BPhO Computational Challenge

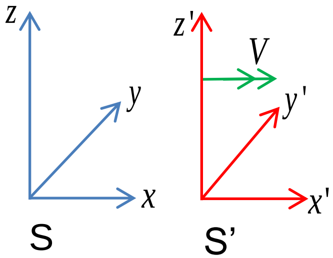
Special Relativity

Dr Andrew French.

December 2023.

BPhO British Physics Olympiad

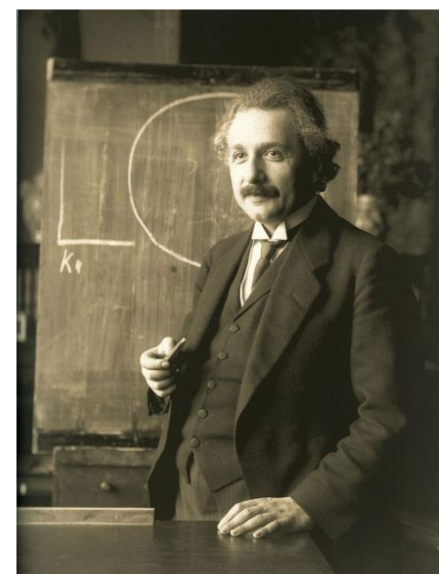
Licht ist keine Ente!



An Introduction to Special Relativity

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Licht ist kein Hamster!



Dr Andrew French. December 2016.

Based on Winchester College lectures of JJLD, notes of JAAB, Morin Classical Mechanics and Schwartz, J. & McGuinness, *Introducing Einstein*.

Some of the key Physicists in this story

Galileo Galilei
1564-1642

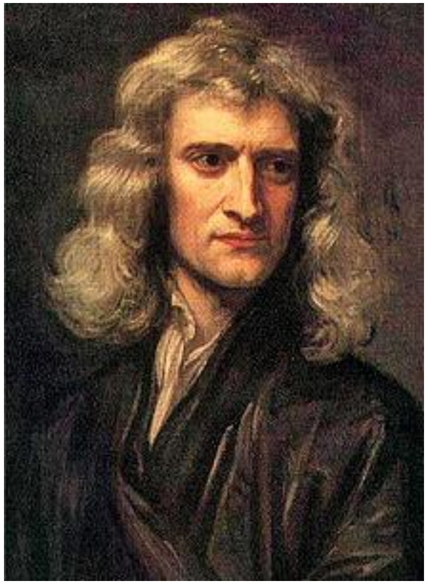
↑

Mechanics
Frames of reference
Relative motion
Scientific method
May have dropped some balls from the tower of Pisa
The *Inquisition* was not too keen on his rather sunny outlook though

Christiaan Huygens
1629-1695

↑

Theory of Waves



Isaac Newton
1642-1726

↑

Mechanics
Calculus
Optics
Thermodynamics
Gravity.....
Alchemy
Wasn't very nice to Hooke

Thomas Young
1773-1829

↑

Young's slits (diffraction)
Young's modulus (elasticity)
Egyptology
Sadly died young as well

Waves

Augustin-Jean Fresnel
1788-1827

Humphry Davy
1778-1829

Michael Faraday
1791-1867

Electromagnetism
Chemistry

Léon Foucault
1819-1868

Measured the speed of light

Hippolyte Fizeau
1819-1896

Electromagnetism

Hermann von Helmholtz
1821-1894

$$\oint_{\text{closed surface}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{s} = \int_{\text{volume}} \rho_{\text{free}} \, \mathrm{d}\tau$$

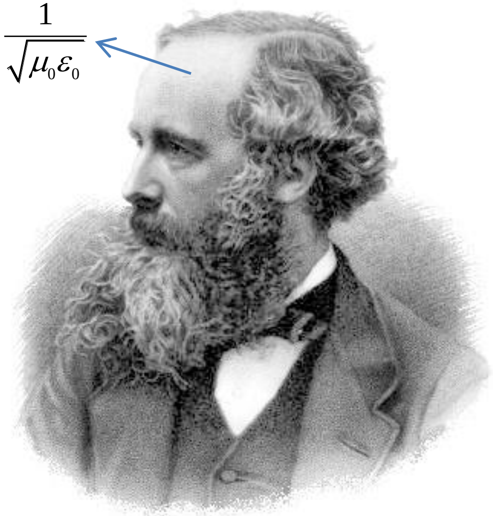
$$\oint_{\text{closed surface}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s} = 0$$

$$\oint_{\text{loop}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

$$\oint_{\text{loop}} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = I_{\text{free}} + \int_{\text{surface}} \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}\boldsymbol{s}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

I prefer to talk in maths



James Clerk Maxwell
1831-1879

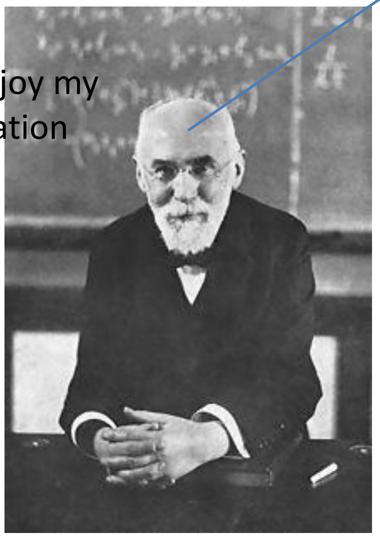
Edward Morley
1838-1923

There is no aether!

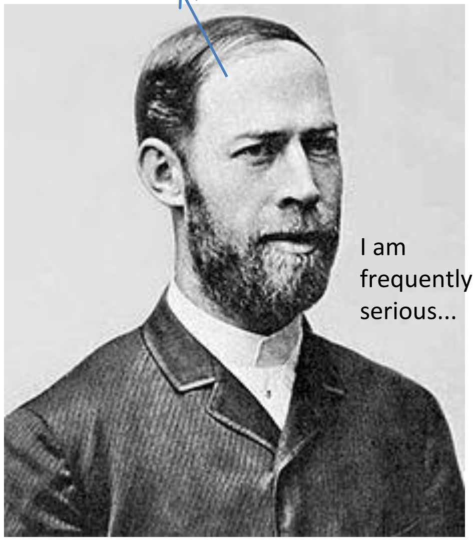
Albert Michelson
1852-1931

Maxwell is correct!

You will enjoy my Transformation



Hendrik Lorentz
1853-1928



I am frequently serious...

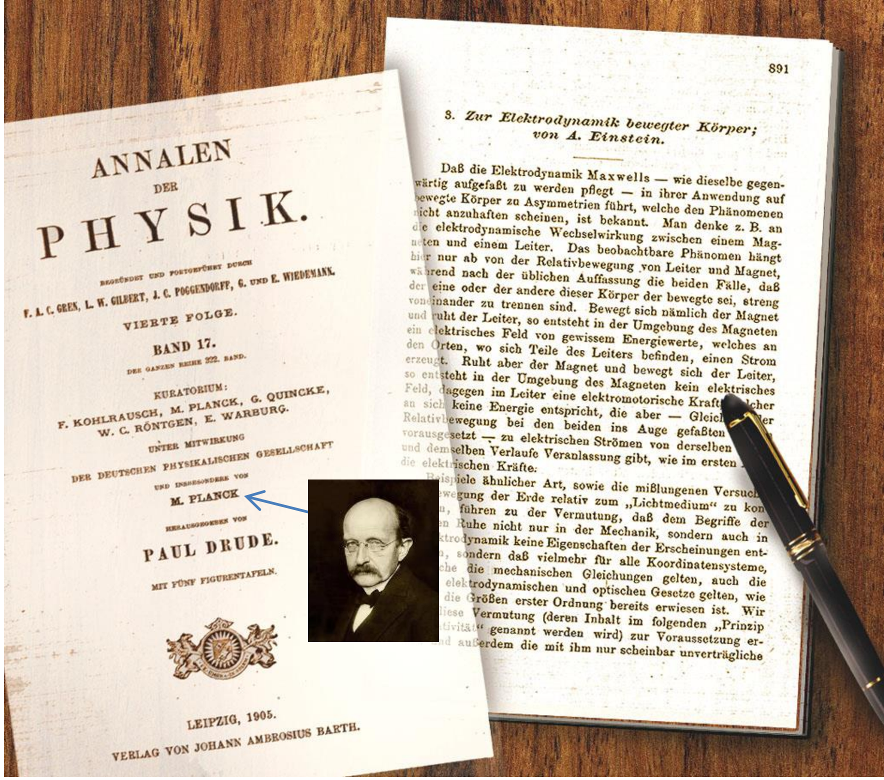
Heinrich Hertz
1857-1894

Developed wireless technology (Radio, Radar ...)

Guglielmo Marconi
(1874-1937)

1905 was a *very* good year for me

Albert Einstein
1879-1955



ANNALEN
DER
PHYSIK.

F. A. C. GREN, L. W. GILBERT, J. C. POGGENDORFF, G. UND E. WIEDEMANN.

VIERTE FOLGE.

BAND 17.

KURATORIUM:
F. KOHLRAUSCH, M. PLANCK, G. QUINCKE,
W. C. RÖNTGEN, E. WARBURG.

UNTER MITWIRKUNG
DER DEUTSCHEN PHYSIKALISCHEN GESELLSCHAFT
M. PLANCK

PAUL DRUDE.

MIT FÜNF FIGURENTAFELN.

LEIPZIG, 1905.
VERLAG VON JOHANN AMBROSIUS BARTH.

891

3. *Zur Elektrodynamik bewegter Körper;*
von A. Einstein.

