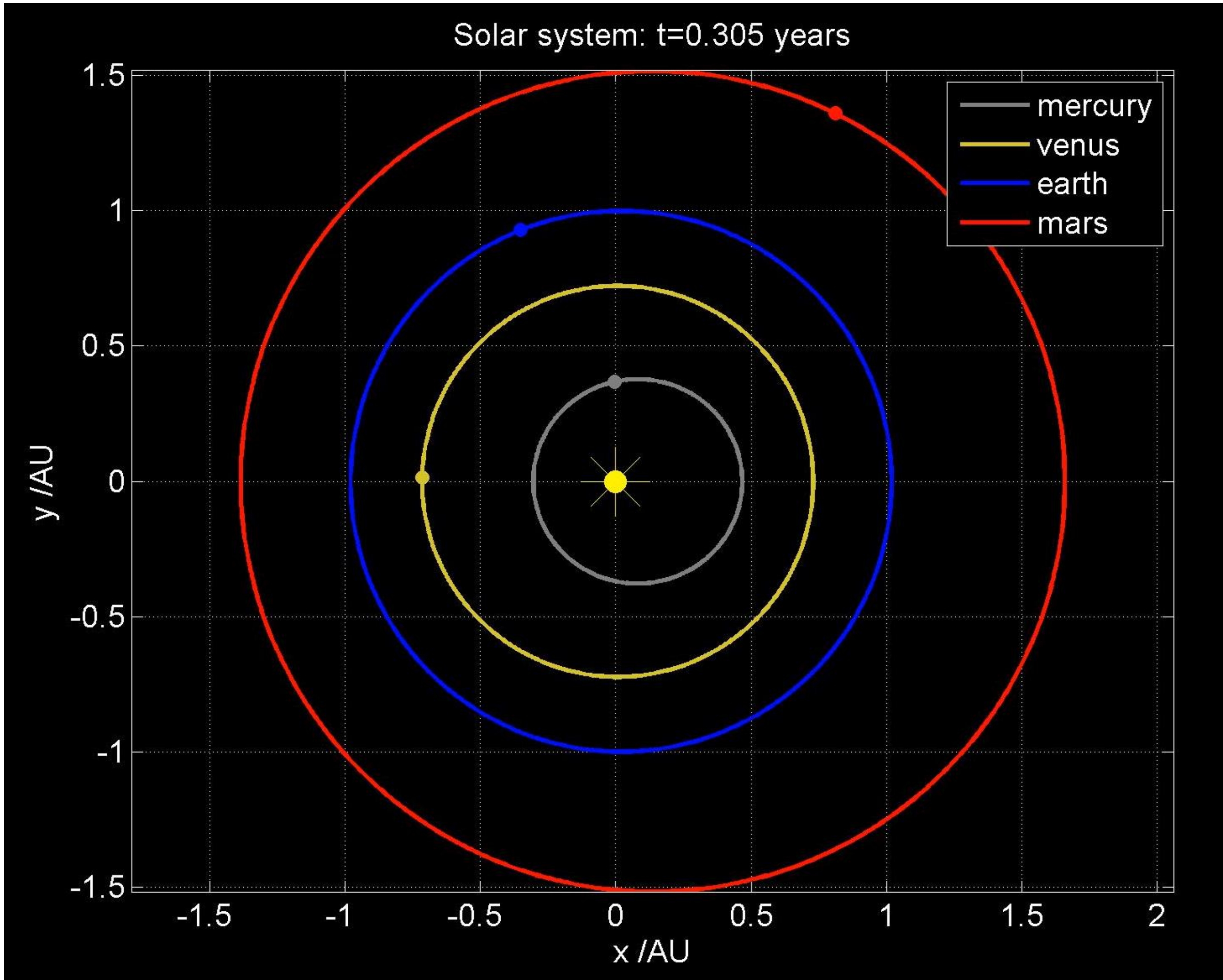
Plot the inner five planets on a separate scale to the outer planets

For the *inner* planets, set a frame rate such that one orbit of the Earth takes a second i.e. **one year is one second.** For the *outer* planets, **set the orbit of *Jupiter* to take one second**.

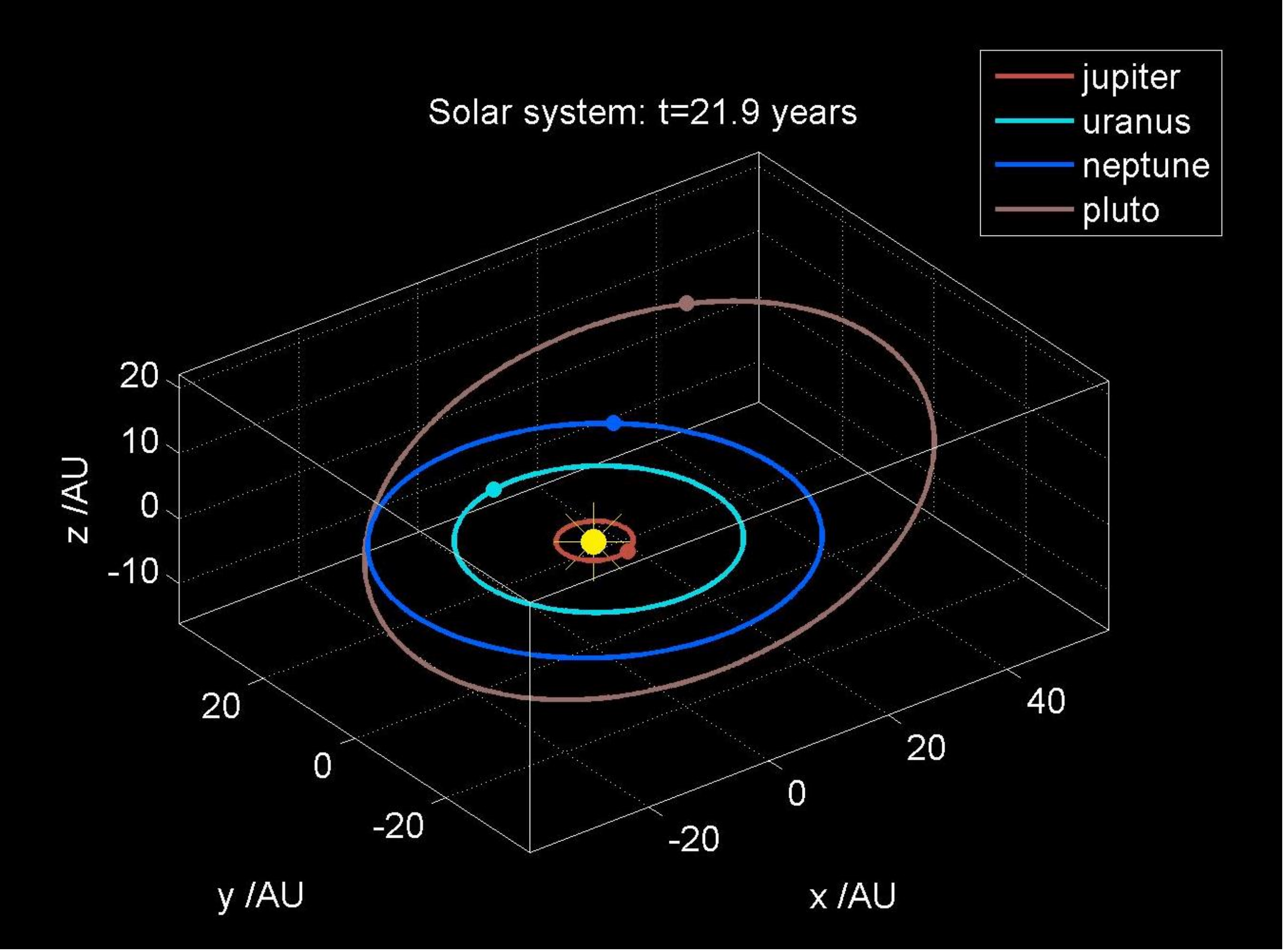
$$x = r\cos\theta, \ \ y = r\sin\theta$$

$$\theta = \frac{2\pi t}{P}$$

Run the simulation till the outermost planet completes at least one orbit.



[YouTube example video]



Challenge #4: Create a 3D animation of the solar system orbits

YouTube example video

Use the elliptical inclination angle β. *(See next slide).* Most orbits won't change much, but Pluto is the exception! The coordinate change is:

$$x' = x\cos\beta \qquad z' = x\sin\beta \qquad y' = y$$

β is the orbital inclination /degrees. In all cases the semi-major axis pointing direction is

$$\mathbf{d} = d_x\hat{\mathbf{x}} + d_y\hat{\mathbf{y}} + d_{\mathbf{y}}\hat{\mathbf{z}} = \cos\beta\hat{\mathbf{x}} + \sin\beta\hat{\mathbf{z}}$$

Object	$M/M_\oplus$	a /AU	ε	θ_0	β
Sun	332,837	-	-	-	-
Mercury	0.055	0.387	0.21	*	7.00
Venus†	0.815	0.723	0.01	*	3.39
Earth	1.000	1.000	0.02	*	0.00
Mars	0.107	1.523	0.09	*	1.85
Jupiter	317.85	5.202	0.05	*	1.31
Saturn	95.159	9.576	0.06	*	2.49
Uranus†	14.500	19.293	0.05	*	0.77
Neptune	17.204	30.246	0.01	*	1.77
Pluto†	0.003	39.509	0.25	*	17.5

$R/R_\oplus$	T_{rot} / days	P /Yr
109.123	-	-
0.383	58.646	0.241
0.949	243.018	0.615
1.000	0.997	1.000
0.533	1.026	1.881
11.209	0.413	11.861
9.449	0.444	29.628
4.007	0.718	84.747
3.883	0.671	166.344
0.187	6.387	248.348

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

$$P^2 = \frac{4\pi^2}{G(m+M_\odot)}a^3$$

β is the orbital inclination /degrees. In all cases the semi-major axis pointing direction is

$$\mathbf{d} = d_x\hat{\mathbf{x}} + d_y\hat{\mathbf{y}} + d_{\mathbf{y}}\hat{\mathbf{z}} = \cos\beta\hat{\mathbf{x}} + \sin\beta\hat{\mathbf{z}}$$

You could begin with all zero, or perhaps a random angle for each planet's orbit.

$$M_\odot = 1.9891 \times 10^{30}\ \text{kg}$$
$$R_\odot = 6.960 \times 10^{8}\ \text{m}$$
$$M_\oplus = 5.9742 \times 10^{24}\ \text{kg}$$
$$R_\oplus = 6.37814 \times 10^{6}\ \text{m}$$
$$1\text{AU} = 1.495979 \times 10^{11}\ \text{m}$$

* For the current orbital polar angle θ_0 (and indeed more accurate values for solar system parameters) see the website of the Jet Propulsion Laboratory (JPL) `http://ssd.jpl.nasa.gov/`

†These planets rotate clockwise about their own internal polar axis. ("Retrograde"). All the other planets rotate anti-clockwise about their own internal axis. All the planets orbit the sun in an anticlockwise direction.

Calculating orbit angle vs time

$$\frac{dA}{dt} = \tfrac{1}{2}\sqrt{G\left(m+M\right)\left(1-\varepsilon^2\right)a}$$

Equal areas swept out in equal times

This is a *constant*

Orbit time can be determined from polar angle using Kepler II:

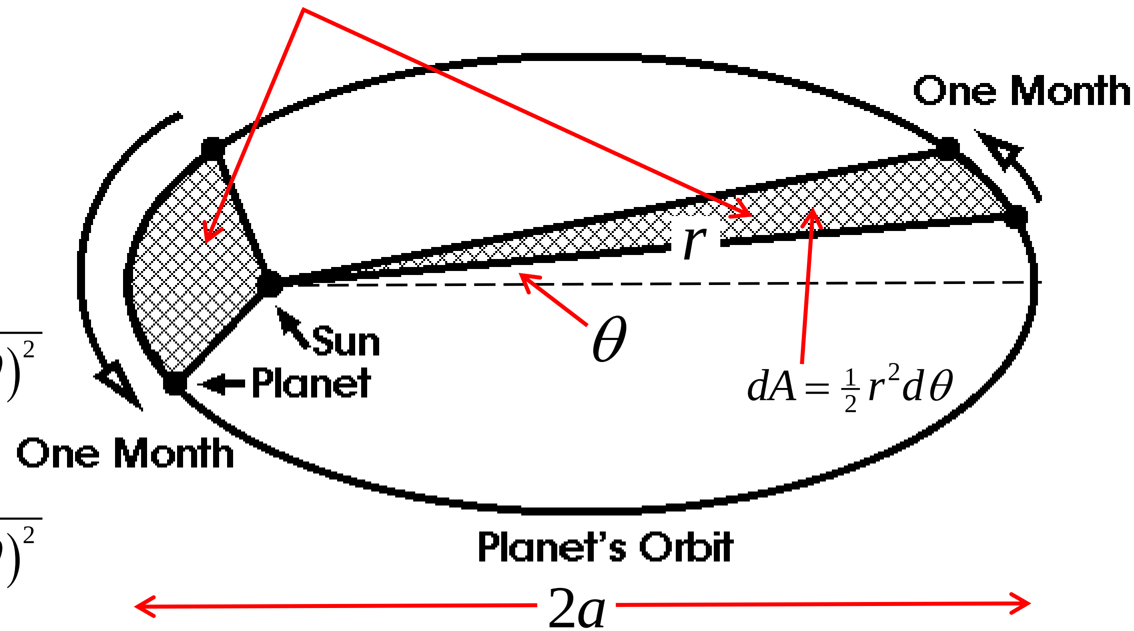
$$r^2\frac{d\theta}{dt} = \sqrt{G\left(m+M\right)\left(1-\varepsilon^2\right)a}$$

$$\therefore \int_{\theta_0}^{\theta} r^2 d\theta = t\sqrt{G\left(m+M\right)\left(1-\varepsilon^2\right)a}$$

$$\therefore t = \frac{a^2\left(1-\varepsilon^2\right)^2}{\sqrt{G\left(m+M\right)\left(1-\varepsilon^2\right)a}}\int_{\theta_0}^{\theta}\frac{d\theta}{\left(1-\varepsilon\cos\theta\right)^2}$$

$$\therefore t = \frac{a^2\left(1-\varepsilon^2\right)^2}{\sqrt{G\left(m+M\right)\left(1-\varepsilon^2\right)a}}\int_{\theta_0}^{\theta}\frac{d\theta}{\left(1-\varepsilon\cos\theta\right)^2}$$

$$\therefore t = \sqrt{\frac{a^3\left(1-\varepsilon^2\right)^3}{G\left(m+M\right)}}\int_{\theta_0}^{\theta}\frac{d\theta}{\left(1-\varepsilon\cos\theta\right)^2}$$



From Kepler III: $P^2 = \dfrac{4\pi^2}{G\left(m+M\right)}a^3$

$$t = P\left(1-\varepsilon^2\right)^{\frac{3}{2}}\frac{1}{2\pi}\int_{\theta_0}^{\theta}\frac{d\theta}{\left(1-\varepsilon\cos\theta\right)^2}$$

Evaluate this *numerically*

Note when:

$$\varepsilon \ll 1$$

$$t \approx P\left(\theta-\theta_0\right)$$

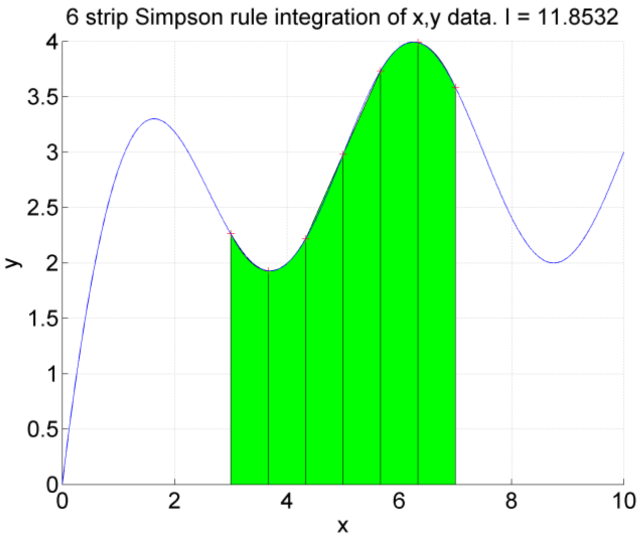
Challenge #5: Calculate orbit angle vs time for an *eccentric* orbit (e.g. pluto) and compare to a circular version with the same period.