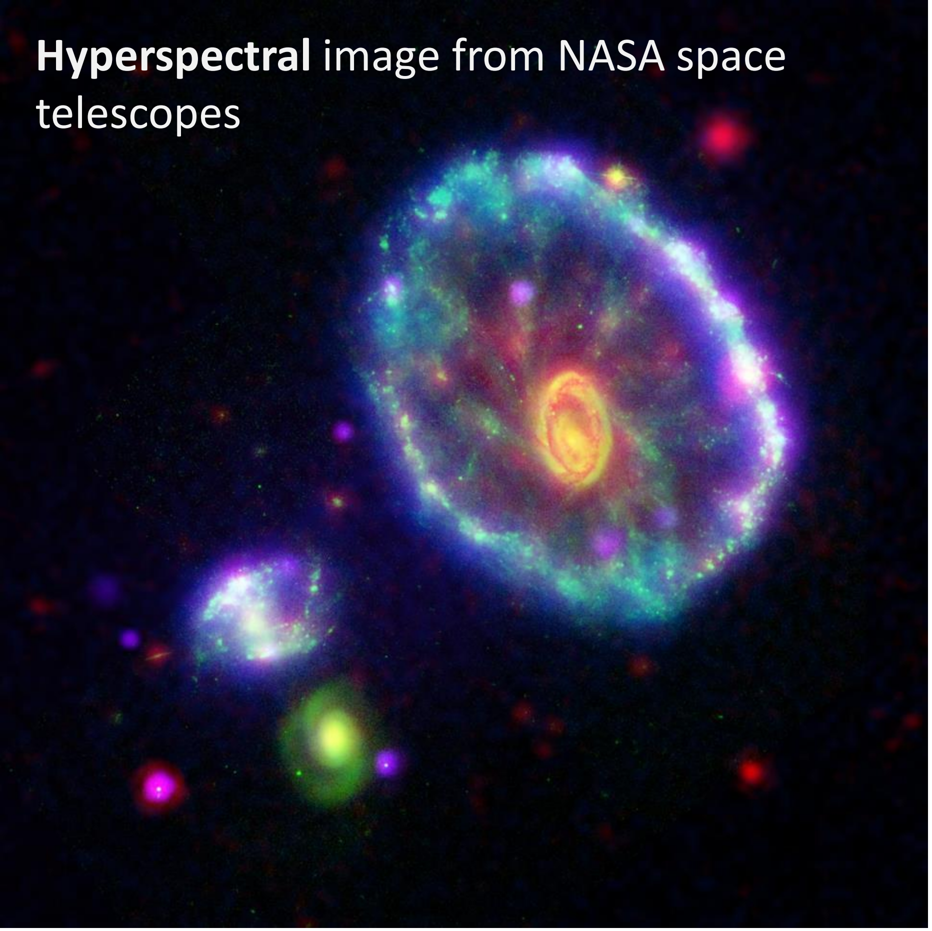
Daß die Elektrodynamik Maxwells — wie dieselbe gegenwärtig aufgefaßt zu werden pflegt — in ihrer Anwendung auf bewegte Körper zu Asymmetrien führt, welche den Phänomenen nicht anzuhaften scheinen, ist bekannt. Man denke z. B. an die elektrodynamische Wechselwirkung zwischen einem Magneten und einem Leiter. Das beobachtbare Phänomen hängt hier nur ab von der Relativbewegung von Leiter und Magnet, während nach der üblichen Auffassung die beiden Fälle, daß der eine oder der andere dieser Körper der bewegte sei, streng voneinander zu trennen sind. Bewegt sich nämlich der Magnet und ruht der Leiter, so entsteht in der Umgebung des Magneten ein elektrisches Feld von gewissem Energiewerte, welches an den Orten, wo sich Teile des Leiters befinden, einen Strom erzeugt. Ruht aber der Magnet und bewegt sich der Leiter, so entsteht in der Umgebung des Magneten kein elektrisches Feld, dagegen im Leiter eine elektromotorische Kraft, welcher an sich keine Energie entspricht, die aber — Gleichheit der Relativbewegung bei den beiden ins Auge gefaßten Fällen vorausgesetzt — zu elektrischen Strömen von derselben Größe und demselben Verlaufe Veranlassung gibt, wie im ersten Falle die elektrischen Kräfte.

Beispiele ähnlicher Art, sowie die mißlungenen Versuche, eine Bewegung der Erde relativ zum „Lichtmedium" zu konstatieren, führen zu der Vermutung, daß dem Begriffe der absoluten Ruhe nicht nur in der Mechanik, sondern auch in der Elektrodynamik keine Eigenschaften der Erscheinungen entsprechen, sondern daß vielmehr für alle Koordinatensysteme, für welche die mechanischen Gleichungen gelten, auch die gleichen elektrodynamischen und optischen Gesetze gelten, wie dies für die Größen erster Ordnung bereits erwiesen ist. Wir wollen diese Vermutung (deren Inhalt im folgenden „Prinzip der Relativität" genannt werden wird) zur Voraussetzung erheben und außerdem die mit ihm nur scheinbar unverträgliche

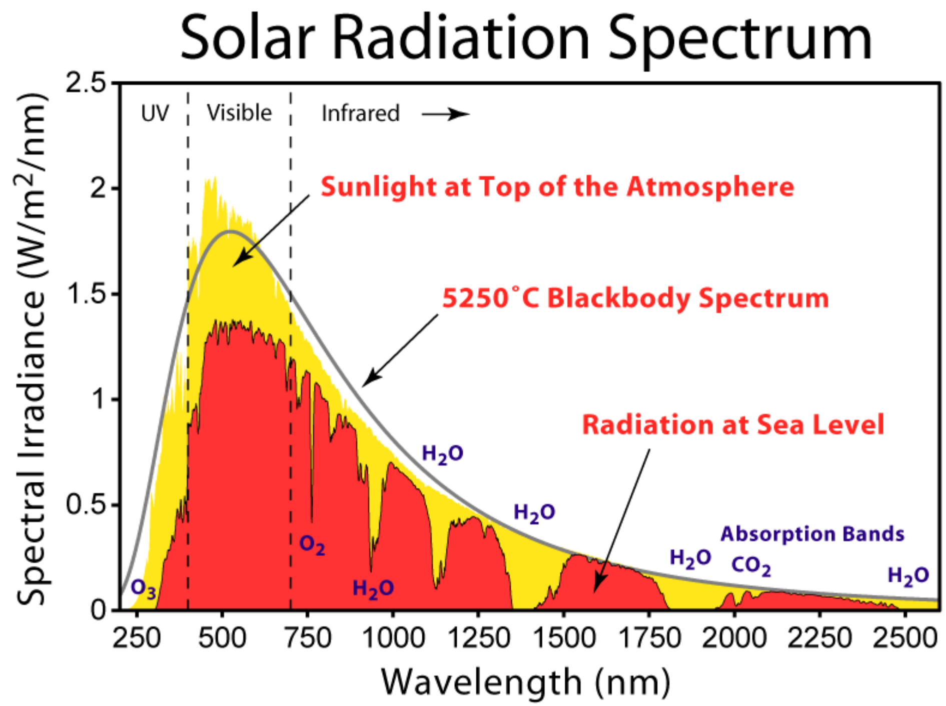
I have a talent for making the complicated make sense and explaining the inexplicable.

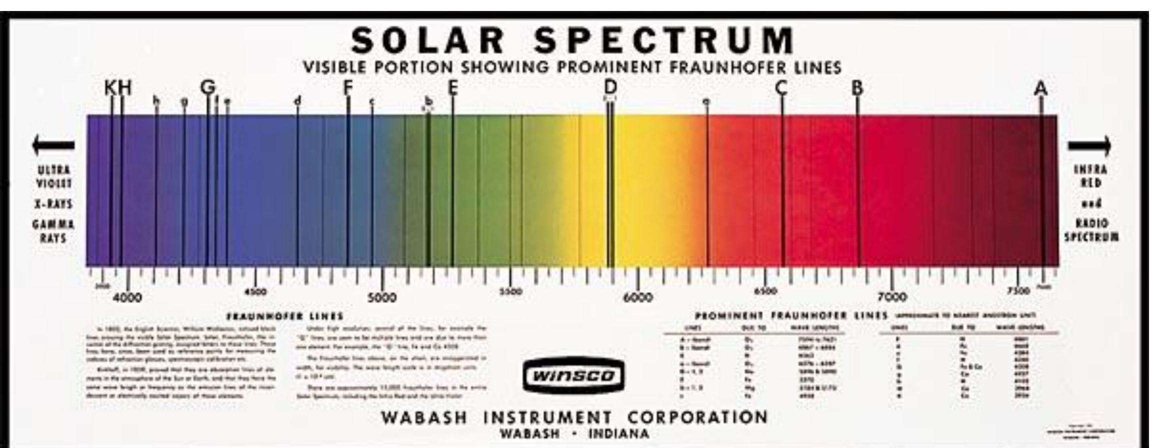
I can also pick locks, paint and play the bongos

Richard Feynman
1918-1988



Hyperspectral image from NASA space telescopes

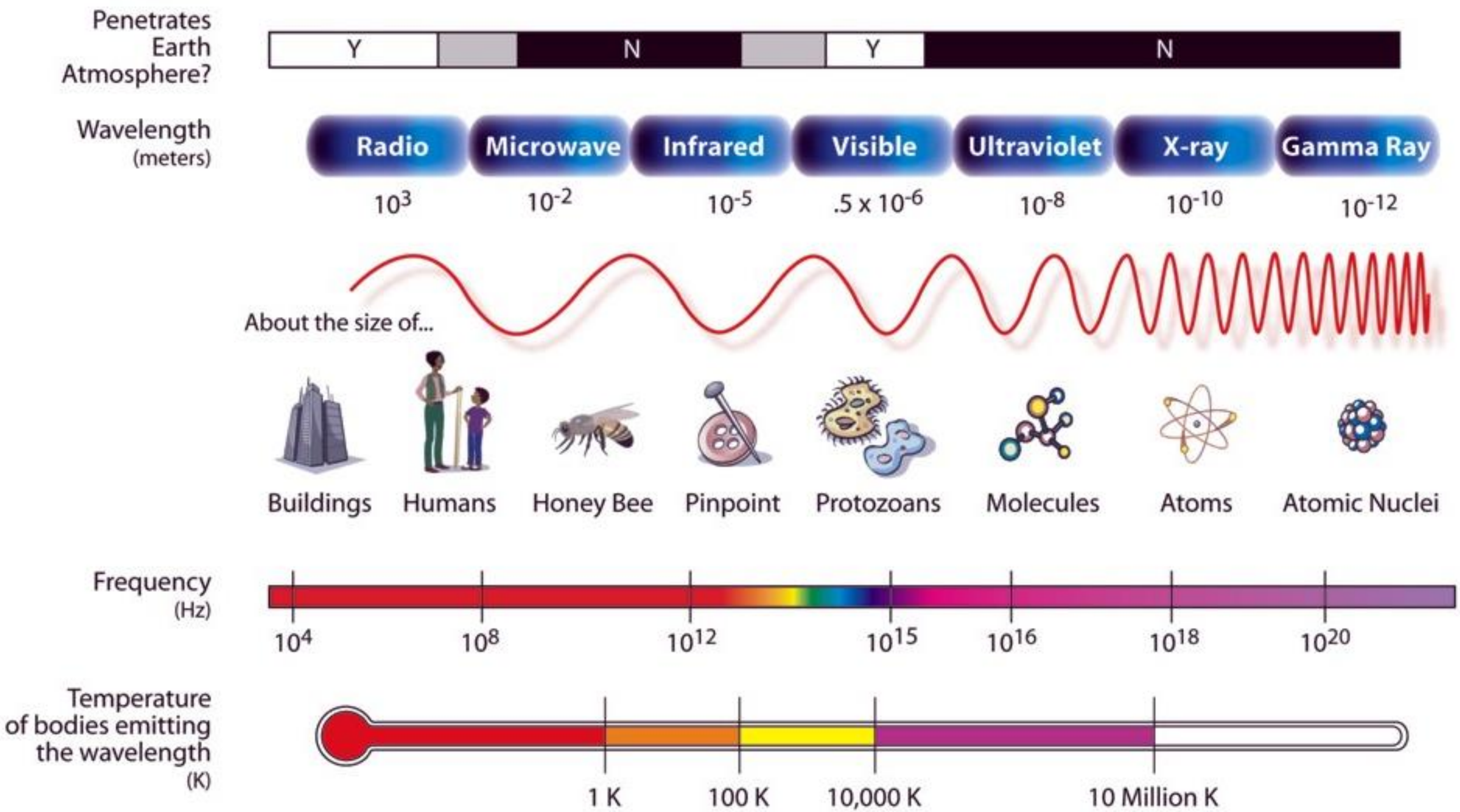




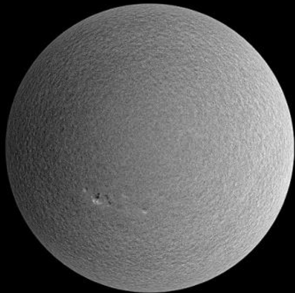
Light – perhaps the best understood of all physical phenomena

It is the *only* means for us to understand the Cosmos well beyond the inner solar system

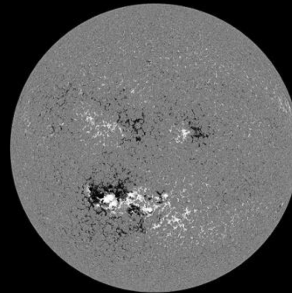
THE ELECTROMAGNETIC SPECTRUM



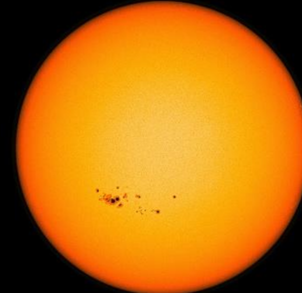
Convert wavelength into frequency using $c = f\lambda$ $\quad$ $c = 2.998 \times 10^{8}\,\mathrm{ms}^{-1}$



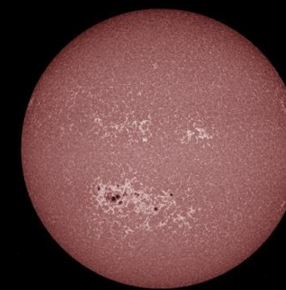
HMI Dopplergram
Surface movement
Photosphere



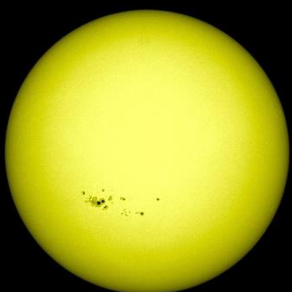
HMI Magnetogram
Magnetic field polarity
Photosphere



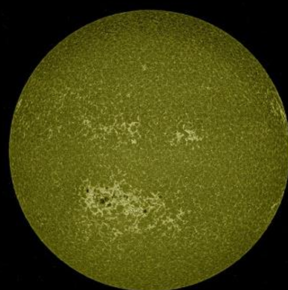
HMI Continuum
Matches visible light
Photosphere



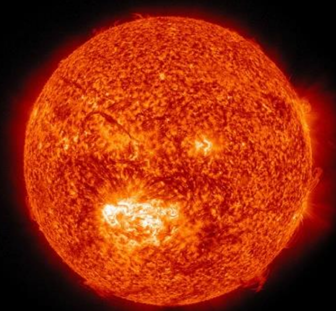
AIA 1700 Å
4500 Kelvin
Photosphere



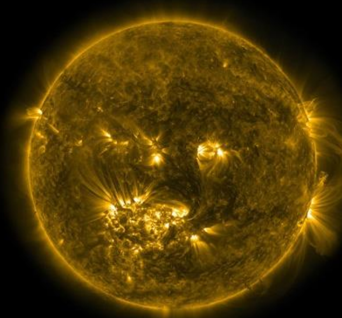
AIA 4500 Å
6000 Kelvin
Photosphere



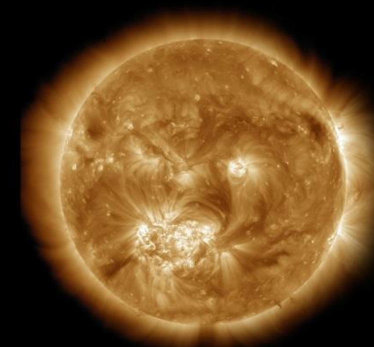
AIA 1600 Å
10,000 Kelvin
Upper photosphere/
Transition region



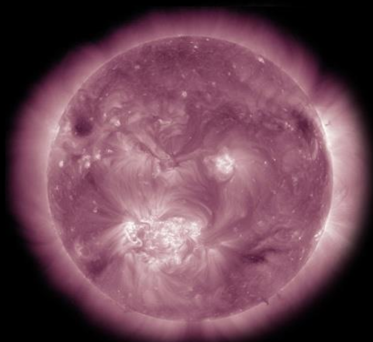
AIA 304 Å
50,000 Kelvin
Transition region/
Chromosphere



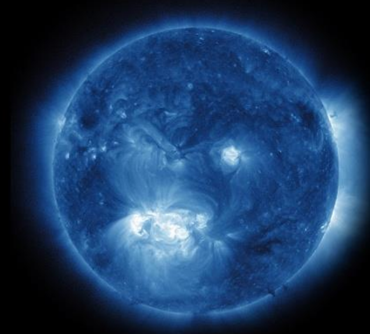
AIA 171 Å
600,000 Kelvin
Upper transition
Region/quiet corona



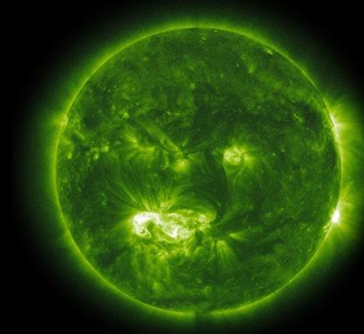
AIA 193 Å
1 million Kelvin
Corona/flare plasma



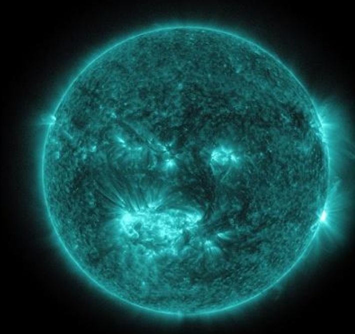
AIA 211 Å
2 million Kelvin
Active regions



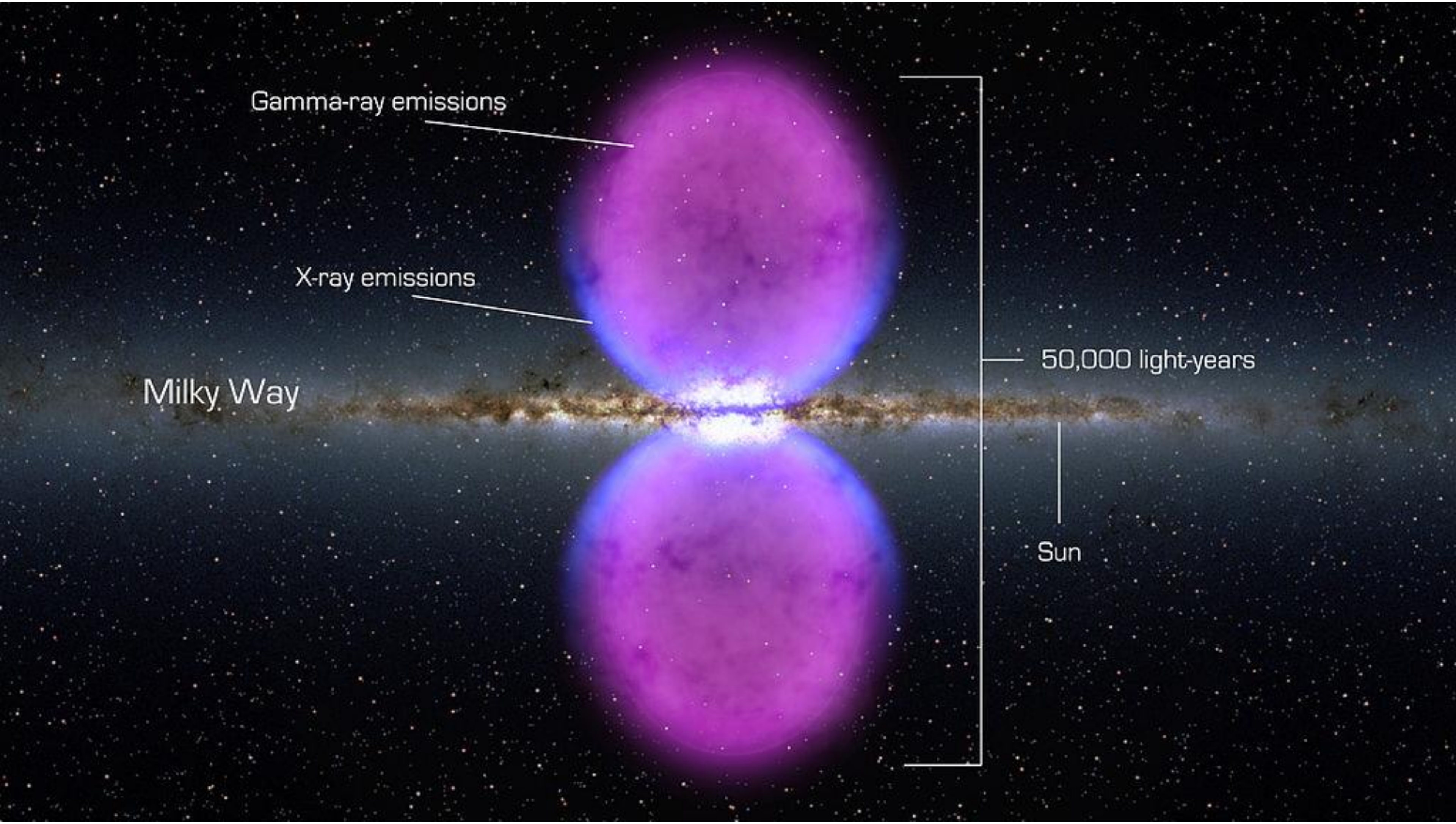
AIA 335 Å
2.5 million Kelvin
Active regions