To evaluate the angle integral, use **Simpson's rule**, which approximates the integrand of an integral with a series of quadratic curve segments.

$$t = P\left(1-\varepsilon^2\right)^{\frac{3}{2}} \frac{1}{2\pi} \int_{\theta_0}^{\theta} \frac{d\theta}{\left(1-\varepsilon\cos\theta\right)^2}$$

$$\varepsilon \ll 1$$

$$t \approx P\left(\theta-\theta_0\right)$$



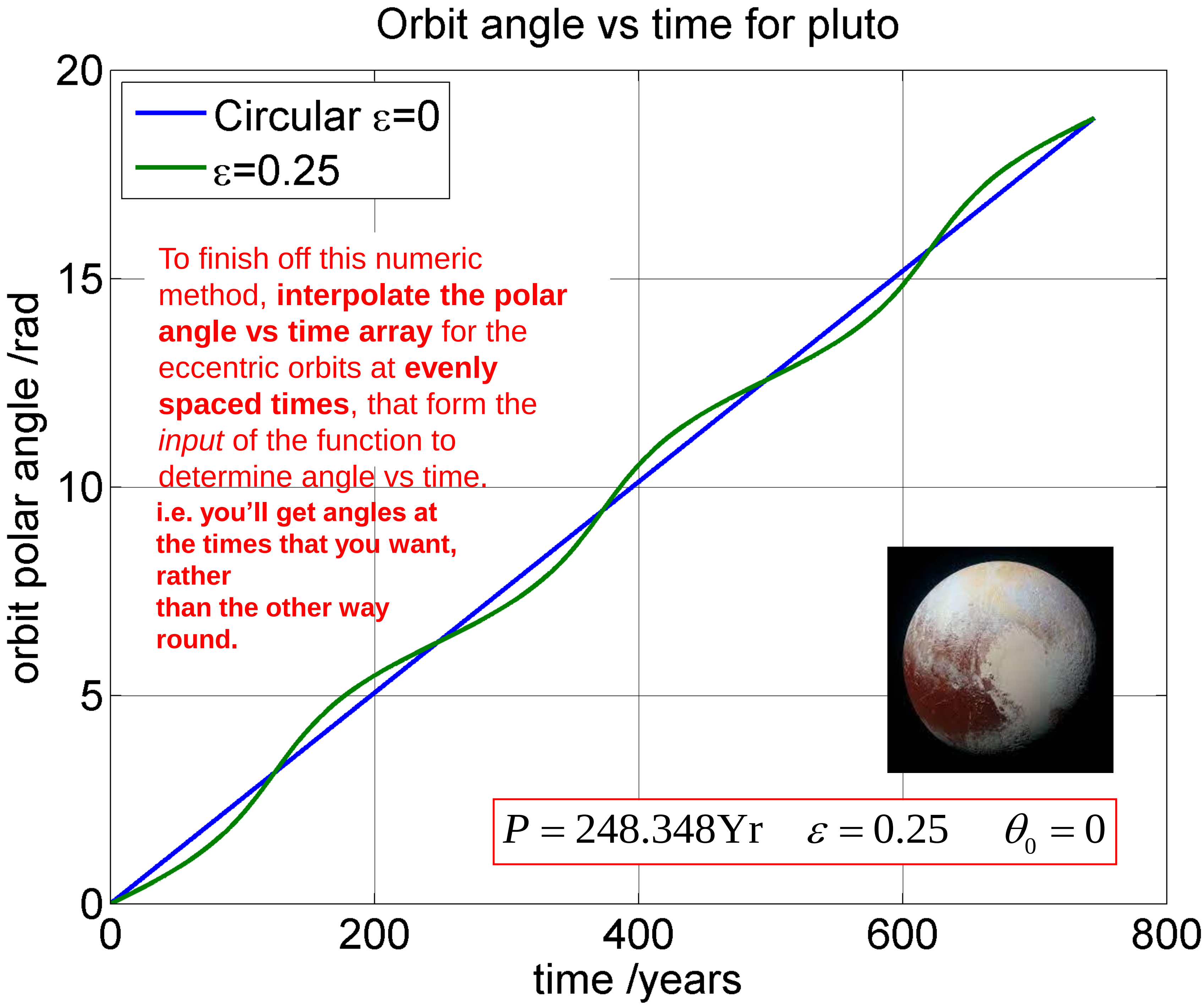
$$\int_a^b f(x)dx \approx \tfrac{1}{3}h\left\{y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + ...4y_{N-1} + y_N\right\}$$

$y_n = f(a+nh)$ h is the strip width $h = \frac{b-a}{N}$

Determine time vs angle for three periods of Pluto's orbit, using $d\theta = h =$ 1/1000.

You'll have to evaluate the integral over a *range* of polar angles, whichamounts to a **cumulative sum**. Many languages have functions (such as `cumsum` in MATLAB) that can perform efficient operations with *arrays*.

$$P = 248.348\,\mathrm{Yr} \quad \varepsilon = 0.25 \quad \theta_0 = 0$$

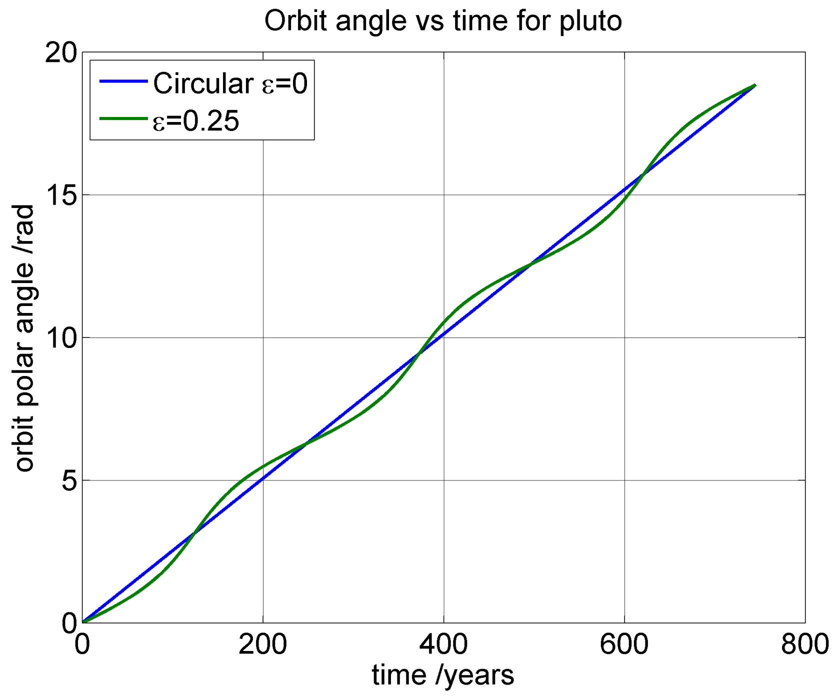


Orbit angle vs time for pluto
Circular ε=0
ε=0.25
To finish off this numeric method, **interpolate the polar angle vs time array** for the eccentric orbits at **evenly spaced times**, that form the *input* of the function to determine angle vs time.
i.e. you'll get angles at the times that you want, rather than the other way round.
$P = 248.348\text{Yr} \quad \varepsilon = 0.25 \quad \theta_0 = 0$
orbit polar angle /rad
time /years
0
5
10
15
20
0
200
400
600
800

MATLAB example code for determining polar angle vs time for an elliptical orbit

```
%Numeric method to compute polar angle vs orbit time
function theta = angle_vs_time( t, P, ecc, theta0 )

%Angle step for Simpson's rule
dtheta = 1/1000;

%

%Number of orbits
N = ceil( t(end)/P );

%Define array of polar angles for orbits
theta = theta0 : dtheta : ( 2*pi*N + theta0 );

%Evaluate integrand of time integral
f = (1 - ecc*cos(theta) ).^(-2);

%Define Simpson rule coefficients  c = [ 1, 4, 2, 4, 2,
L = length(theta);
isodd = rem( 1:(L-2),2 ); isodd( isodd==1 ) = 4; isodd( isodd==0 ) = 2;
c = [1, isodd, 1];

%Calculate array of times
tt = P*( (1-ecc^2)^(3/2) )*(1/(2*pi))*dtheta*(1/3).*cumsum( c.*f );

%Interpolate the polar angles for the eccentric orbit at the circular orbit
%times
theta = interp1( tt, theta, t, 'spline' );
```

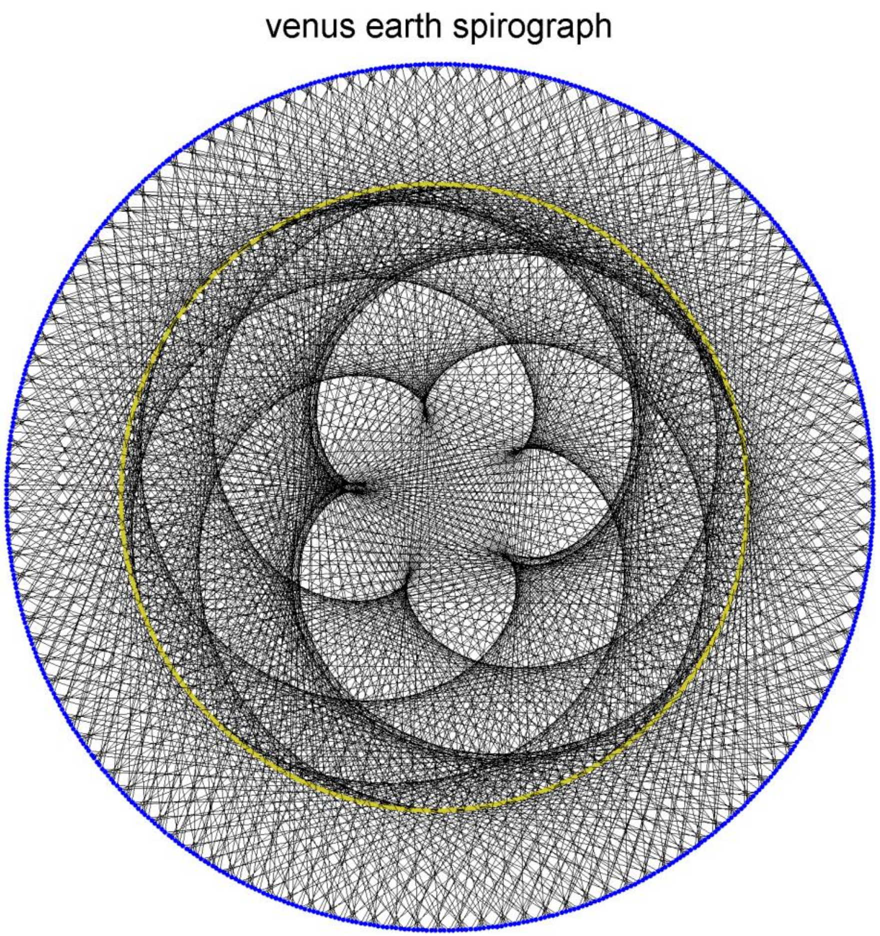
Note time is an input

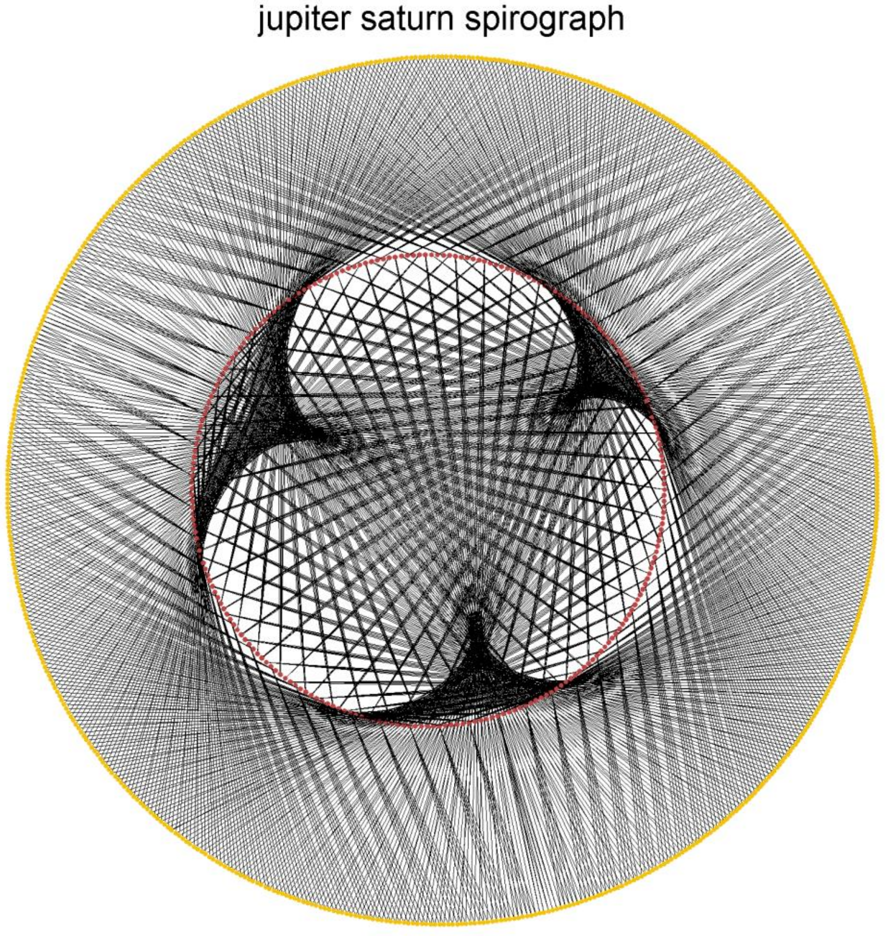


Challenge #6: Solar system spirograph! inspired by: https://engaging-data.com/planetary-spirograph

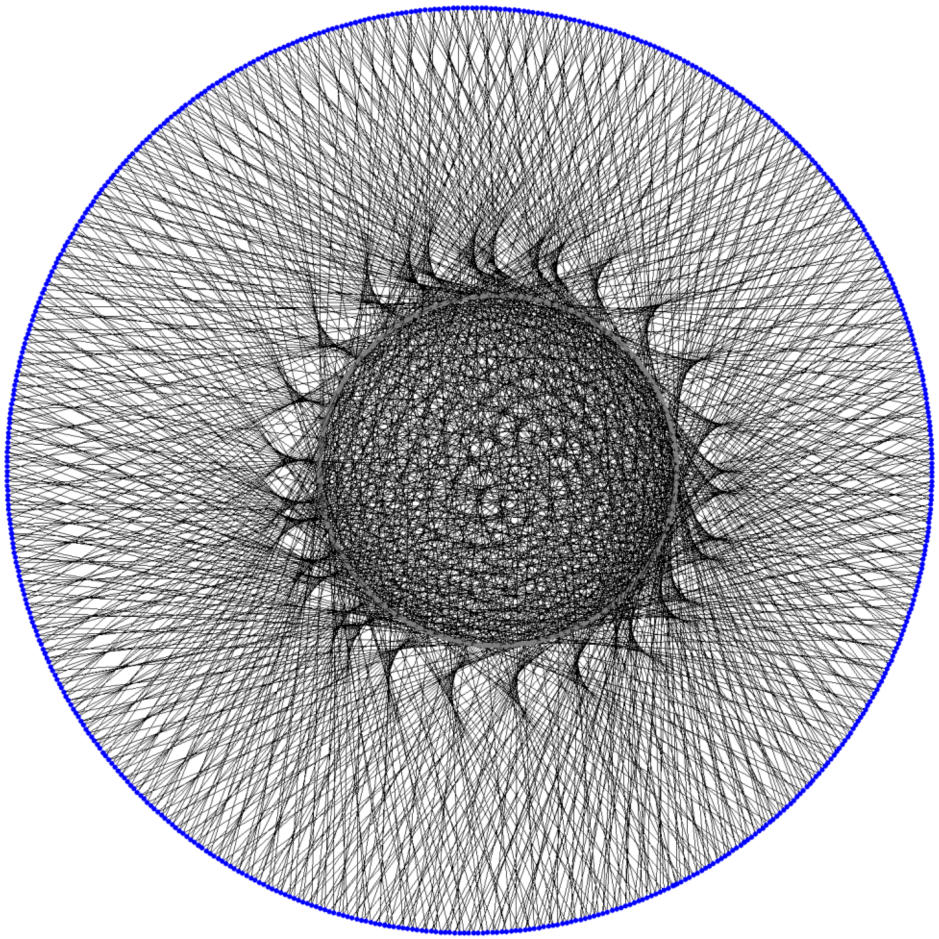
Choose a pair of planets and determine their orbits vs time. At time intervals of Δt, draw a line between the planets and plot this line. Keep going for N orbits of the outermost planet.

N = **10 ,** Δt **=** N x **maximum orbital period /1234***,* might be sensible parameters.

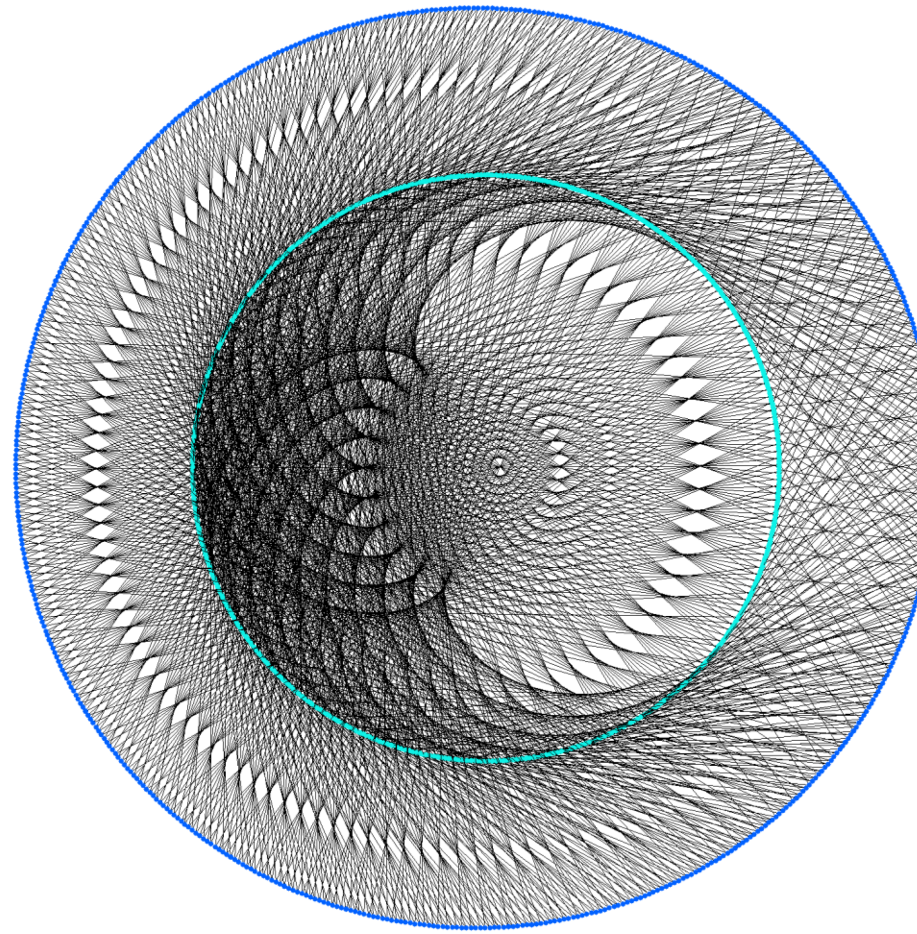




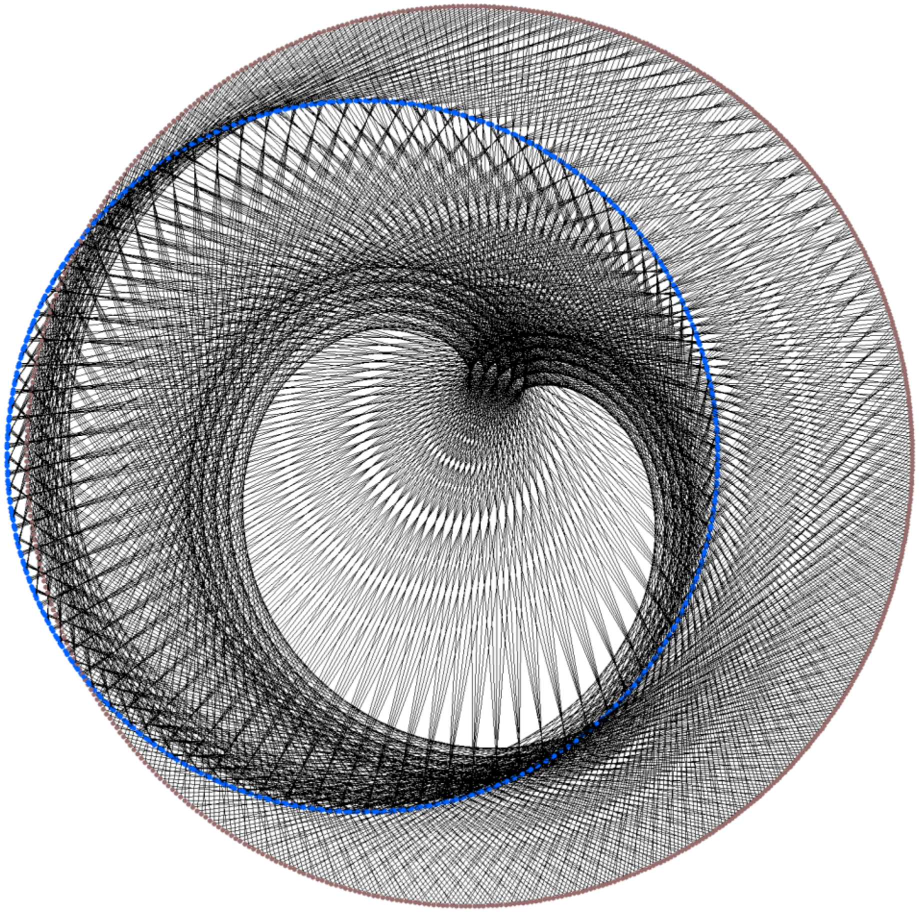
mercury earth spirograph



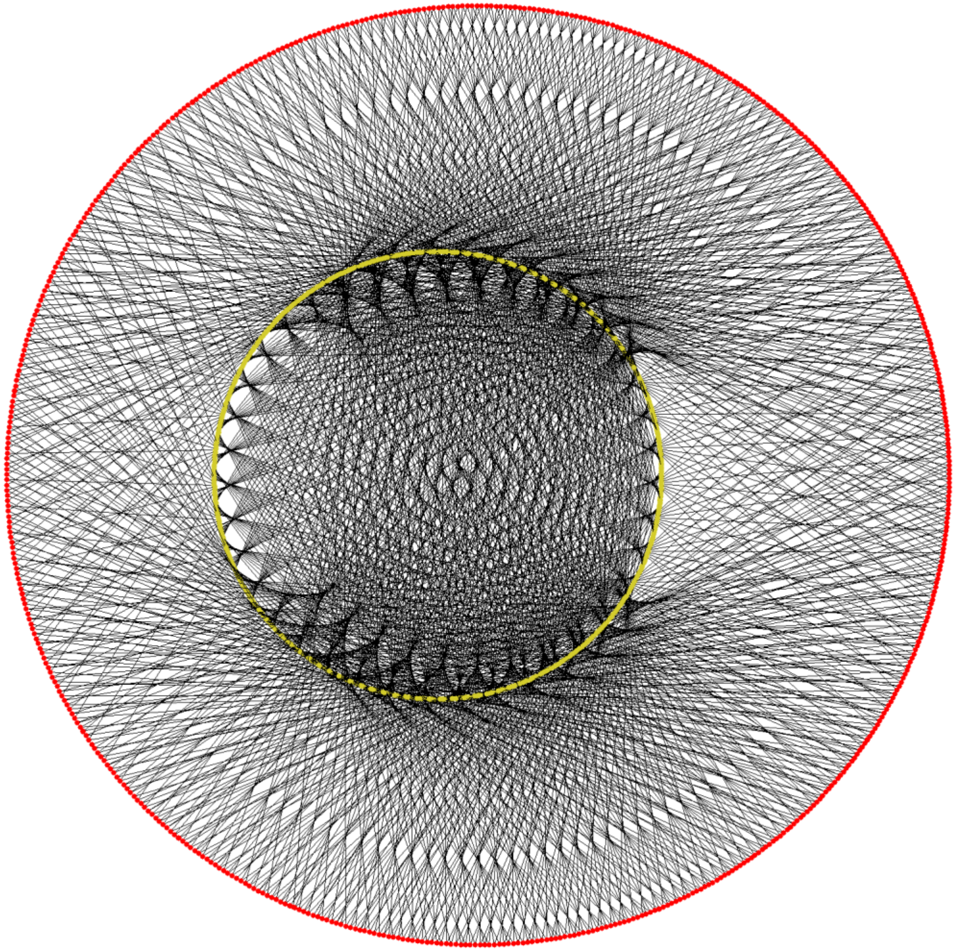
uranus neptune spirograph



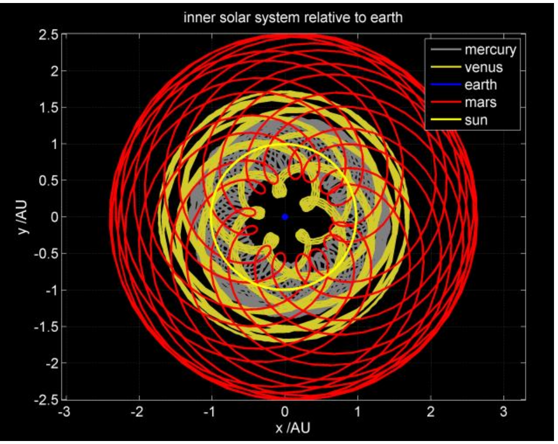
neptune pluto spirograph

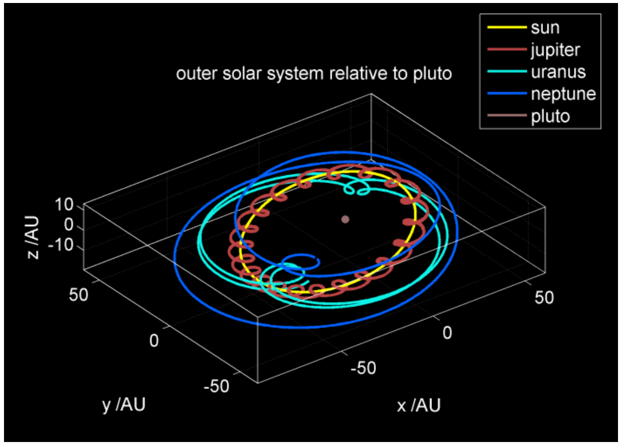


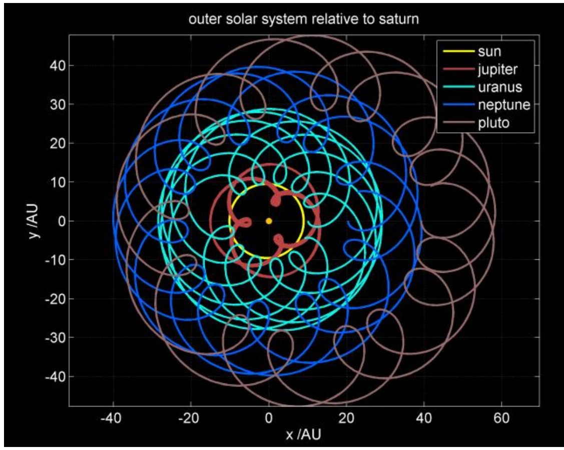
venus mars spirograph

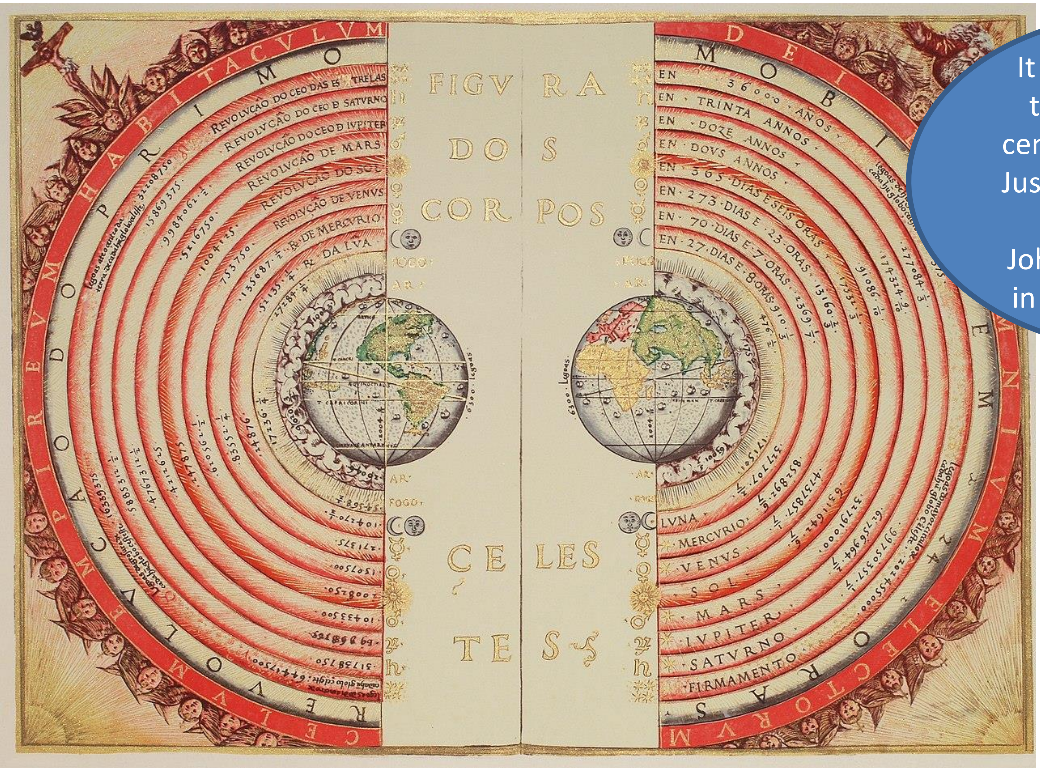


Challenge #7: Use your orbital models to plot the orbits of the other bodies in the solar system, with a chosen object (e.g. Earth) at a *fixed position at the origin of a Cartesian coordinate system.* i.e. choose a coordinate system where your chosen object is at (0,0,0).