AIA 094 Å
6 million Kelvin
Flaring regions



AIA 131 Å
10 million Kelvin
Flaring regions


Gamma-ray emissions
X-ray emissions
Milky Way
50,000 light-years
Sun

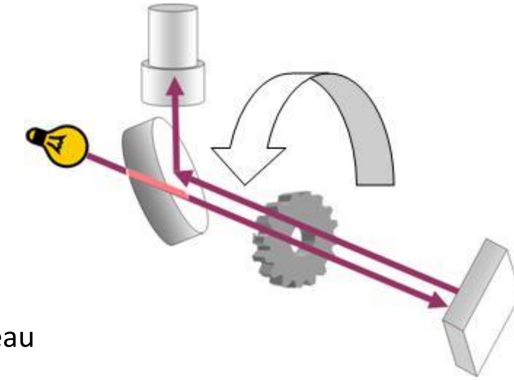
Christiaan Huygens
1629-1695

Thomas Young
1773-1829

Augustin-Jean Fresnel
1788-1827

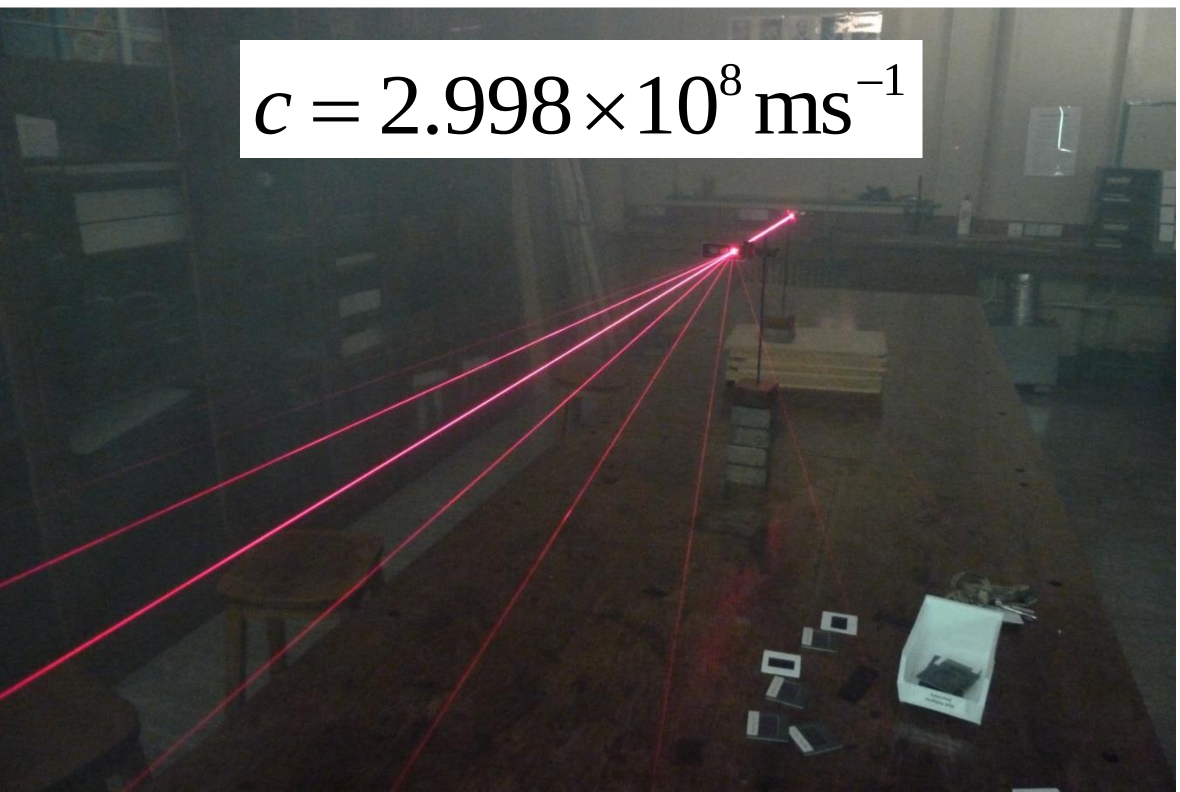
Léon Foucault
1819-1868

Hippolyte Fizeau
1819-1896

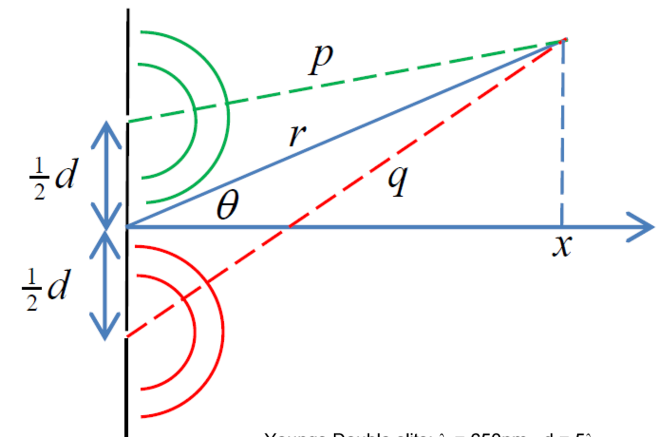


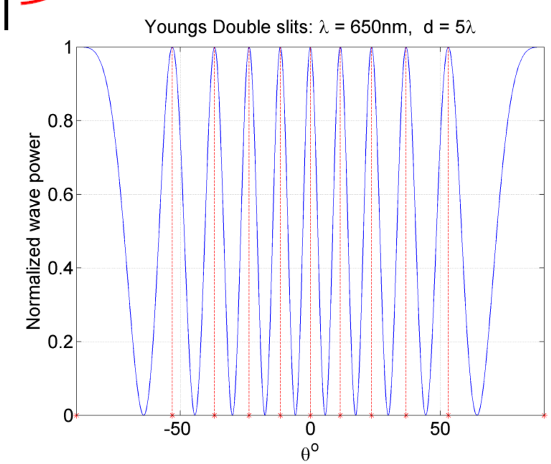
Light is a wave – it **reflects**, **refracts** and **diffracts.**
Its speed of propagation is:

$$c = 2.998 \times 10^{8}\,\mathrm{ms}^{-1}$$



Two infinitesimally thin slits
'Young's double slits'

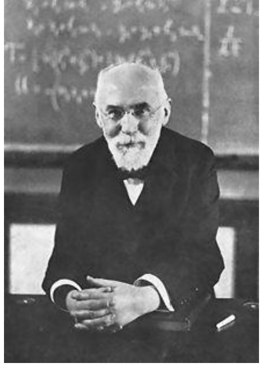




Michael Faraday
1791-1867

Hermann von
Helmholtz
1821-1894

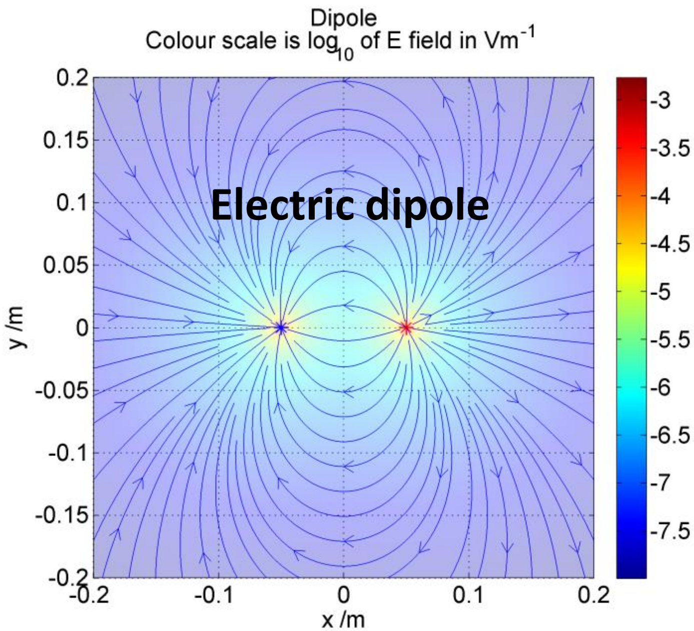
James Clerk
Maxwell
1831-1879

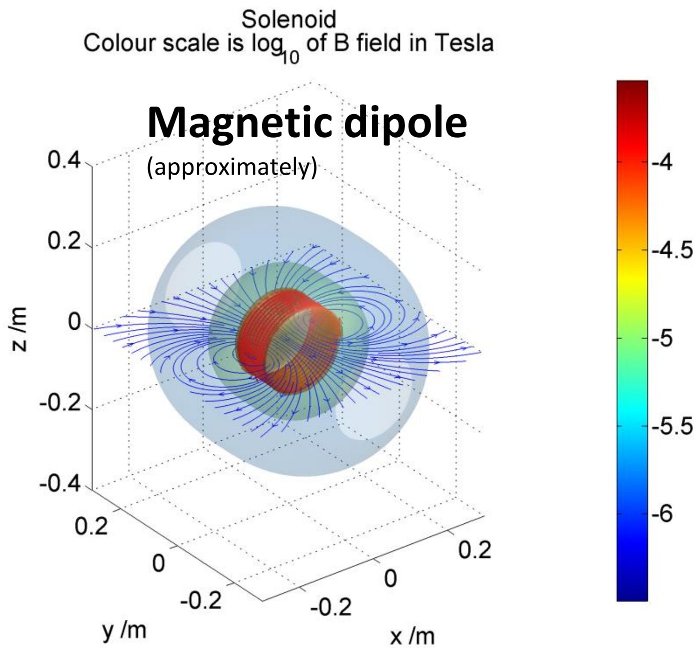


Hendrik Lorentz
1853-1928

Heinrich Hertz
1857-1894

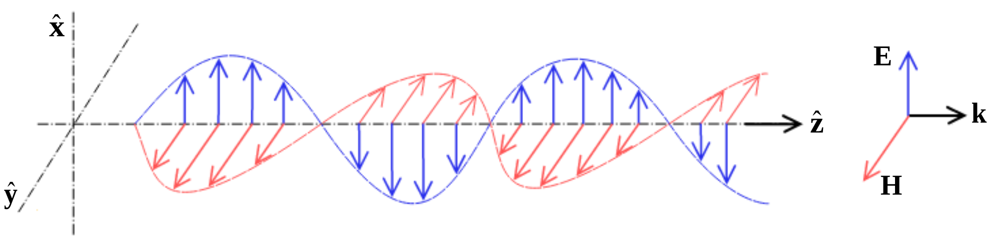
Guglielmo Marconi
1874-1937

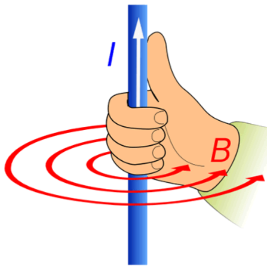




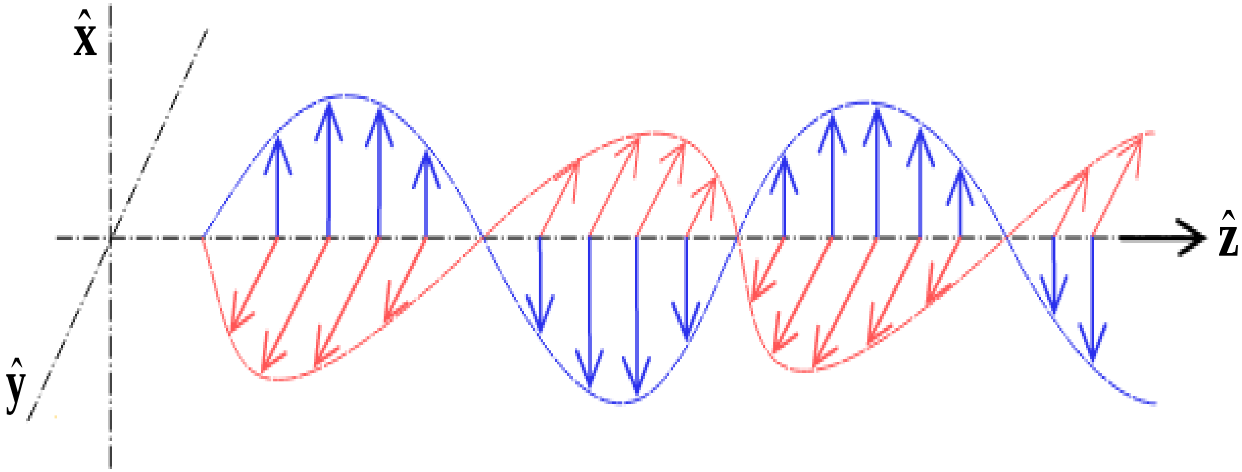
Electricity and
Magnetism
are *linked*.

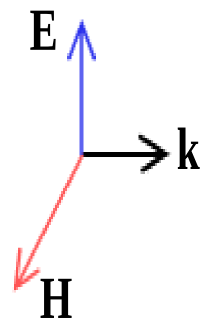
Electromagnetism





Hans Christian
Ørsted
1777-1851





James Clerk Maxwell 1831-1879

Maxwell's Equations predict **Electromagnetic Waves**, which consist of electric and magnetic **fields** at right angles to each other, *both transverse* to the direction of wave propagation.