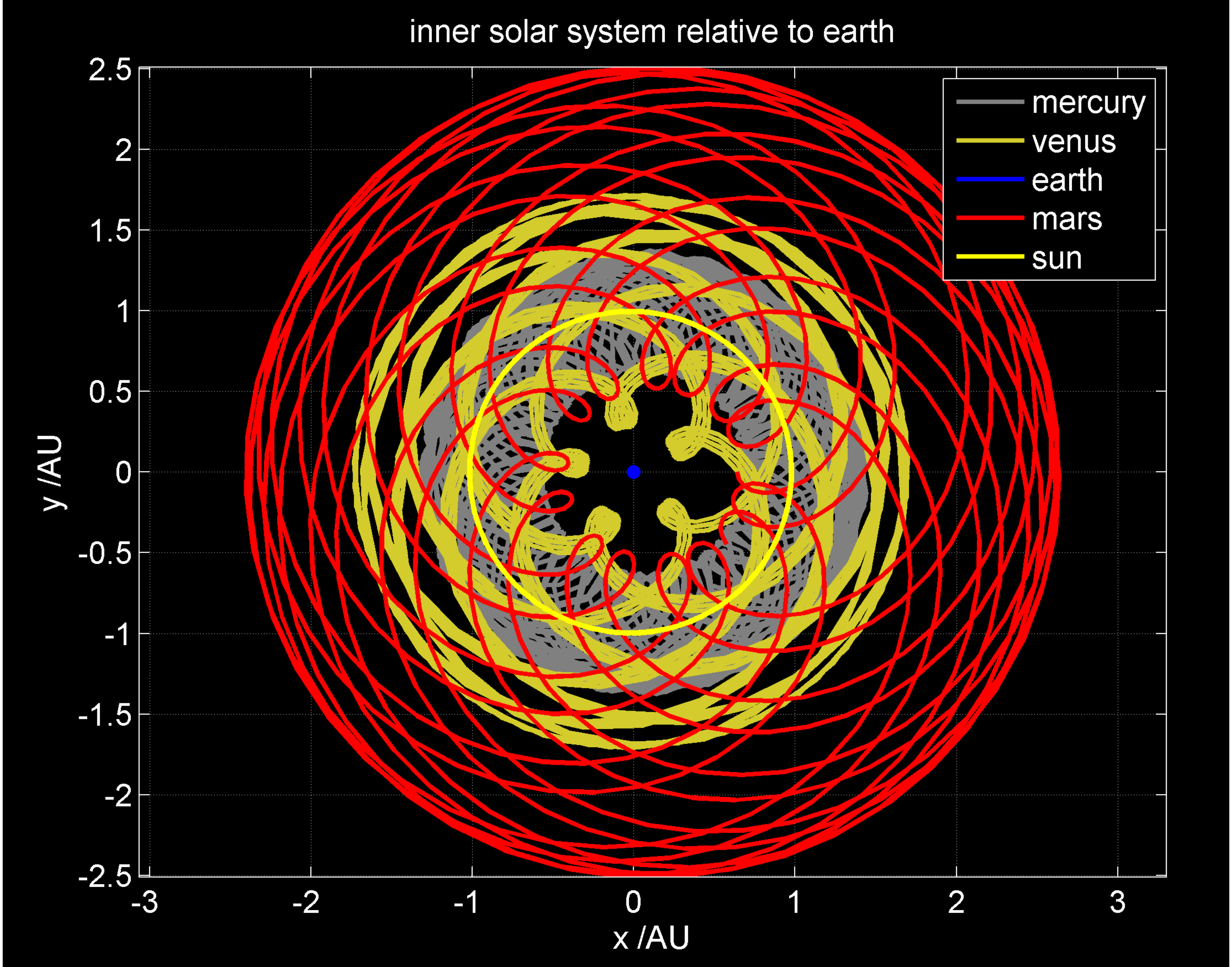


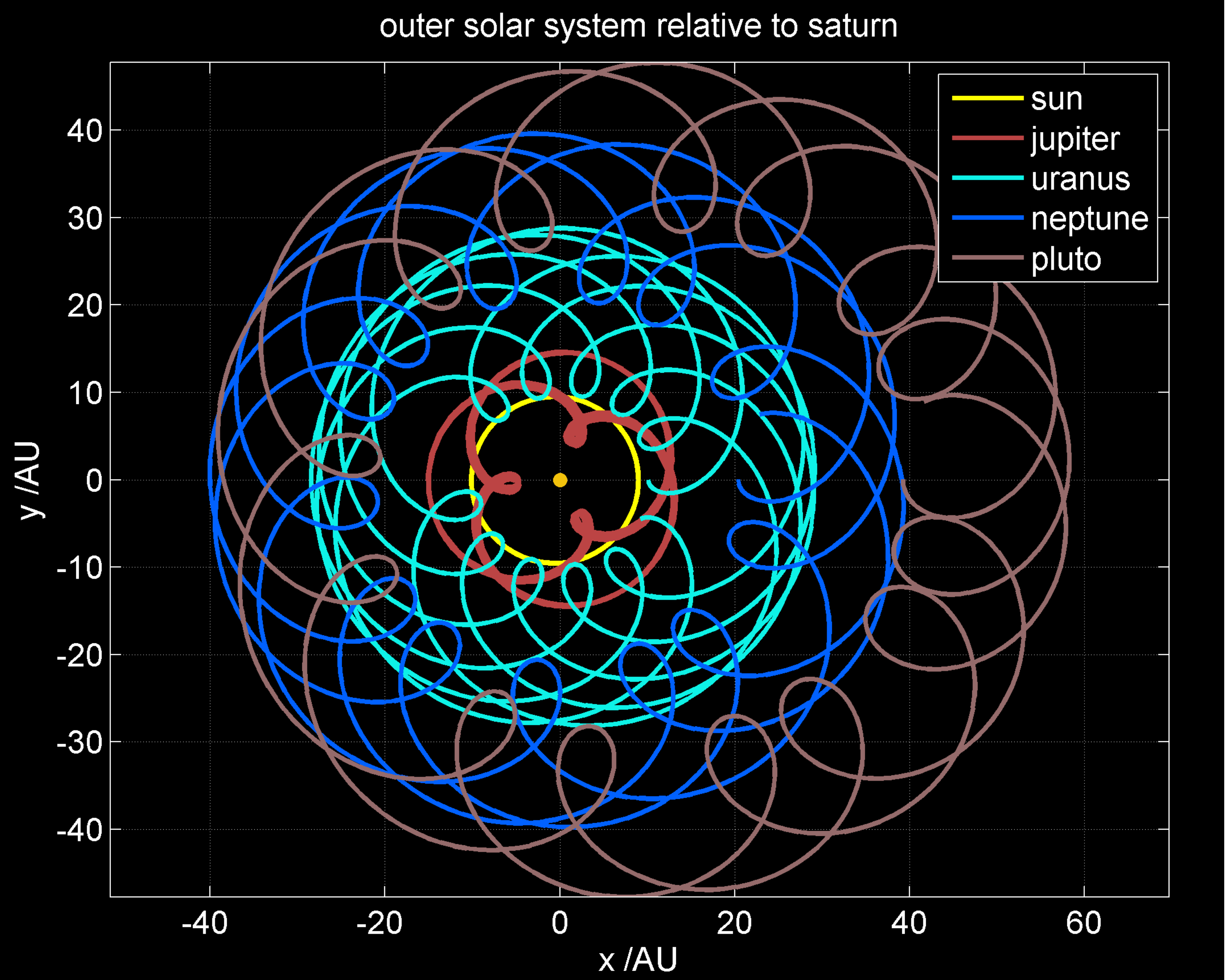


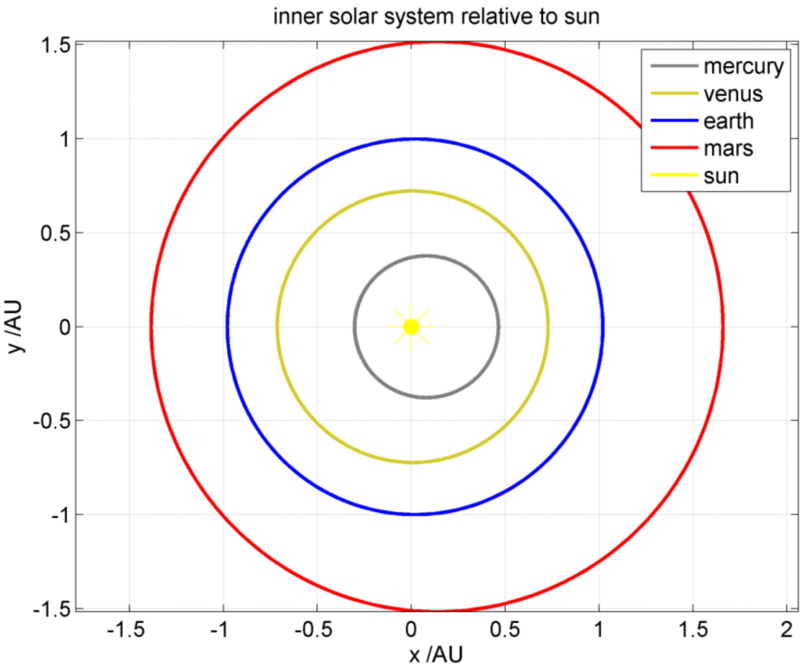


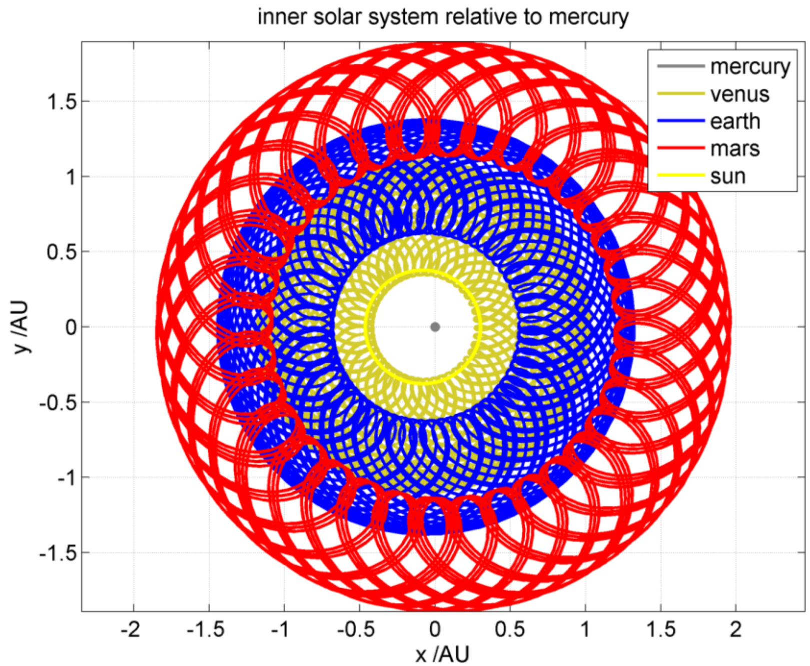
It is perfectly fine for the Earth to be the centre of the Universe! Just don't expect those nice ellipses that Johannes will discover in about 1500 years...

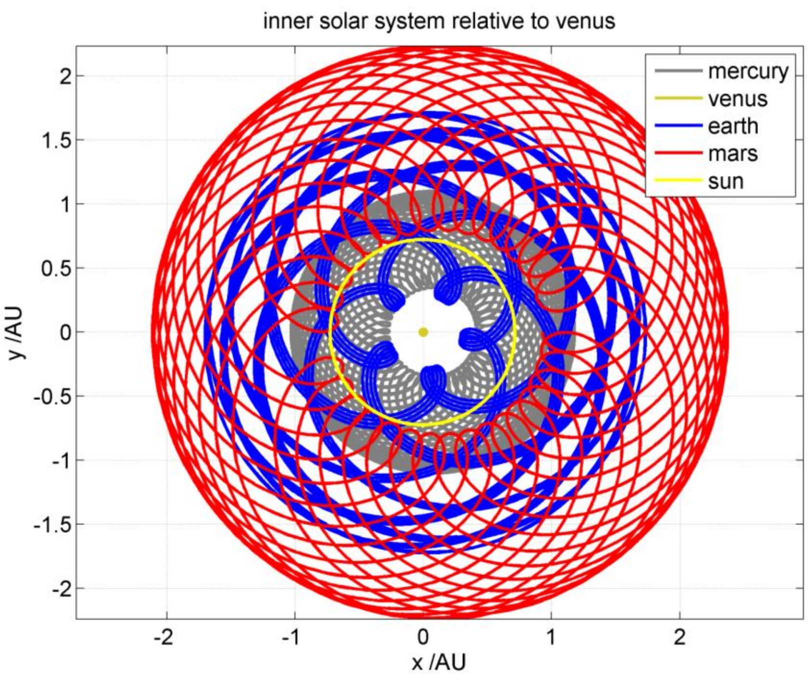
Claudius Ptolemy
(100-170 AD)

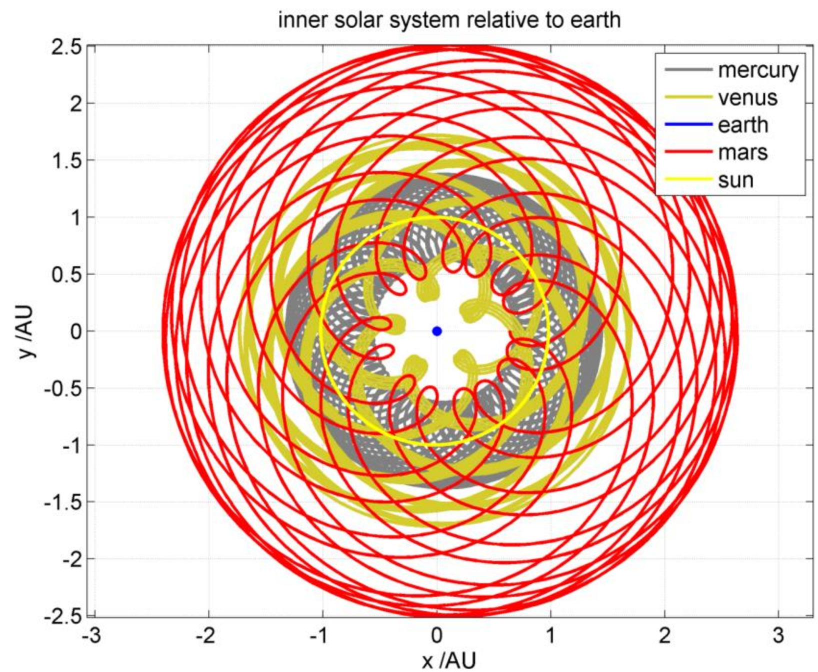

inner solar system relative to earth
mercury
venus
earth
mars
sun
y /AU
x /AU
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-2
-1
0
1
2
3

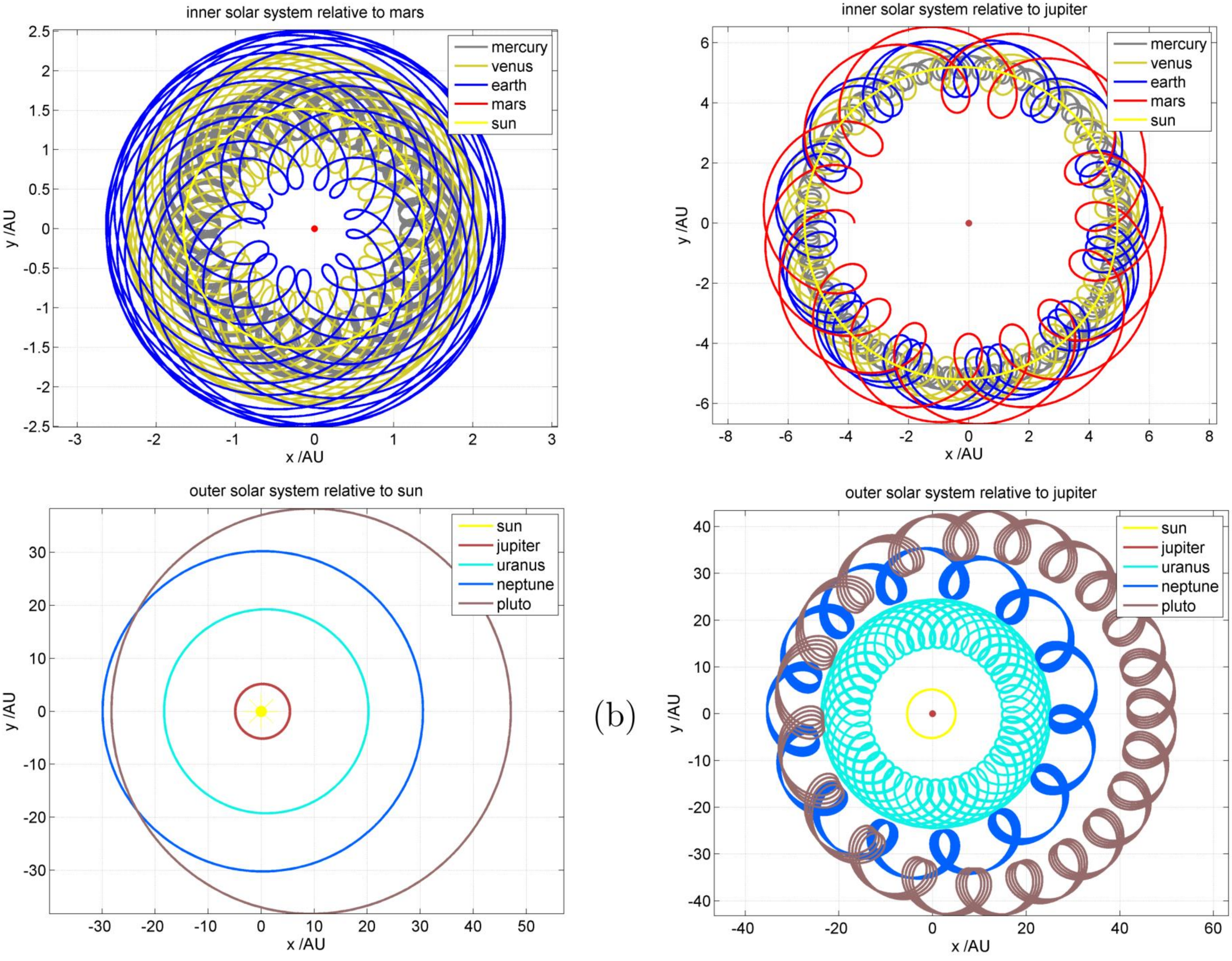

outer solar system relative to saturn
sun
jupiter
uranus
neptune
pluto
y /AU
x /AU
40
30
20
10
0
-10
-20
-30
-40
-40
-20
0
20
40
60


inner solar system relative to sun
mercury
venus
earth
mars
sun
y /AU
x /AU
1.5
1
0.5
0
-0.5
-1
-1.5
2


inner solar system relative to mercury
mercury
venus
earth
mars
sun
y /AU
x /AU
1.5
1
0.5
0
-0.5
-1
-1.5
-2
2

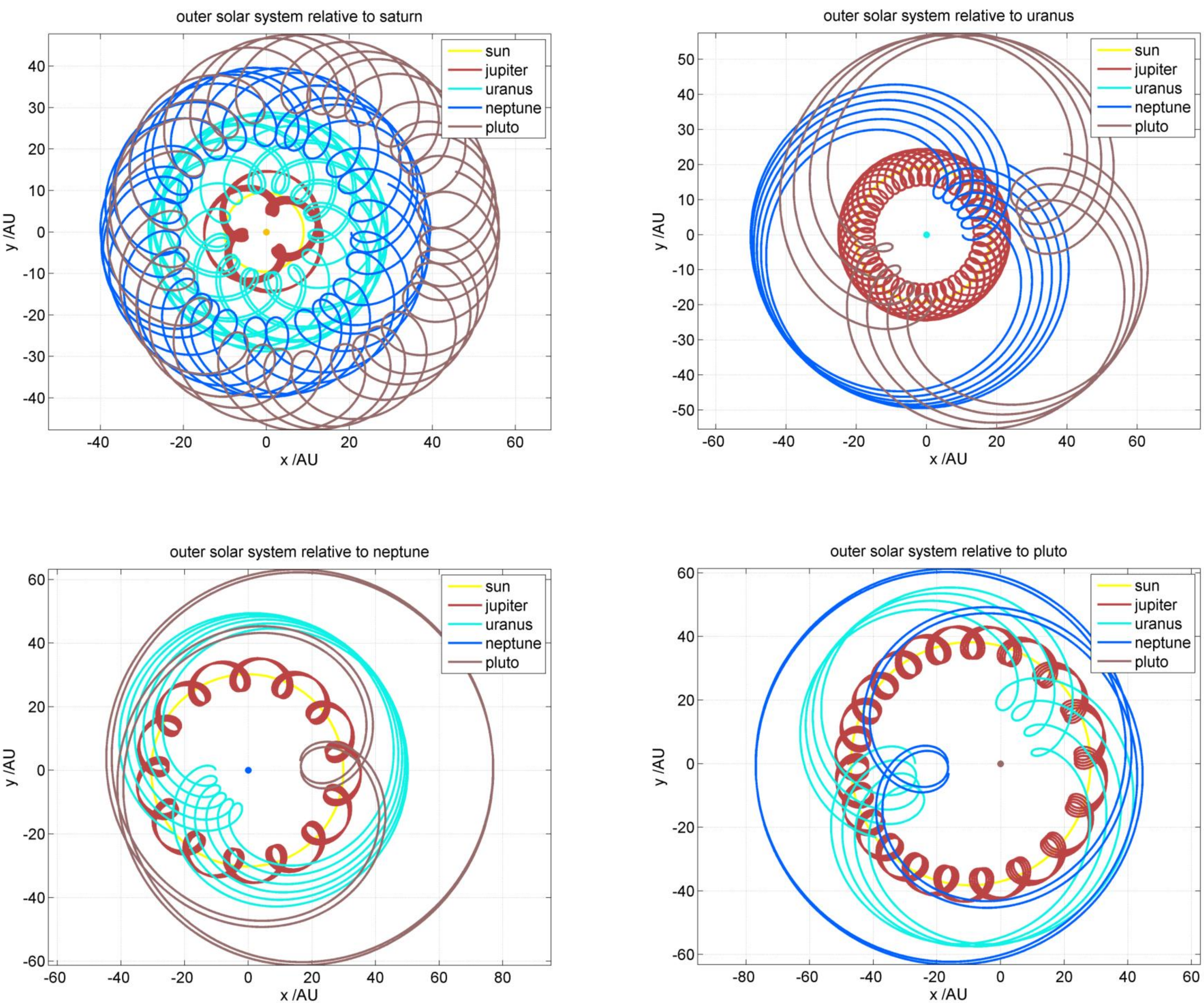

inner solar system relative to venus
mercury
venus
earth
mars
sun
y /AU
x /AU
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2


inner solar system relative to earth
mercury
venus
earth
mars
sun
y /AU
x /AU
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
3



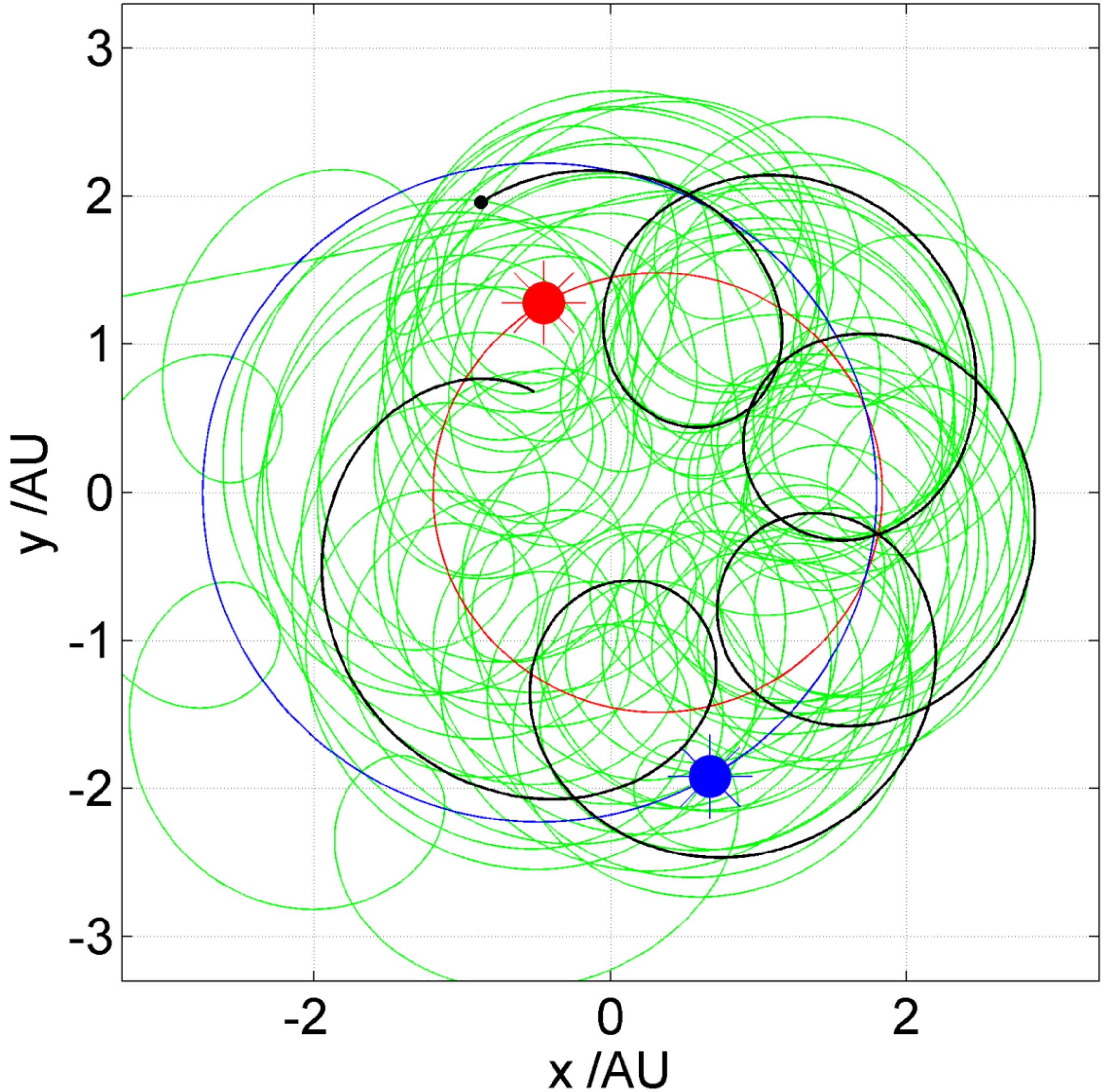
inner solar system relative to mars
mercury
venus
earth
mars
sun
x /AU
y /AU
inner solar system relative to jupiter
mercury
venus
earth
mars
sun
x /AU
y /AU
outer solar system relative to sun
sun
jupiter
uranus
neptune
pluto
x /AU
y /AU
outer solar system relative to jupiter
sun
jupiter
uranus
neptune
pluto
x /AU
y /AU

(b)


outer solar system relative to saturn
sun
jupiter
uranus
neptune
pluto
y /AU
x /AU
outer solar system relative to uranus
sun
jupiter
uranus
neptune
pluto
y /AU
x /AU
outer solar system relative to neptune
sun
jupiter
uranus
neptune
pluto
y /AU
x /AU
outer solar system relative to pluto
sun
jupiter
uranus
neptune
pluto
y /AU
x /AU

What about systems of *more than* two stars or planets?
We need a numeric method!

M1=3, M2=2 T=2.32 years, a=3AU, k=1.1, a_p=0.965AU.



What about systems of *more than* two stars or planets?
We need a numeric method!

The **Verlet Method** implies *constant acceleration motion* between fixed timesteps.

$$\mathbf{a}_n = f(t_n, \mathbf{r}_n, \mathbf{v}_n)$$

Assume we can always calculate acceleration

$$t_{n+1} = t_n + \Delta t$$

Fixed timestep

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{v}_n \Delta t + \tfrac{1}{2}\mathbf{a}_n \Delta t^2$$

Update position based upon constant acceleration motion between timesteps

$$\mathbf{V} = \mathbf{v}_n + \mathbf{a}_n \Delta t$$

$$\mathbf{A} = f(t_{n+1}, \mathbf{r}_{n+1}, \mathbf{V})$$

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \tfrac{1}{2}\left(\mathbf{a}_n + \mathbf{A}\right)\Delta t$$

Acceleration may depend upon velocity, so for greater precision we work out an intermediate velocity $\mathbf{V}$, update acceleration ($\mathbf{A}$) and perform an *average* to calculate the velocity update.

Verlet method

$$\mathbf{a}_n = f(t_n, \mathbf{r}_n, \mathbf{v}_n)$$

$$t_{n+1} = t_n + \Delta t$$

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{v}_n \Delta t + \tfrac{1}{2}\mathbf{a}_n \Delta t^2$$

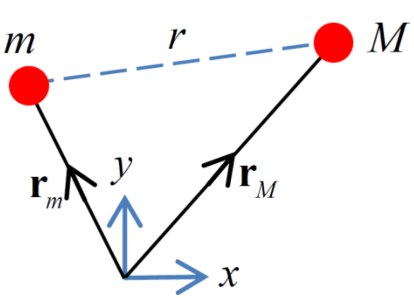
$$\mathbf{V} = \mathbf{v}_n + \mathbf{a}_n \Delta t$$

$$\mathbf{A} = f(t_{n+1}, \mathbf{r}_{n+1}, \mathbf{V})$$

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \tfrac{1}{2}\left(\mathbf{a}_n + \mathbf{A}\right)\Delta t$$

Newton's Law of Gravitation

$$\mathbf{a}_{n,i} = -G\sum_{j \neq i}^{N} M_j \frac{\mathbf{r}_i - \mathbf{r}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3}$$



$$\mathbf{r} = \mathbf{r}_M - \mathbf{r}_m$$

$$r = |\mathbf{r}|$$

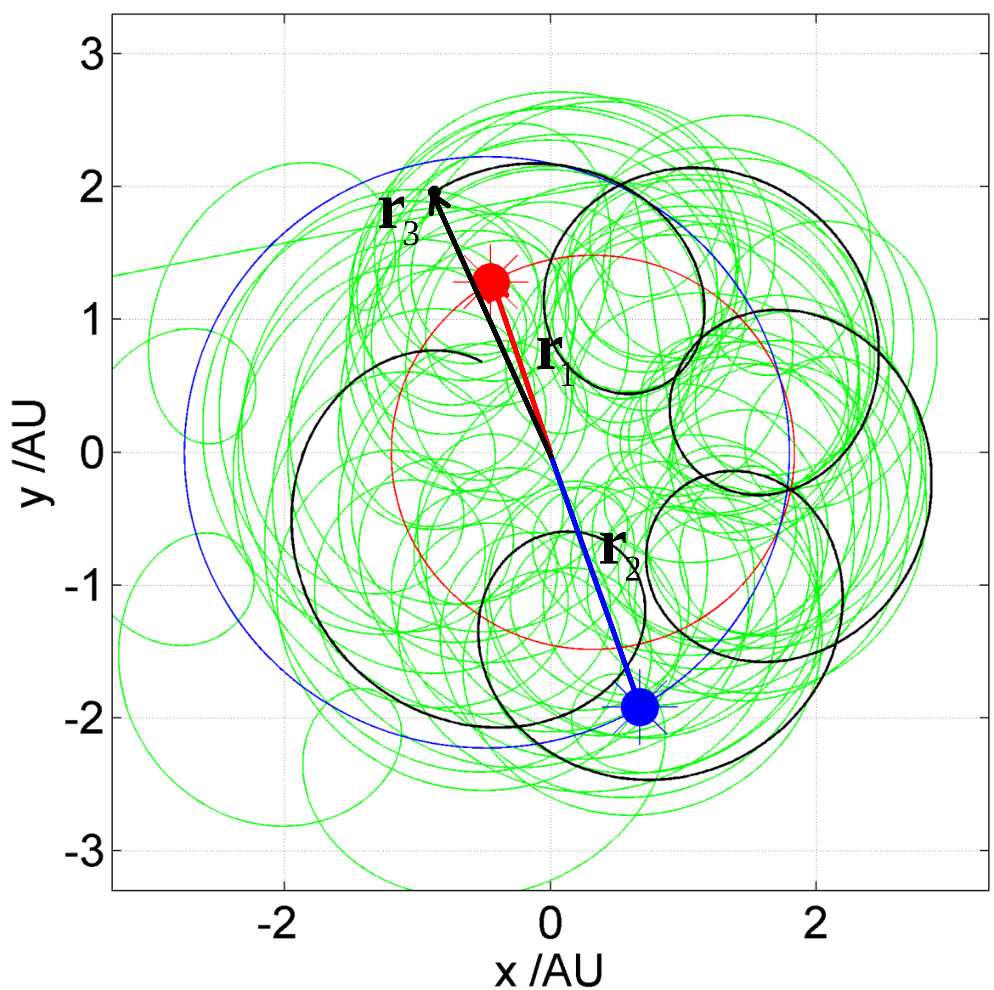
In this simulation:

$$M_1 = 3M_\odot$$

$$M_2 = 2M_\odot$$

$$M_3 \ll M_\odot$$

M1=3, M2=2 T=2.32 years, a=3AU, k=1.1, a_p=0.965AU.



$$\mathbf{a}_n = f(t_n, \mathbf{r}_n, \mathbf{v}_n)$$

$$t_{n+1} = t_n + \Delta t$$

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{v}_n \Delta t + \tfrac{1}{2}\mathbf{a}_n \Delta t^2$$

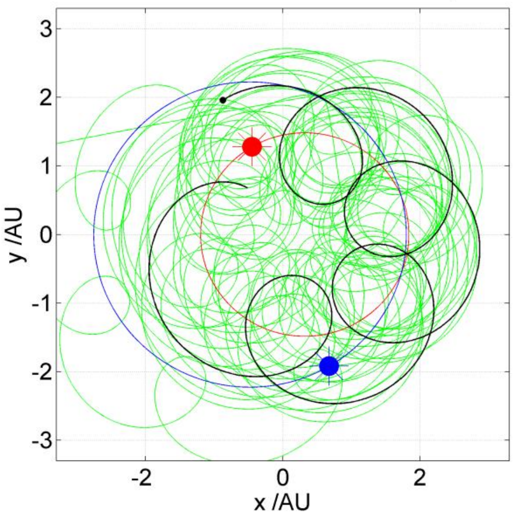
$$\mathbf{V} = \mathbf{v}_n + \mathbf{a}_n \Delta t$$

$$\mathbf{A} = f(t_{n+1}, \mathbf{r}_{n+1}, \mathbf{V})$$

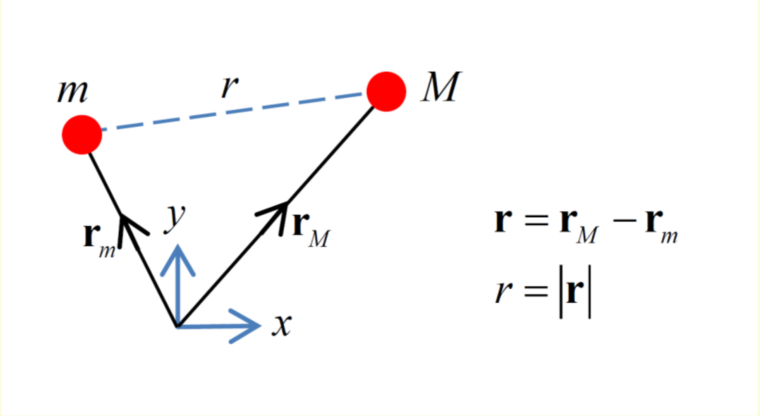
$$\mathbf{v}_{n+1} = \mathbf{v}_n + \tfrac{1}{2}(\mathbf{a}_n + \mathbf{A})\Delta t$$

$$\mathbf{a}_{n,i} = -G\sum_{j \neq i}^{N} M_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

M1=3, M2=2 T=2.32 years, a=3AU, k=1.1, a_p=0.965AU.