Intriguingly, these waves always propagate through a *vacuum* at speed:

$$B = \frac{\mu_0 I}{2\pi r}$$

Magnetic field at a radius r from a wire carrying current I

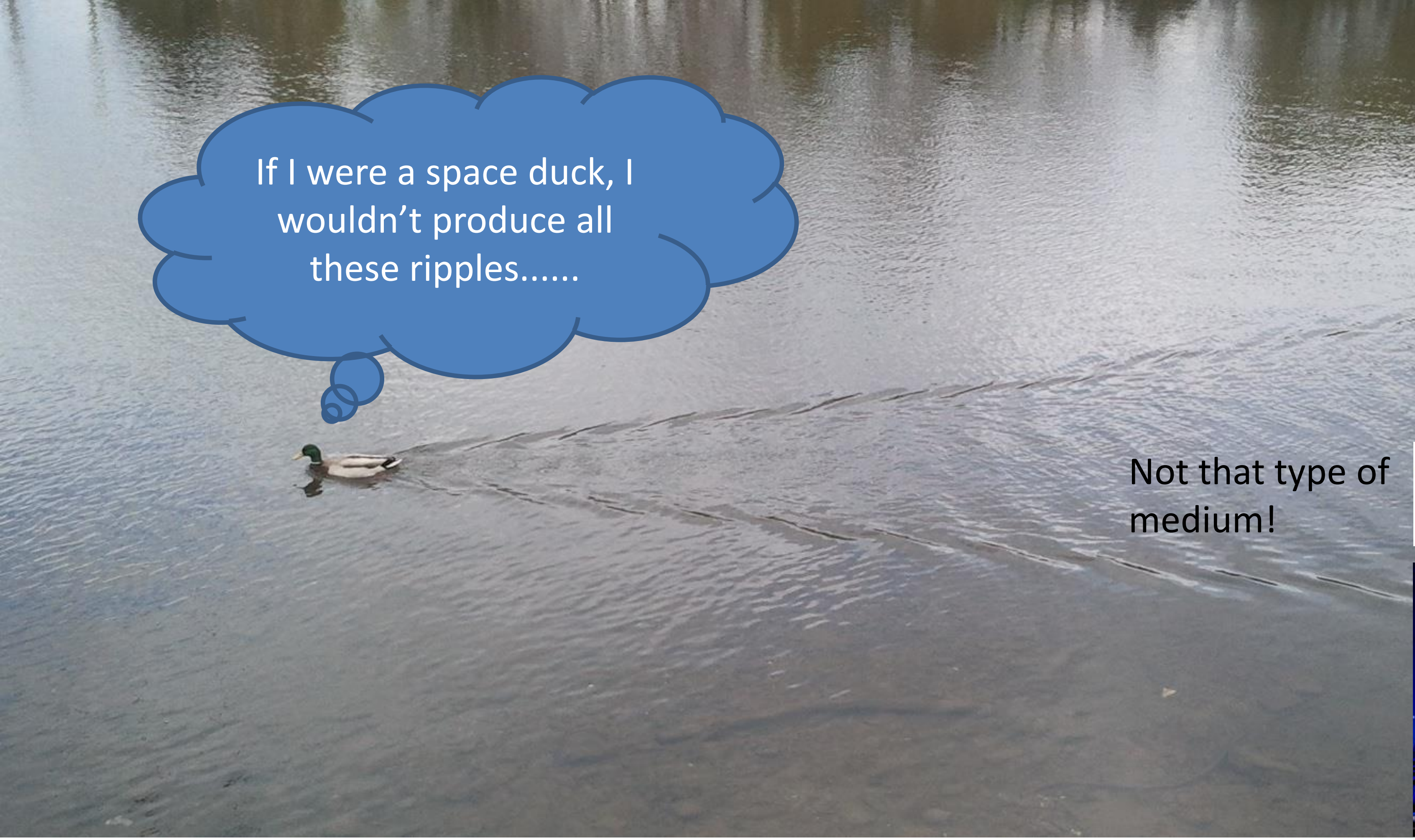
$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.998 \times 10^8 \,\mathrm{ms}^{-1}$$

$$F_E = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{d^2}$$

Force on two electrons of charge e separated by distance d

These are **fundamental constants**. So the wave speed is **independent** of the relative speed of EM wave source and receiver!

Hang on a minute....



Not that type of medium!

Sound waves, **surface waves** etc are the vibration of a ***medium*** (e.g. air or water molecules). They have a **characteristic speed** depending on **density**, and **stiffness** of molecular bonds.

So for an **electromagnetic wave** passing from the Sun to Earth, what medium is vibrating?

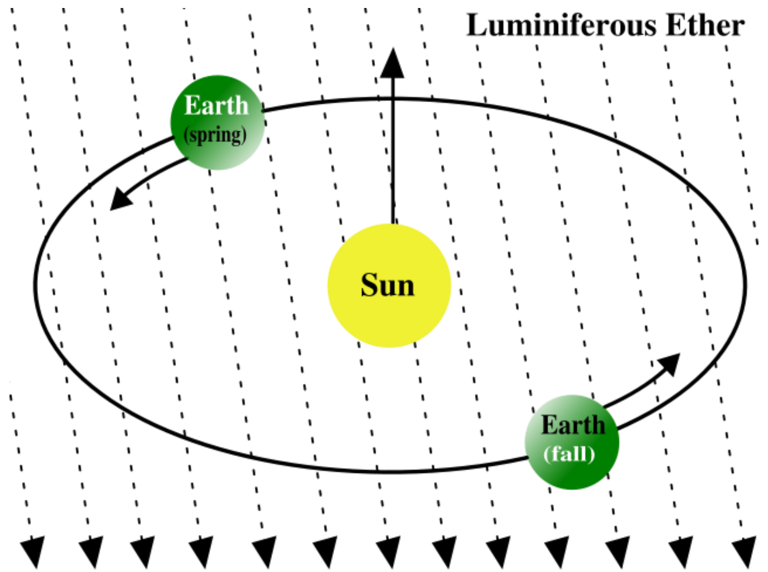
The **Luminiferous Aether** of course!

WRONG

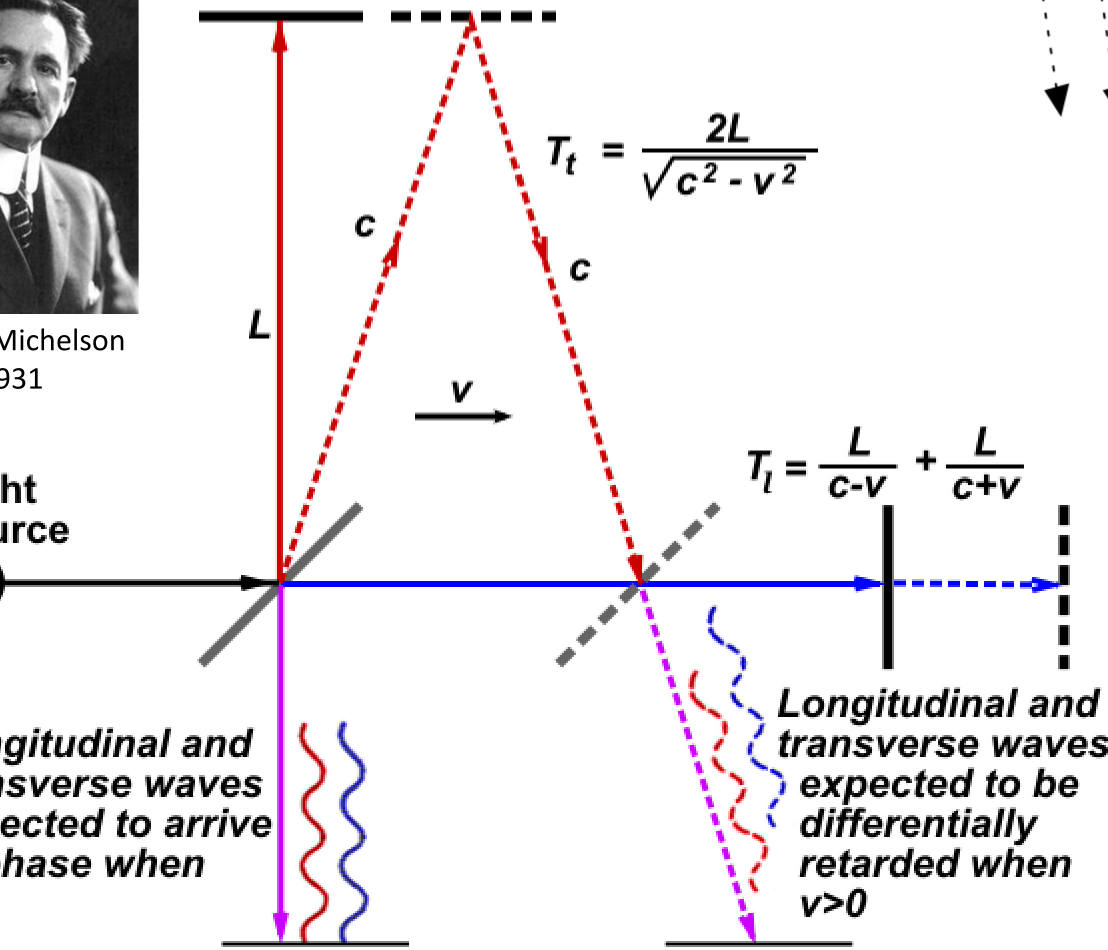
Edward Morley
1838-1923

Albert Michelson
1852-1931

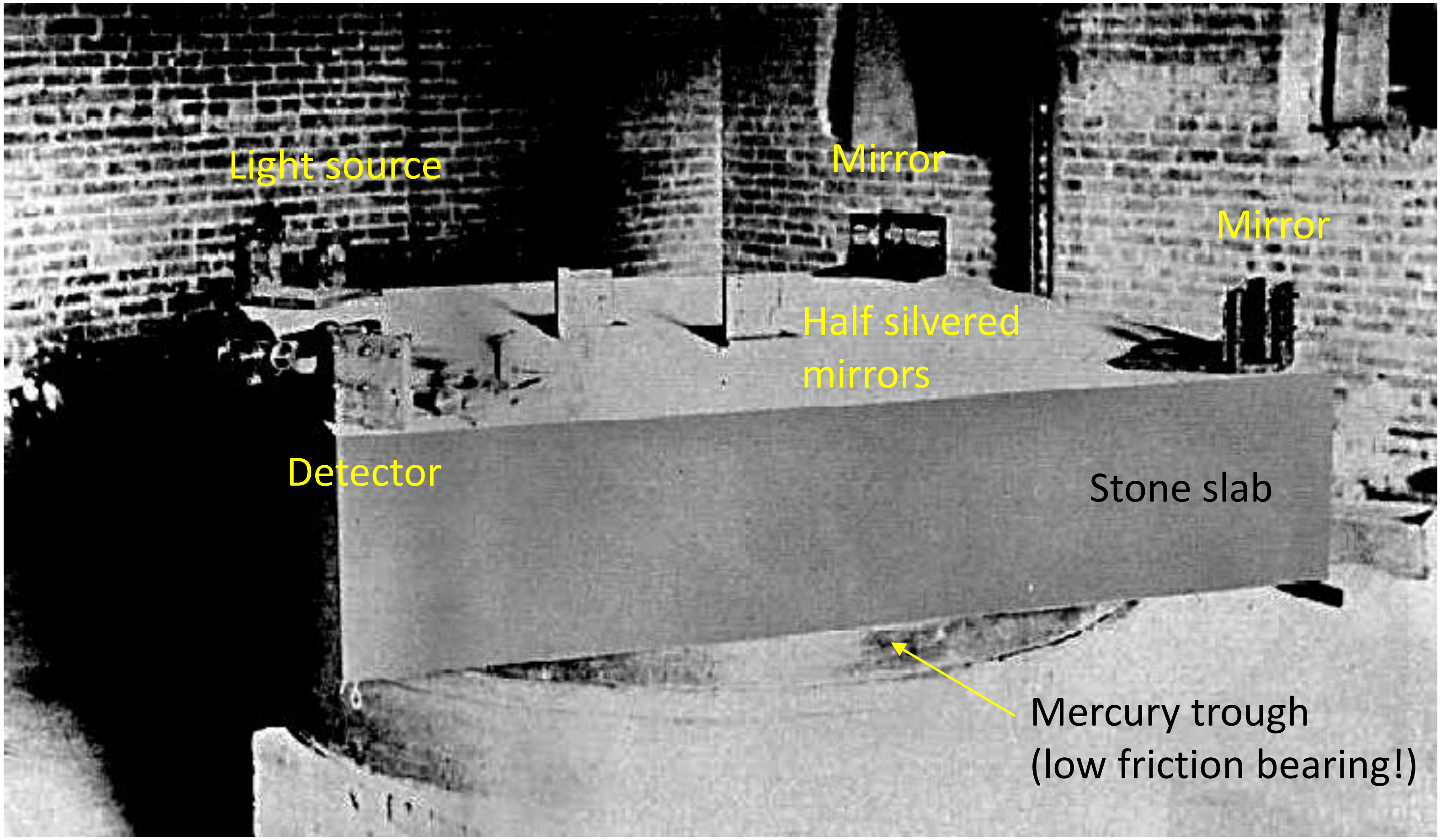
If there is an aether, we should be able to measure the effect of moving towards it.....