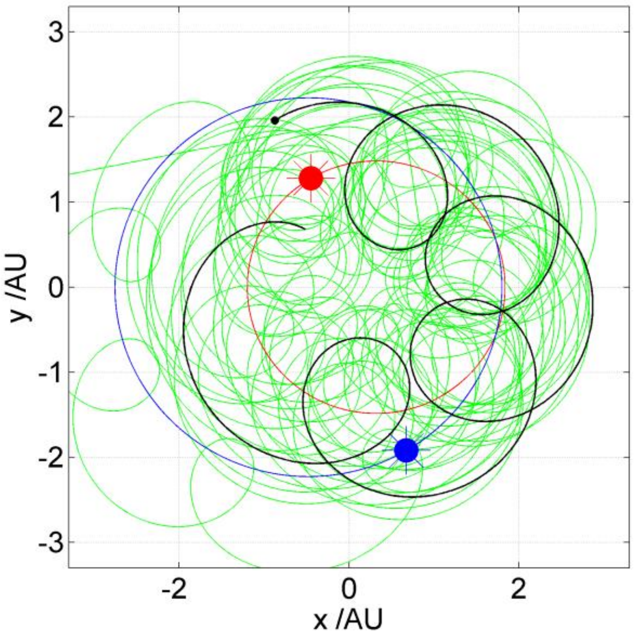

```
function gravity_sim_2_binary_stars_and_planet

%% INPUTS %%

%Semi-major axis of mutual star orbit in AU
a = 3;

%Planet (initial) circular orbit radius about star 1
ap = a/3.11;

%Initial angle from x axis (anticlockwise) of planet /radians
theta0 = pi/4;

%Masses of stars in solar masses
M1 = 3; M2 = 2;

%Initial vy velocity multiplier from mutually circular of stars
k = 1.1;

%Number of orbital periods
num_periods = 50;

%Timestep in years
dt = 0.001;

%Fontsize
fsize = 18;

%Axes limits
limit = 1.1*a;

%Starting period for plot
Pstart = 1.23;
```



$$
\begin{aligned}
\mathbf{a}_n &= f(t_n, \mathbf{r}_n, \mathbf{v}_n) \\
t_{n+1} &= t_n + \Delta t \\
\mathbf{r}_{n+1} &= \mathbf{r}_n + \mathbf{v}_n \Delta t + \tfrac{1}{2}\mathbf{a}_n \Delta t^2 \\
\mathbf{V} &= \mathbf{v}_n + \mathbf{a}_n \Delta t \\
\mathbf{A} &= f(t_{n+1}, \mathbf{r}_{n+1}, \mathbf{V}) \\
\mathbf{v}_{n+1} &= \mathbf{v}_n + \tfrac{1}{2}(\mathbf{a}_n + \mathbf{A})\Delta t
\end{aligned}
$$

$$
\mathbf{a}_{n,i} = -G \sum_{j \neq i}^{N} M_j \frac{\mathbf{r}_i - \mathbf{r}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3}
$$

M1=3, M2=2 T=2.32 years, a=3AU, k=1.1, a_p=0.965AU.



Initial positions

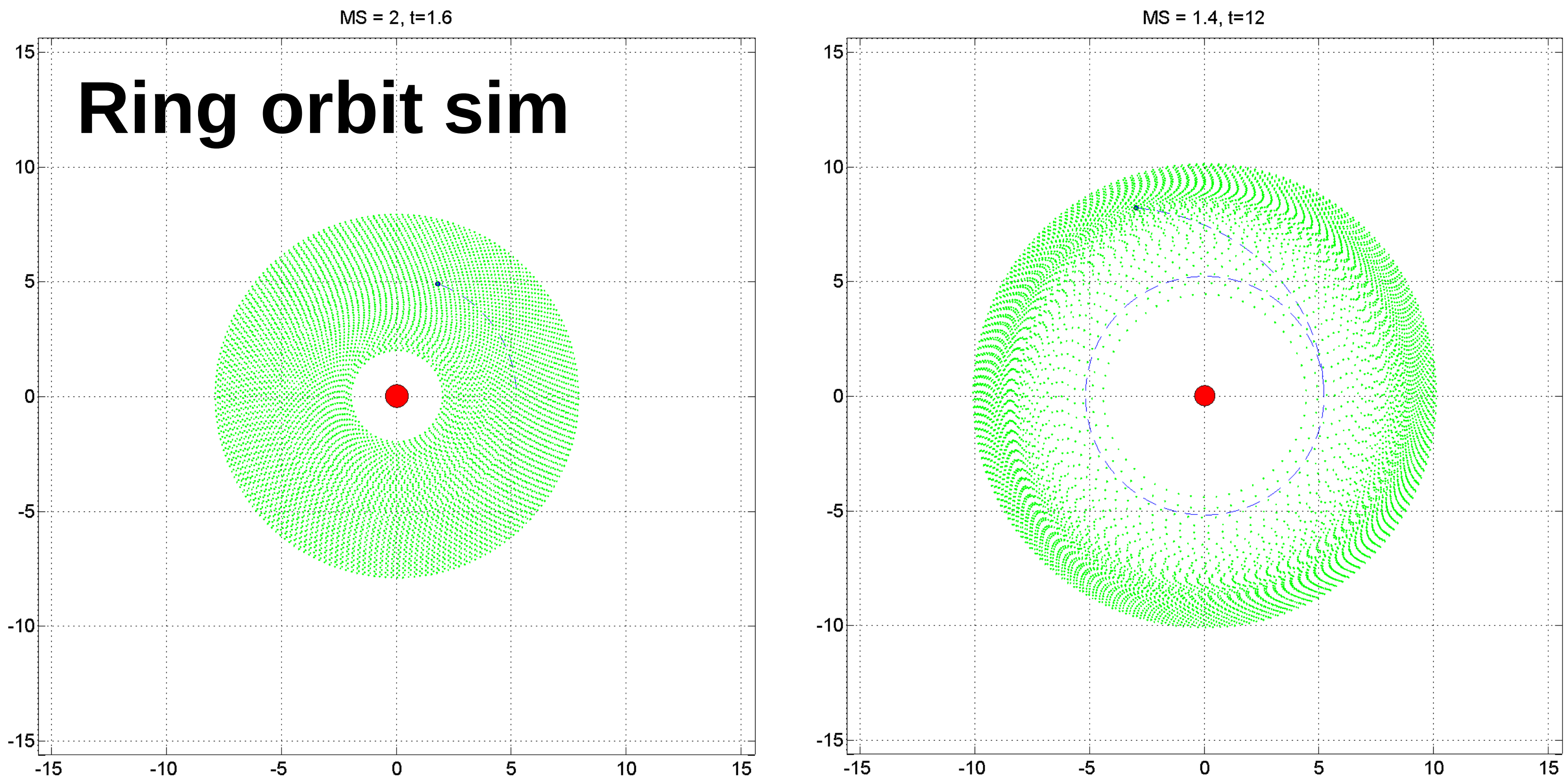
$$
\begin{aligned}
X_1(0) &= -\frac{M_2 a}{M_1 + M_2} \\
Y_1(0) &= 0 \\
X_2(0) &= \frac{M_1 a}{M_1 + M_2} \\
Y_2(0) &= 0 \\
x(0) &= X_1(0) + a_p \cos\theta_0 \\
y(0) &= a_p \sin\theta_0
\end{aligned}
$$

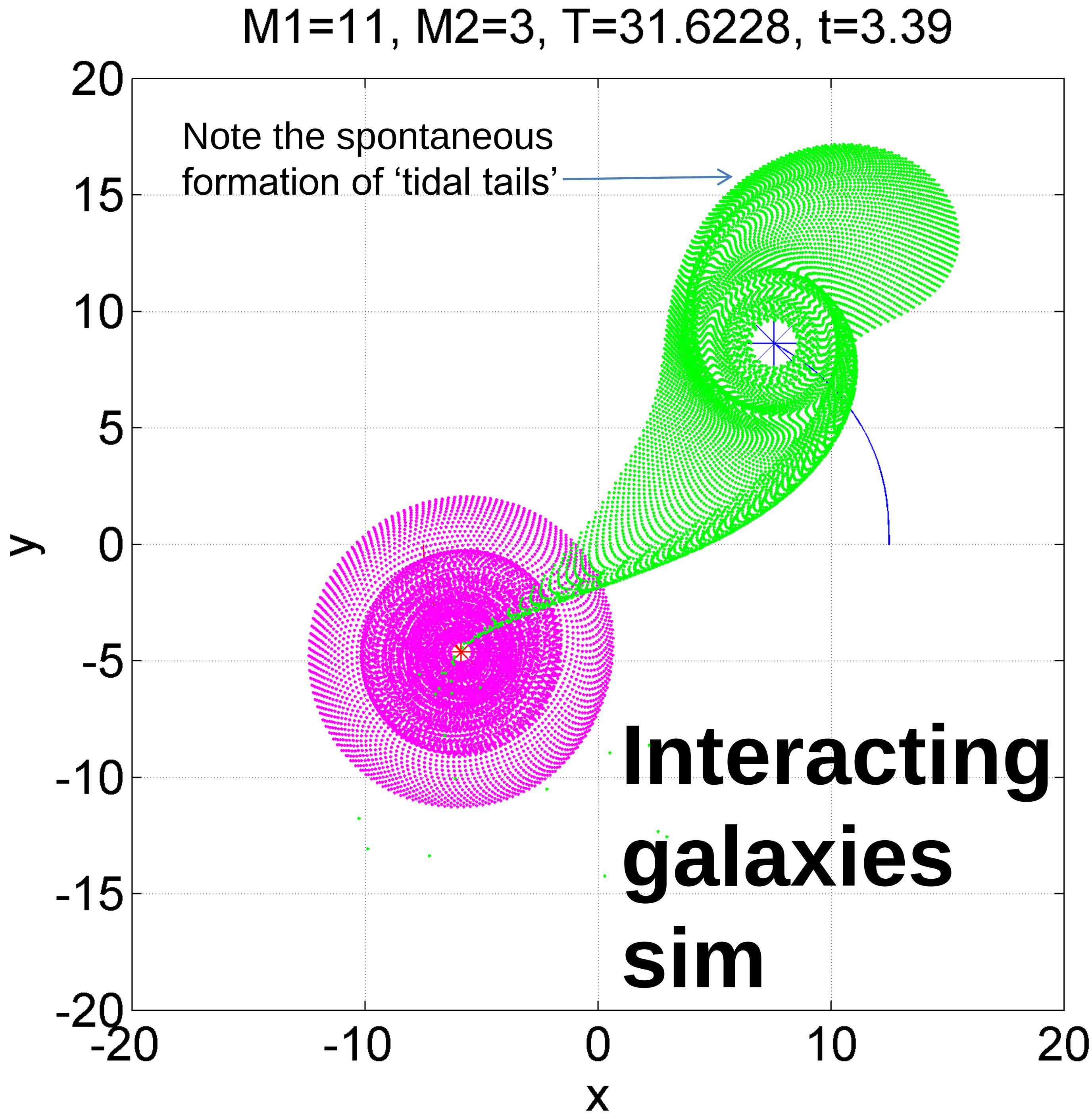
Initial velocities

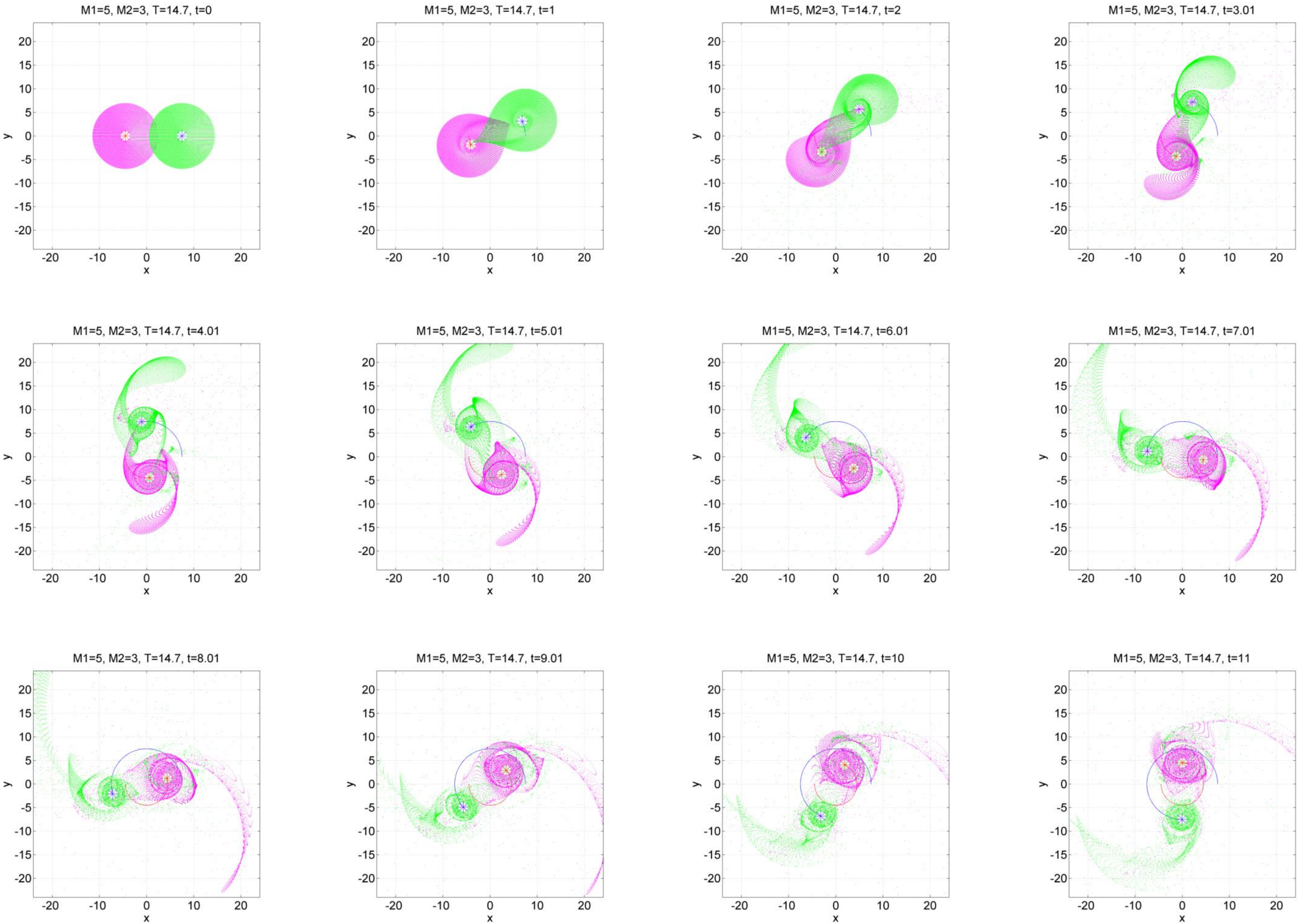
$$
\begin{aligned}
\dot{X}_1(0) &= 0 \\
\dot{Y}_1(0) &= \frac{2\pi k X_1(0)}{P} \\
\dot{X}_2(0) &= 0 \\
\dot{Y}_2(0) &= \frac{2\pi k X_2(0)}{P} \\
\dot{x}(0) &= -2\pi \sqrt{\frac{M_1}{a_p}} \sin\theta_0 \\
\dot{y}(0) &= -2\pi \sqrt{\frac{M_1}{a_p}} \cos\theta_0 + \dot{Y}_1(0)
\end{aligned}
$$

```
% Gravity simulation which begins with a single Jupiter-like planet
% orbiting a sun-like star, plus concentric circles of 'masslets' that act
% like an accretion disc or dust cloud around the star. The planet and
% masslets don't interact, and the star mass is assumed to be much larger
% than then mass of the planet, even after it has shed mass.
%
% After N planet rotations, the star loses fraction f of it's mass. The simulation
% uses Verlet integration to determine the subsequent dynamics for another
% M planet periods, before resetting.
```

Ring orbit sim



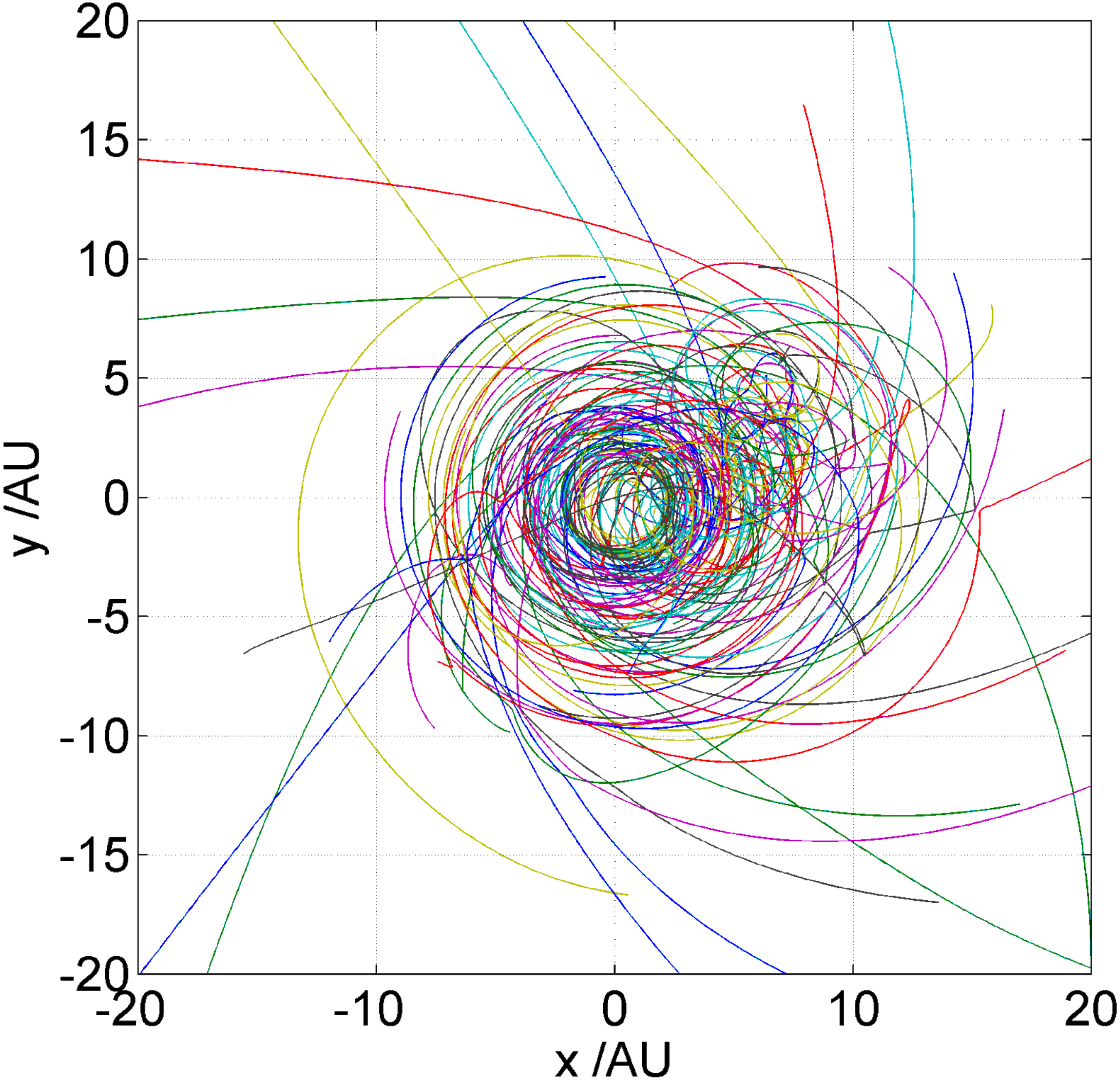

M1=11, M2=3, T=31.6228, t=3.39
Note the spontaneous
formation of 'tidal tails'
Interacting
galaxies
sim
y
x
20
15
10
5
0
-5
-10
-15
-20
-20
-10
0
10
20



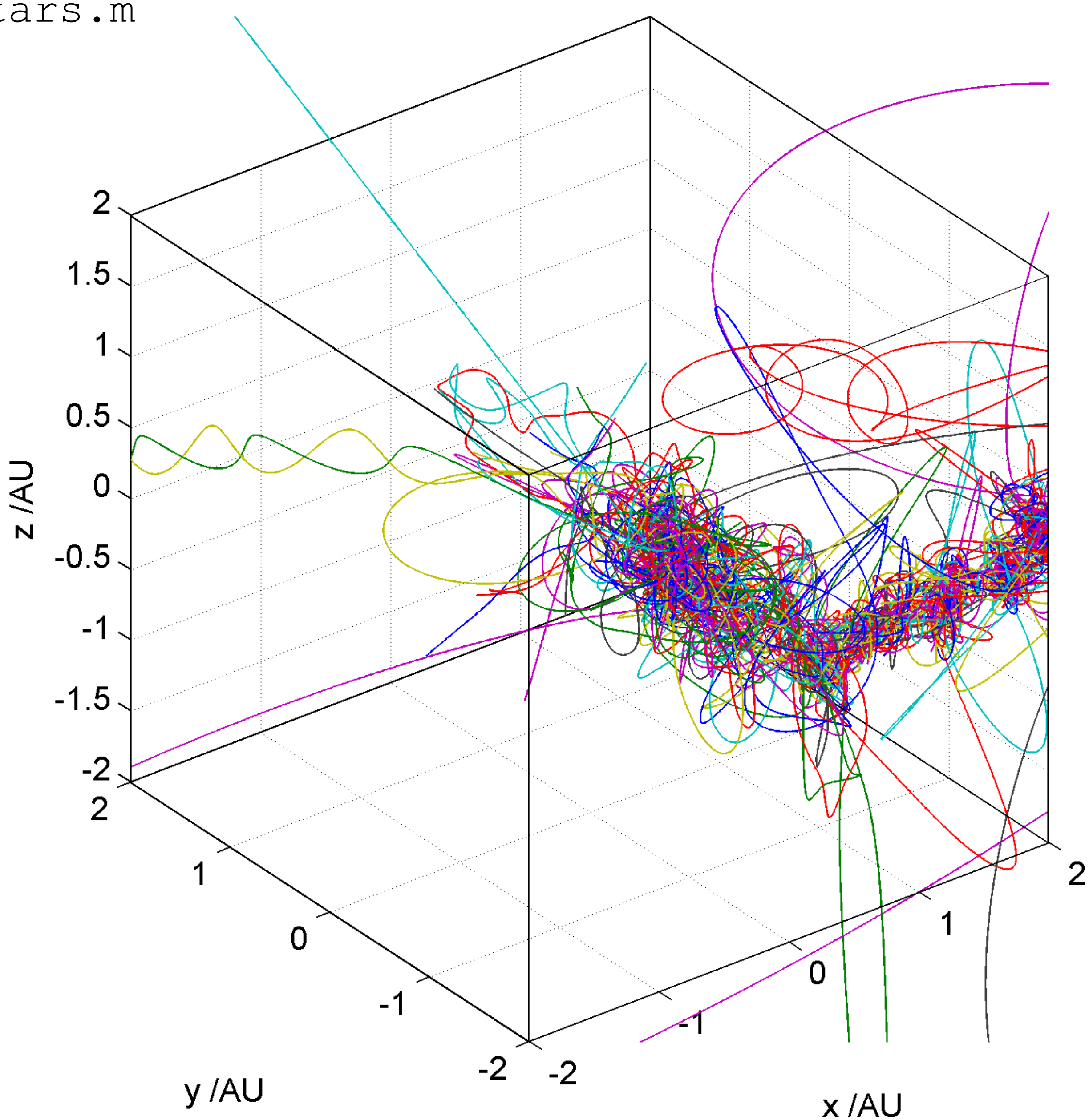
A possible explanation for common spiral galactic forms

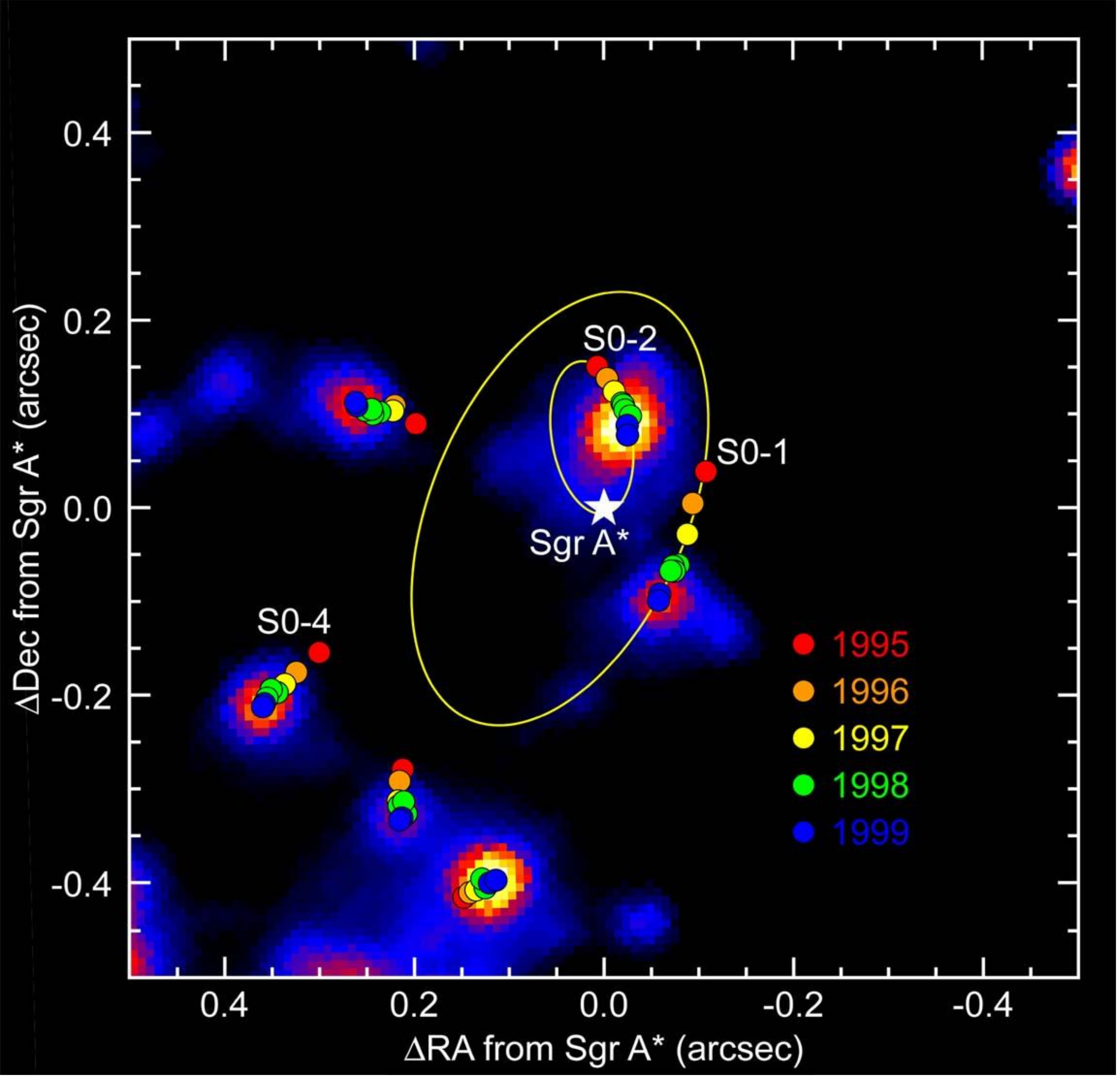

Messier 83 galaxy

Gravity simulator using Verlet method: 52 masses

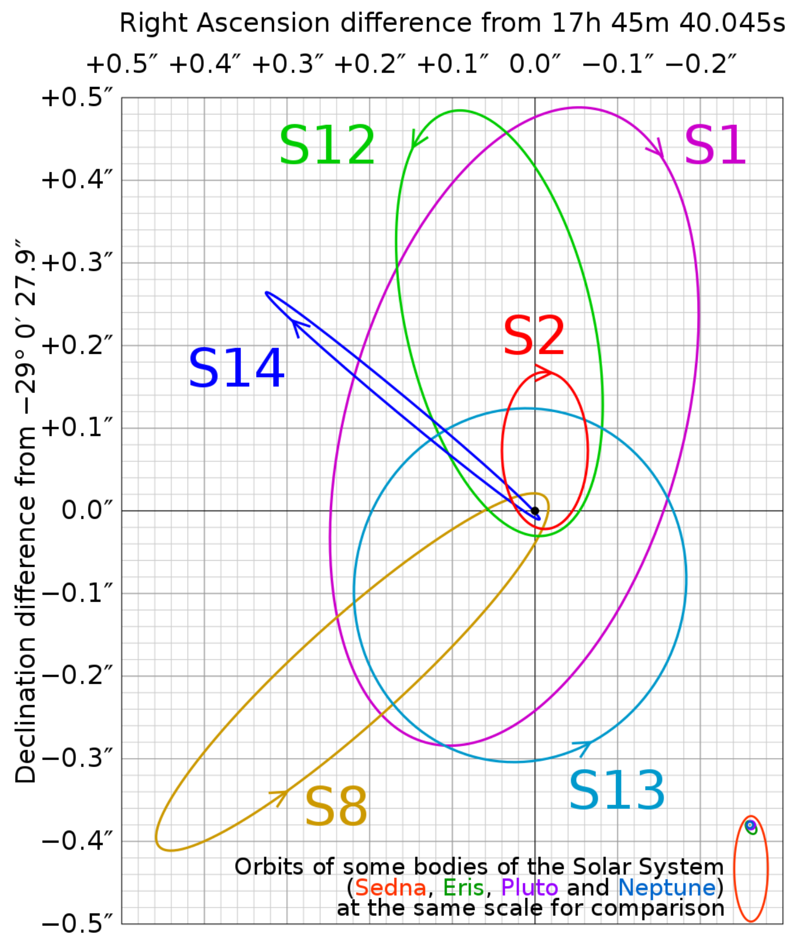


random_stars.m



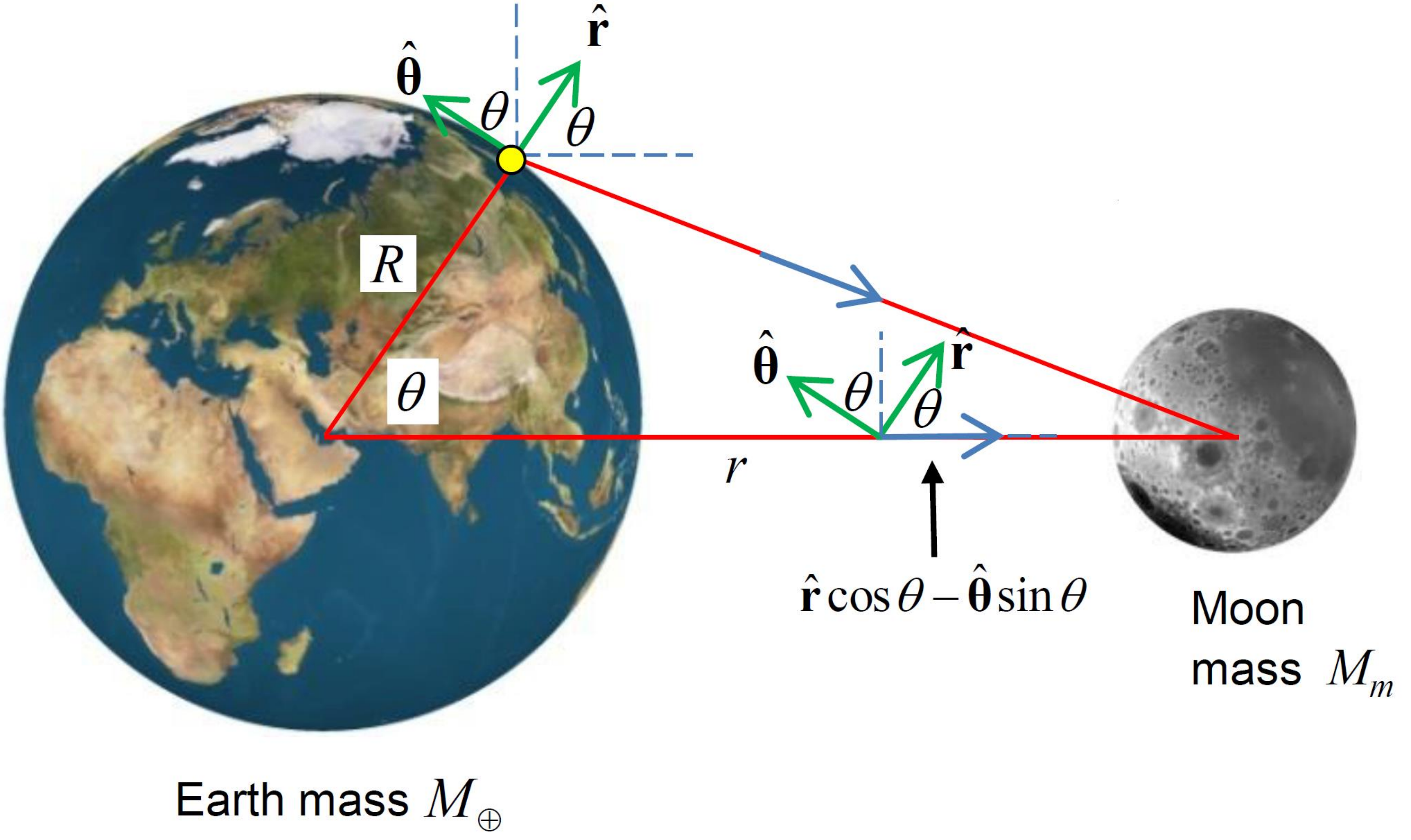


Although nearby star orbits look complex, the distances involved (and the relative mass of the black hole) mean you can model each as an elliptical orbit in a two-body system.