Earth orbital speed is about 30 km/s



If there is a relative motion between the Earth and the aether, we should expect to see a **difference in phase** between the longitudinal and transverse beams in the **Michelson-Morley interferometer**



Michelson and Morley's *interferometer*, mounted on a stone slab that floats in an annular trough of mercury.

Conducted over the spring and summer of 1887 at what is now Case Western University, Cleveland Ohio, USA.

https://en.wikipedia.org/wiki/Michelson%E2%80%93Morley_experiment

So did Michelson & Morely observe any phase shift due to relative motion between the Earth and the Aether?

Conclusion: **There is no aether.** Light can propagate in vacuum. It itself moves

Light is not like a duck

Back to Maxwell's discovery ...

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.998 \times 10^8 \,\mathrm{ms}^{-1}$$

These are **fundamental constants**. So the wave speed is **independent** of the relative speed of EM wave source, and receiver!

James Clerk Maxwell 1831-1879

Let's assume Maxwell is correct... c is always the same

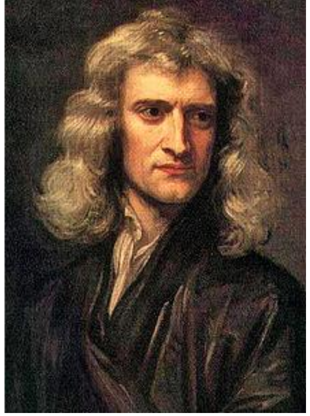
What happens to my image in a shaving mirror if I were to travel at the speed of light? Would it disappear? Could I go faster than the speed of light? What would happen then?

The mirror might crack if you bumped into it at the speed of light. Now that would be unlucky

Albert Einstein 1879-1955

This is a *Gedanken* (thought) experiment

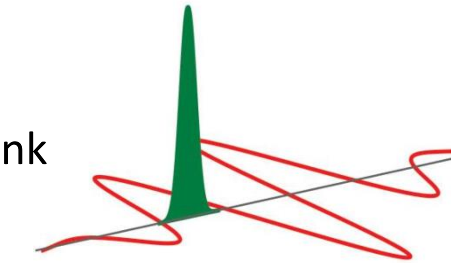
Galileo Galilei
1564-1642



Isaac Newton
1642-1726

Let's use the **mechanics** of Galileo and Newton to work out what will happen.

To keep things simple we'll think about a short pulse of light.

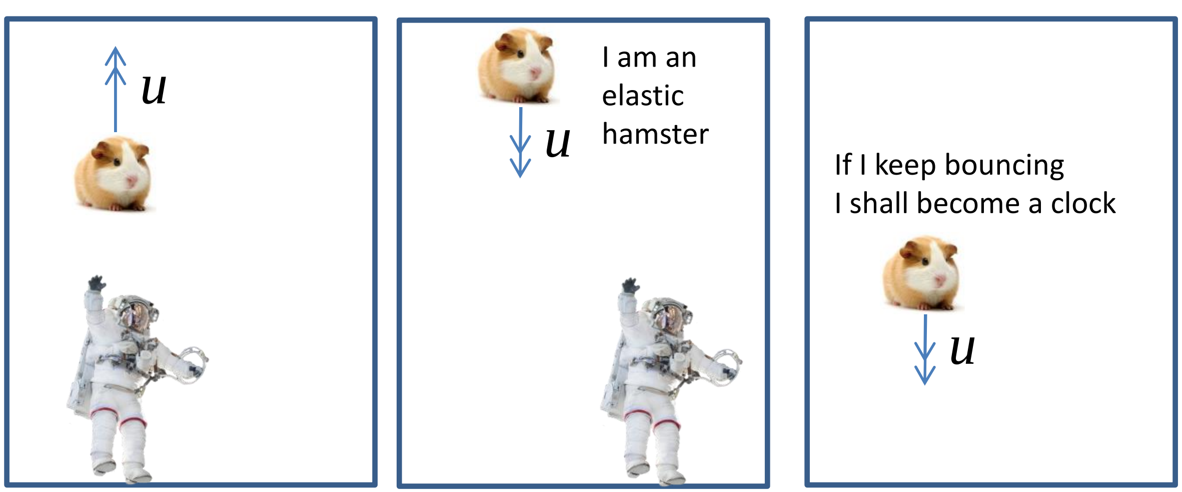


Is the dynamics of the light pulse *just like that of a projectile?*

Is light like a hamster?

Oh no!

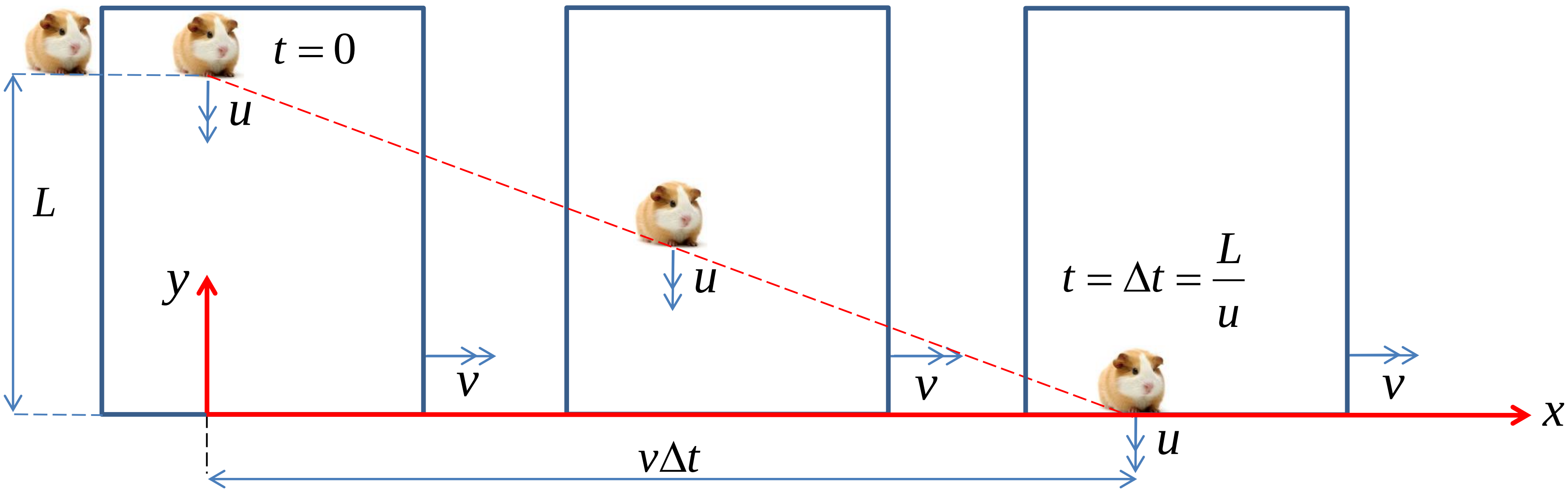
To keep things even simpler, let's consider hurling a hamster vertically upwards in a box in **zero gravity**



To fall (or rise) distance L the hamster takes time

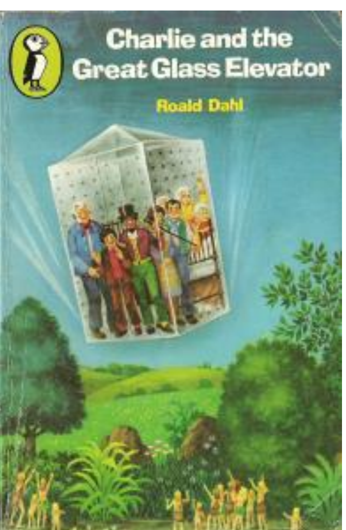
$$\Delta t = \frac{L}{u}$$

Let's have a spacewalk



Mr Wonka supplies a glass elevator for the experiment.
Prof. Feynman observes it translating at speed *v* to his right.
From *his perspective*, the hamster moves along the red dotted line path.