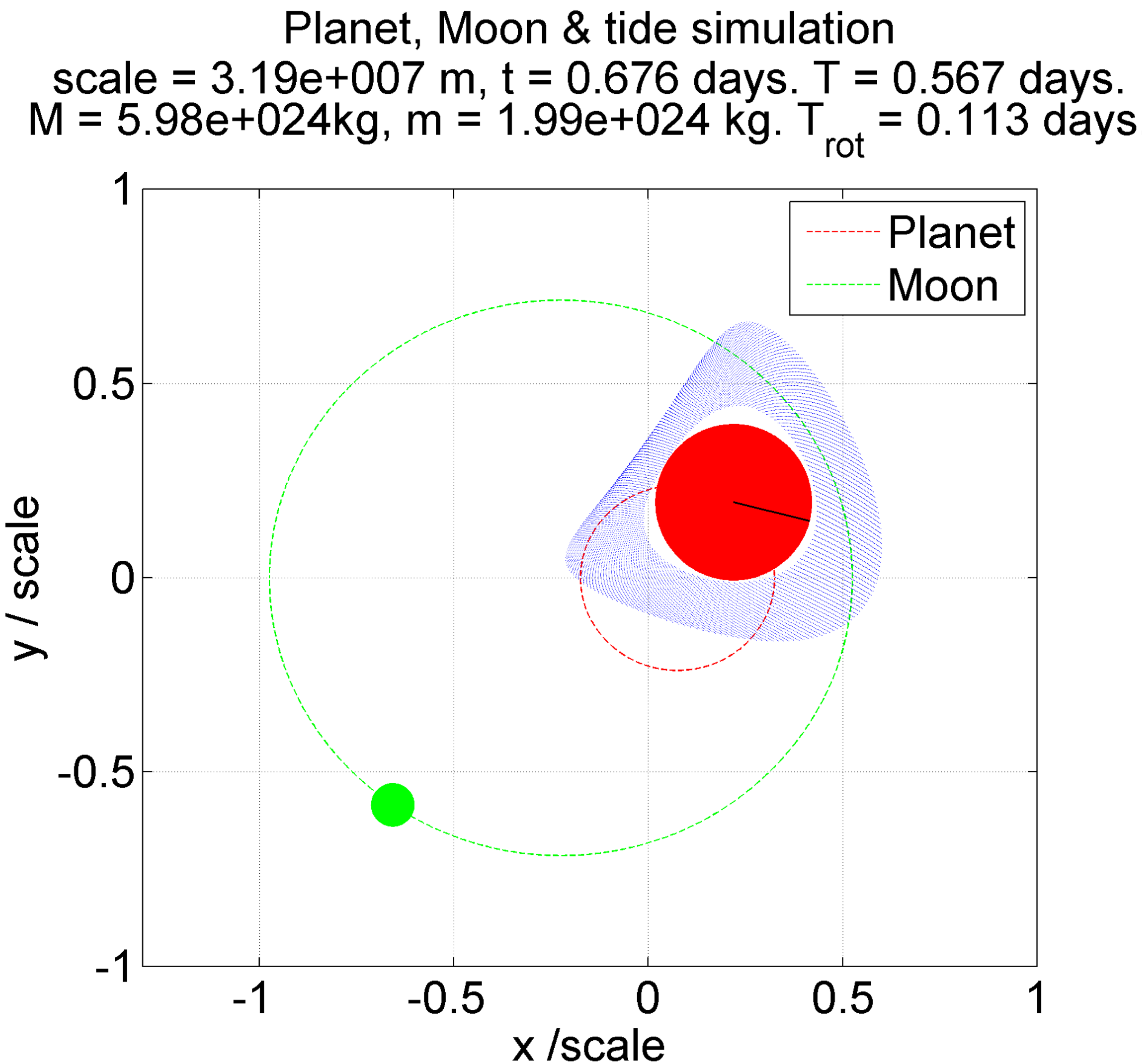


Sagittarius A* is a *supermassive black hole* in the centre of the Milky Way galaxy. It has a mass of about 4.2 **million** solar masses.

Tidal forces



$$\mathbf{g} \approx -\frac{GM_\oplus}{R}\hat{\mathbf{r}} + \hat{\mathbf{r}}\frac{GM_m}{r^2}\left(\cos\theta - \frac{R}{r}\right)\left(1 - \frac{3R}{r}\cos\theta\right) - \hat{\boldsymbol{\theta}}\frac{GM_m \sin\theta}{r^2}\left(1 - \frac{3R}{r}\cos\theta\right)$$


Planet, Moon & tide simulation
scale = 3.19e+007 m, t = 0.676 days. T = 0.567 days.
M = 5.98e+024kg, m = 1.99e+024 kg. T_{rot} = 0.113 days.
Planet
Moon
y / scale
x /scale
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

$$F_T = \frac{\frac{4}{3}G\pi R^3 \rho \delta m}{(r-R_m)^2} - \frac{G\frac{4}{3}\pi R^3 \rho \delta m}{r^2}$$

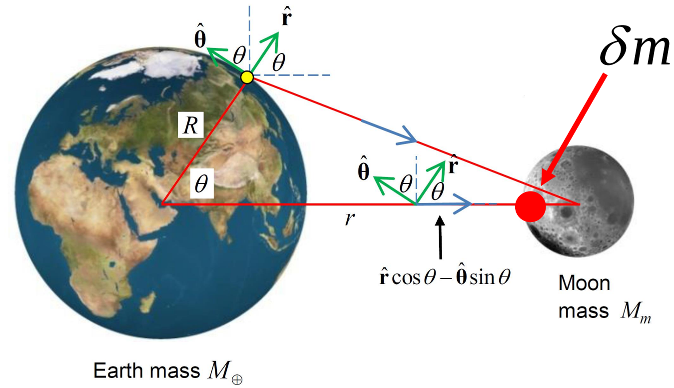
$$F_T = \tfrac{4}{3}G\pi R^3 \rho \delta m \left(\frac{1}{(r-R_m)^2} - \frac{1}{r^2} \right)$$

$$F_T = \frac{4}{3}\frac{G\pi R^3 \rho \delta m}{r^2} \left(\frac{1}{\left(1-\frac{R_m}{r}\right)^2} - 1 \right)$$

'Extra' gravity compared to centre of moon

$$F_T \approx \frac{4}{3}\frac{G\pi R^3 \rho \delta m}{r^2} \left(1 + \frac{2R_m}{r} - 1 \right)$$

$$F_T \approx \frac{8}{3}\frac{G\pi R^3 R_m \rho \delta m}{r^3}$$



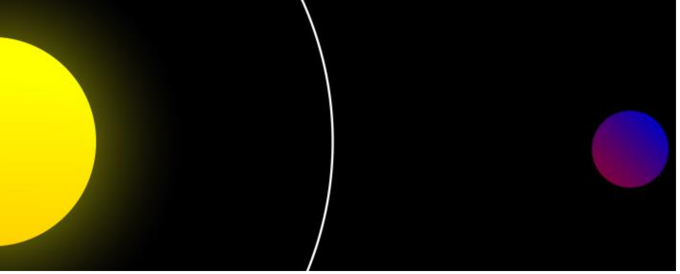
Édouard Roche
1820-1883

$$F_M = \frac{4}{3}\frac{G\pi R_m^3 \rho_m \delta m}{R_m^2} = \tfrac{4}{3}G\pi R_m \rho_m \delta m$$

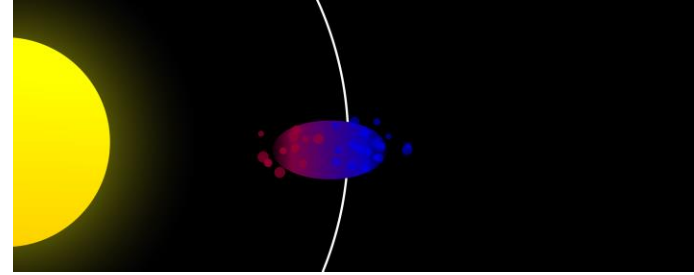
Therefore define the Roche limit to be when $F_T > F_M$:

$$\frac{8}{3}\frac{G\pi R^3 R_m \rho \delta m}{r^3} > \tfrac{4}{3}G\pi R_m \rho_m \delta m$$

PTO

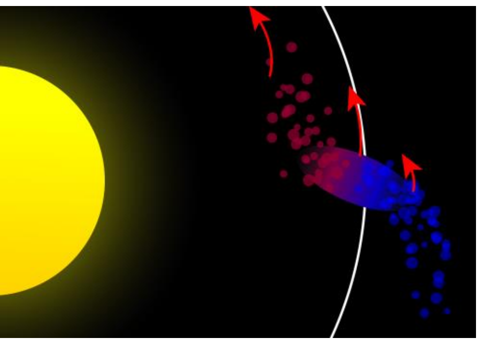


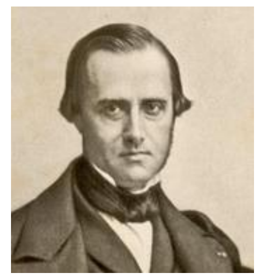
$$\therefore r \lesssim 1.26R\left(\frac{\rho}{\rho_m}\right)^{\frac{1}{3}}$$



This is not too far from Roche's actual calculation, which considered the moon to be a 'tidally locked fluid satellite', with the effect of distorting the moon from a spherical shape. In Roche's analysis

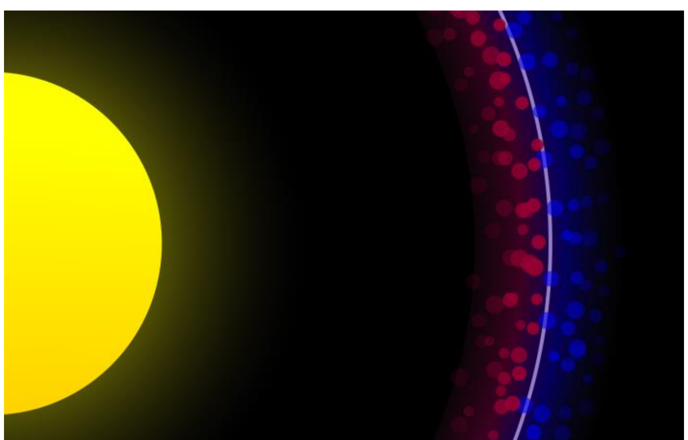
$$r \lesssim 2.44R\left(\frac{\rho}{\rho_m}\right)^{\frac{1}{3}}$$





Édouard Roche
1820-1883

This means the Roche limit for the Earth Moon system is:

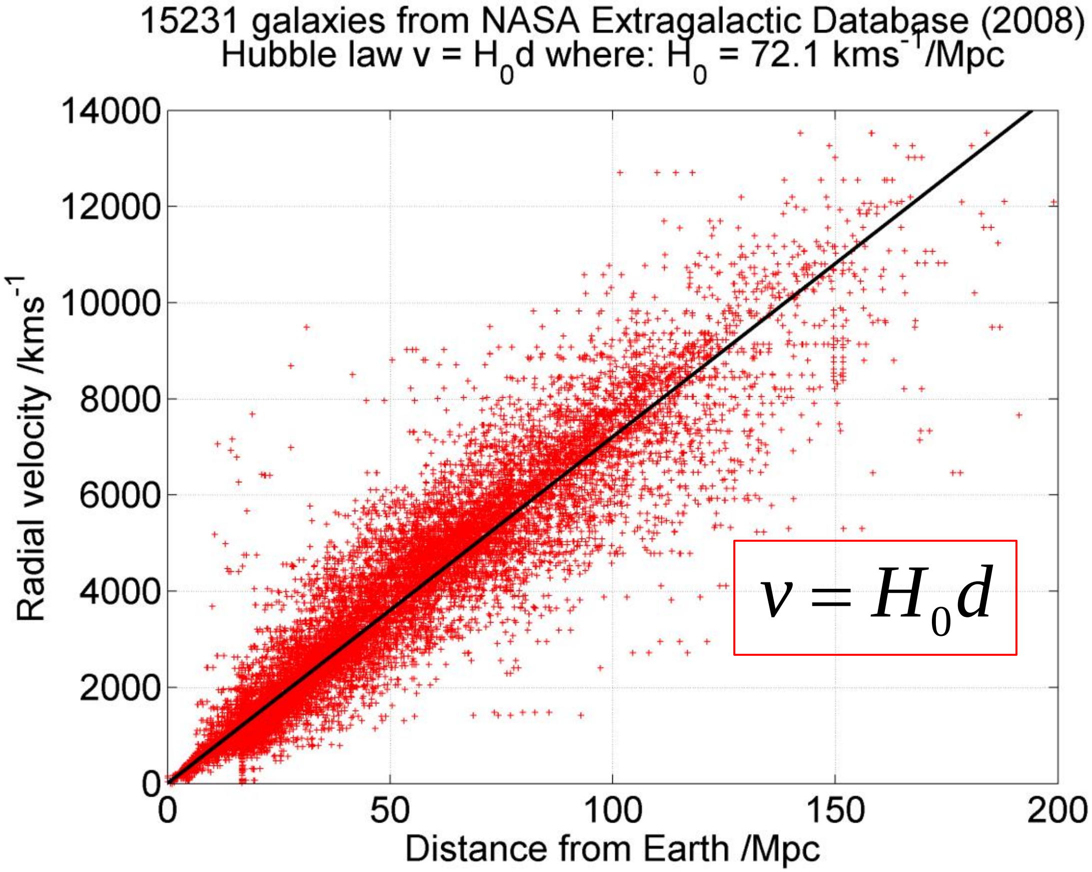


$$r \lesssim 2.44R_{\oplus}\left(\frac{5,513}{3,347}\right)^{\frac{1}{3}}$$

$$r \lesssim 2.44R_{\oplus} \times 1.181$$

$$r \lesssim 2.88R_{\oplus}$$

The age of an expanding Universe



1Mega-parsec (Mpc)
= 3.086 x10^{22} m

Edwin Hubble 1889-1953

$$v = \frac{d}{t} \qquad v = H_0 d$$

$$\therefore H_0 d = \frac{d}{t} \qquad \therefore t = \frac{1}{H_0}$$

$$t = \left(\frac{71.9 \times 10^3 \mathrm{ms}^{-1}}{3.086 \times 10^{22} \mathrm{m}} \right)^{-1} = \boxed{13.6 \text{ billion years}}$$

As of 2017, the best estimate for the age of the Universe is **13.799 +/- 0.021 billion years** using the **Lambda-CDM model** and observations of the **Cosmic Microwave Background** (CMB) radiation via **Planck** and **Wilkinson Microwave Anisotropy (WMAP)** probe (and others).