Richard Feynman
1918-1988



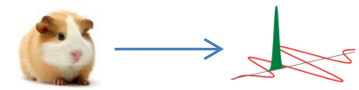
The total distance travelled according to Prof. Feynman is:

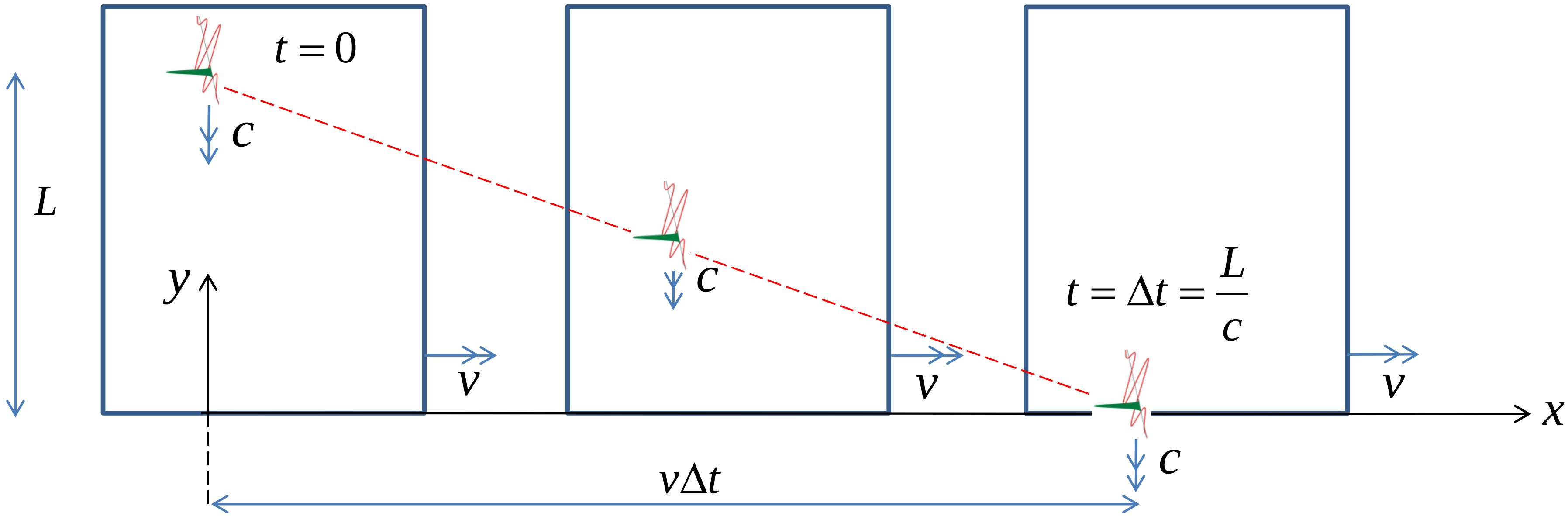
$$d = \sqrt{L^2 + v^2 \Delta t^2} = \sqrt{L^2 + \frac{v^2 L^2}{u^2}} = L\sqrt{1 + \frac{v^2}{u^2}}$$

Therefore the hamster speed is:

$$w = \frac{d}{\Delta t} = d\,\frac{u}{L} = \boxed{u\sqrt{1 + \frac{v^2}{u^2}}}$$

Now what if we replace the hamster with a light pulse?





Prof. Feynman's Hamster speed

$$w = u\sqrt{1 + \frac{v^2}{u^2}}$$

Hence Prof. Feynman's light pulse speed?

$$w = c\sqrt{1 + \frac{v^2}{c^2}}$$

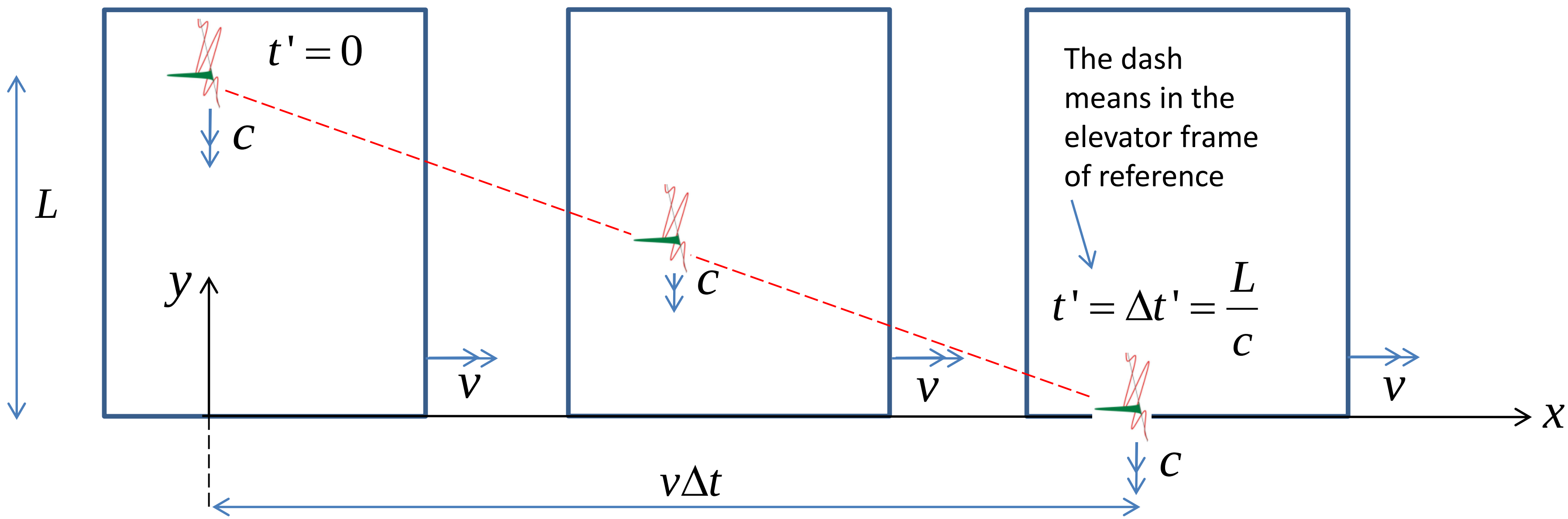
This cannot be correct

But according to Maxwell, this *cannot be correct*, since the speed of light is always c regardless of the **frame of reference** it is measured in......

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.998 \times 10^8 \,\mathrm{ms}^{-1}$$

TIME FOR A BOLD LEAP OF THE IMAGINATION!

Light is not like me



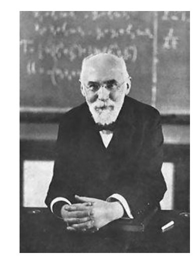
Let time progress at a **different rate**, depending on the relative motion of two frames of reference

$$w = \frac{d}{\Delta t} = \frac{\sqrt{L^2 + v^2 \Delta t^2}}{\Delta t}$$

Maxwell's assertion

$$w = c$$

$$\therefore c^2 \Delta t^2 = L^2 + v^2 \Delta t^2 \qquad \therefore L = c\Delta t \sqrt{1 - \frac{v^2}{c^2}}$$



The Lorentz factor

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

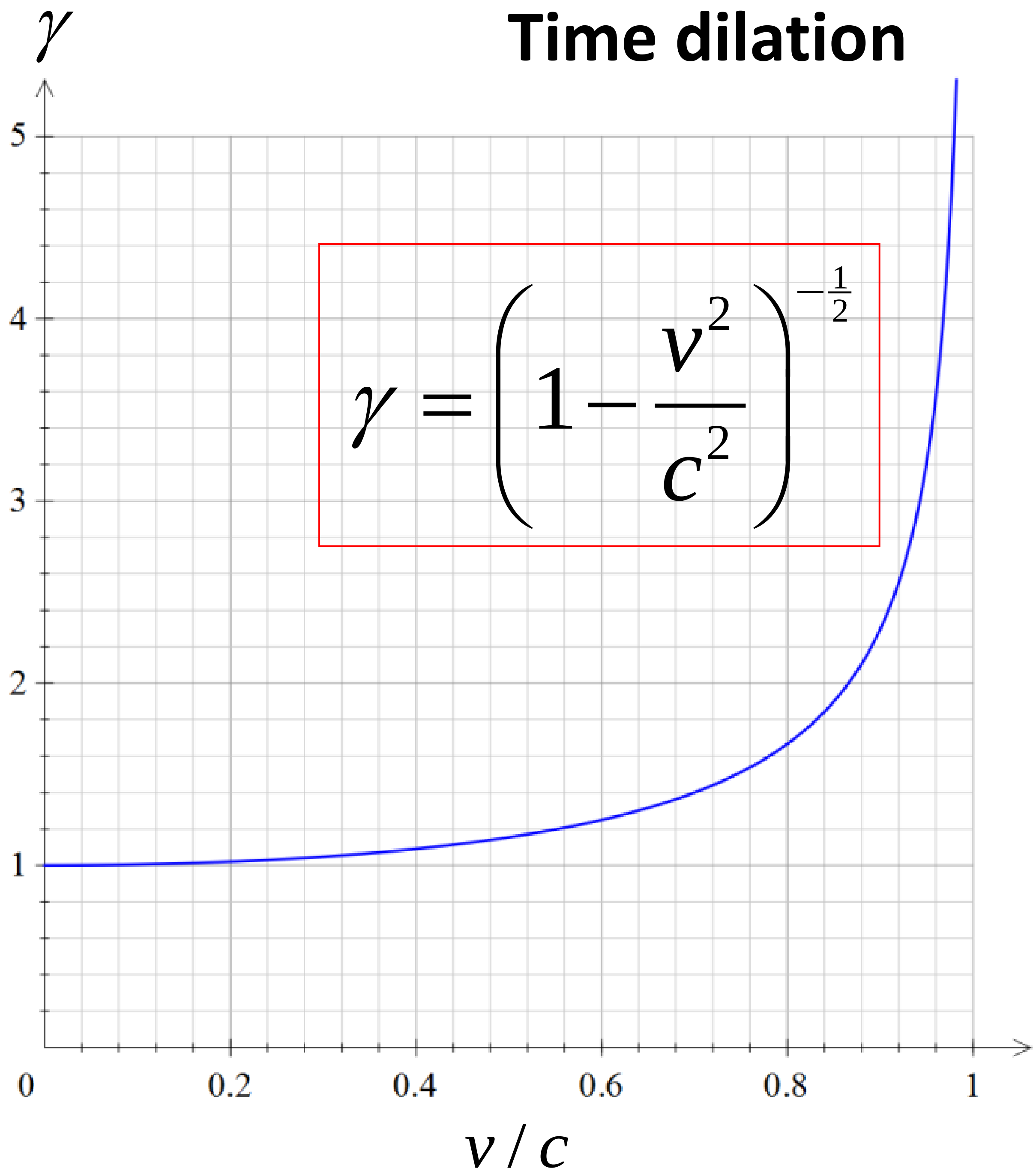
Now $L = c\Delta t' \quad \therefore \quad \Delta t' = \Delta t \sqrt{1 - \frac{v^2}{c^2}}$

i.e. we relate L to the time passed in the *elevator frame*

$$\therefore \Delta t' = \frac{\Delta t}{\gamma}$$

So **MOVING CLOCKS RUN SLOW**

Time dilation



$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Moving frame → $$\Delta t' = \frac{\Delta t}{\gamma}$$ ← 'Lab' frame

MOVING CLOCKS RUN SLOW

Note $\gamma \approx 1,\ v \ll c$

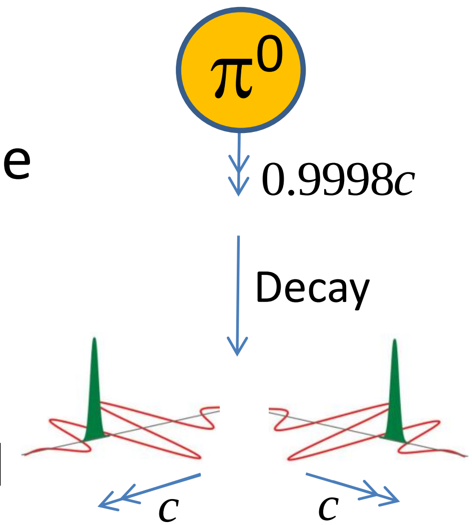
so for speeds much less than the speed of light

$$\Delta t' \approx \Delta t$$

Thank goodness

Well this **Special Relativity** stuff is all well in theory, but can we do an **experiment** to confirm?

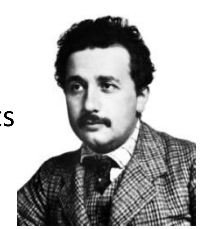
In 1964 Alväger and co-workers at CERN fired protons at a Beryllium target to produce fast-moving **neutral pions** (π^0), travelling at 0.9998c. These pions quickly decayed into two gamma **photons**. They measured the speed of these photons in the laboratory rest frame and found the speed to be c to within 0.005%.



A similar experiment by Filippas & Fox was conducted in 1963 with neutral pions at a speed of 0.2c. This also confirmed the hypothesis that light travels at c, *regardless* of the relative velocity of the source and detector.

(Adapted from JAAB's Special Relativity notes)

My theory has passed all the tests so far ...



A **photon** is essentially a 'light pulse'
More in the **Quantum Mechanics** course!

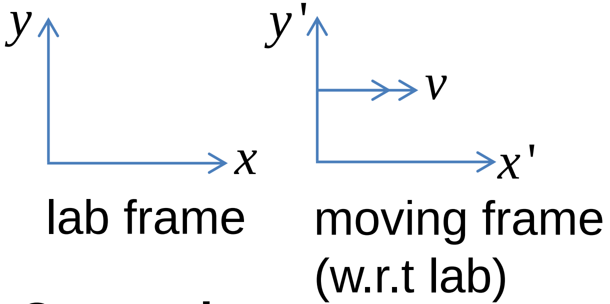
Lots more tests of Special Relativity are described at:
http://math.ucr.edu/home/baez/physics/Relativity/SR/experiments.html

Recap of the main results so far:

- **Speed of light is invariant for all frames of reference**
- **Time dilation**: **"Moving clocks run slow"**

Moving frame → $\Delta t' = \dfrac{\Delta t}{\gamma}$ ← 'Lab' frame

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$



Time dilation example 1: **A return trip to Alpha Centauri**
Time dilation example 2: **Mysterious muon decay!**

The **length** of a moving object ***contracts*** in the direction of motion, relative to its length measured at rest

Length of moving object → $l = \dfrac{L}{\gamma}$ ← Length at rest

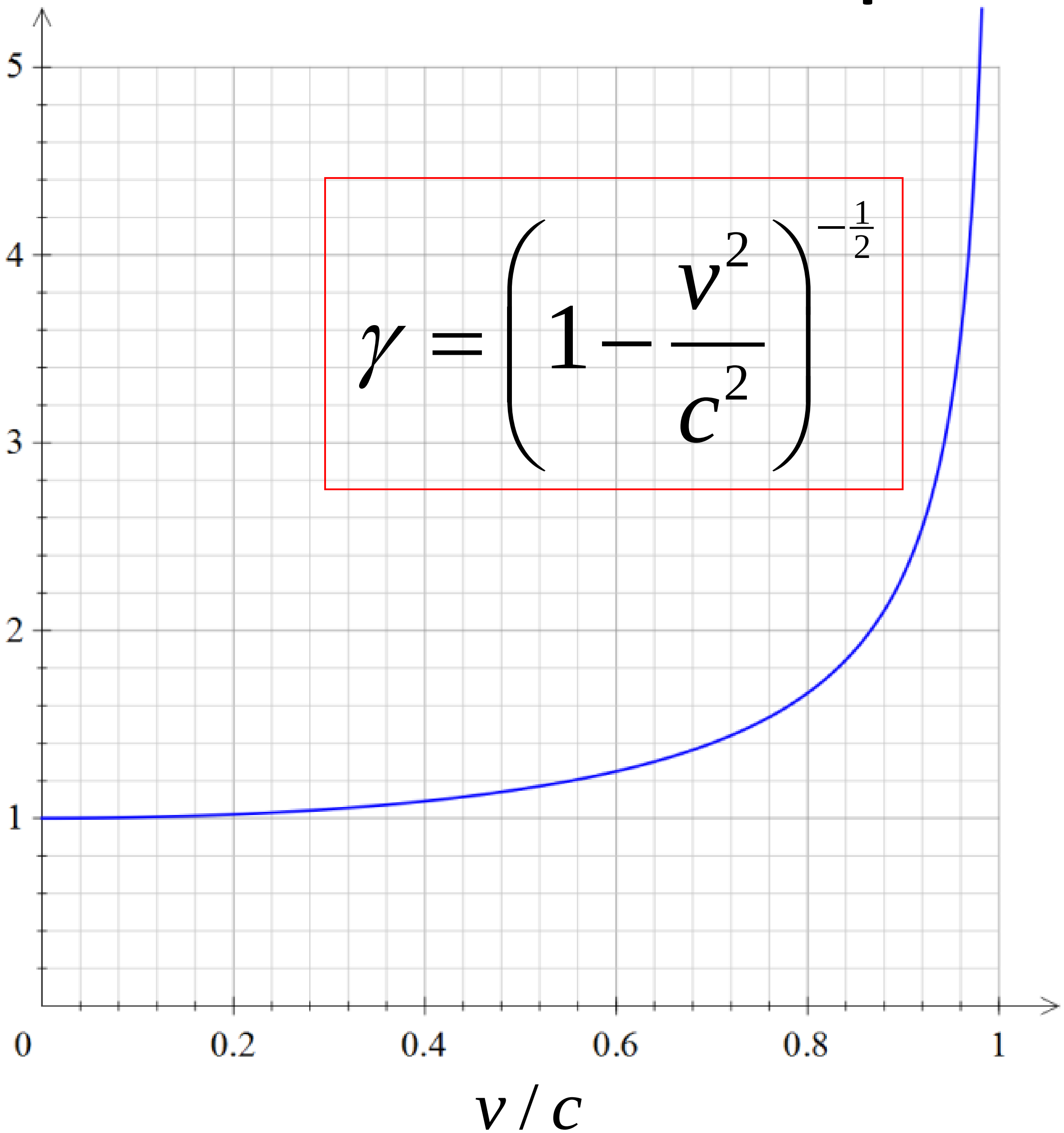
This explains the Muon mystery from the Muon's perspective!

Relative motion close to the speed of light causes a **loss in simultaneity** i.e. time is **offset** as well as **dilated**.

Explanation of the Twins 'Paradox' (We'll need the **Lorentz Transform** for this)

↑ i.e. 'putting all the relativistic effects together'

Time dilation recap



$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Moving frame → $$\Delta t' = \frac{\Delta t}{\gamma}$$ ← 'Lab' frame

MOVING CLOCKS RUN SLOW

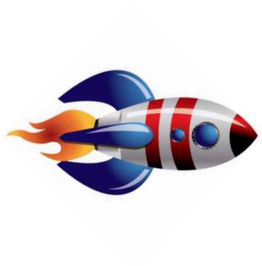
Note $\gamma \approx 1,\ v \ll c$

so for speeds much less than the speed of light

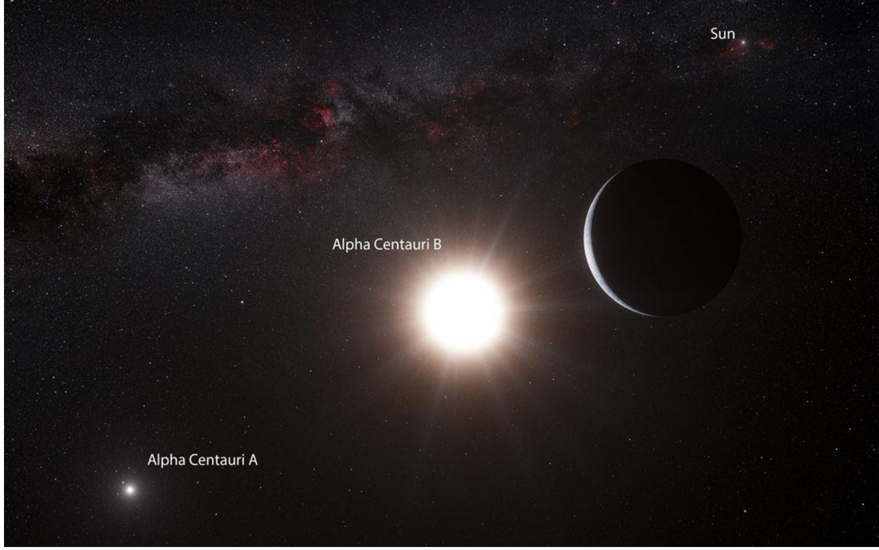
$$\Delta t' \approx \Delta t$$

Thank goodness

From the perspective of the astronauts, how long will the return journey to Alpha Centauri last? (Assume they stay there for a year)



$\rightarrow\!\!\rightarrow v = \frac{4}{5}c$



4 light-years $= 4cT_{year}$

$$T_{year} \approx 365 \times 24 \times 3600\text{s}$$

$$T_{year} \approx \pi \times 10^7\,\text{s}$$

Earth perspective:

$$\Delta t_{\oplus} = 2 \times \frac{4cT_{year}}{\frac{4}{5}c} + 1 = \boxed{11\,\text{years}}$$

Only time dilate the moving part of the journey

Astronaut perspective:

$$\Delta t' = \frac{\Delta t_{\oplus} - 1}{\gamma} + 1$$

$$\Delta t' = \frac{10}{\frac{5}{3}} + 1 = \boxed{7\,\text{years}}$$

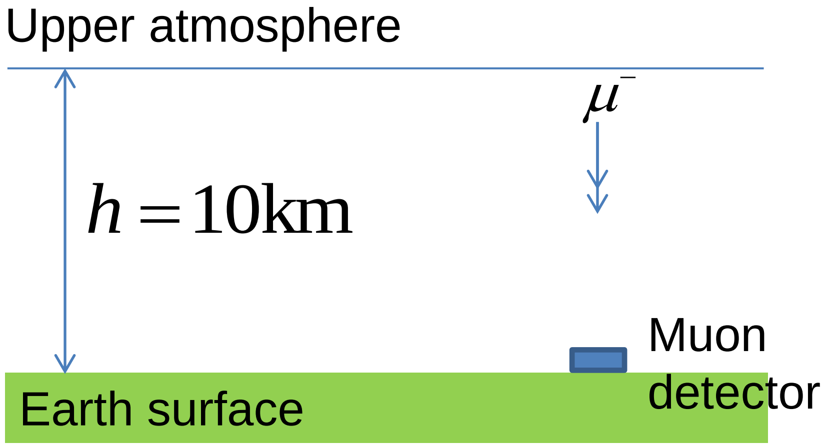
$$\boxed{\Delta t' = \frac{\Delta t}{\gamma}}$$

$$\gamma = \left(1 - \frac{\left(\frac{4}{5}c\right)^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\gamma = \left(1 - \tfrac{16}{25}\right)^{-\frac{1}{2}} = \left(\tfrac{9}{25}\right)^{-\frac{1}{2}} = \tfrac{5}{3}$$

Muon mystery! μ^-

One of the effects of cosmic radiation is to create **muons** in the upper atmosphere



$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$$

electron — electron neutrino — muon antineutrino

Muons have a half-life of 2.2μs and they travel at about 0.98c

To travel the 10km from upper atmosphere to a detector should take about:

$$\Delta t = \frac{h}{0.98c} = \frac{10^4\,\text{m}}{0.98 \times 2.998 \times 10^8\,\text{ms}^{-1}} \approx 34\mu s$$

This mystery is elementary, my dear Watson!

i.e. ≈ 15.5 half lives

We should therefore expect about $2^{-15.5} \approx \frac{1}{45,000}$

of the atomospheric muons to be detected on Earth....

But it is more like an eighth...

The explanation, **from the detector perspective**, is that the *muon's moving clock runs slow*. The half life is not 2.2μs but **2.2μs multiplied by γ**

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\therefore \gamma = \left(1 - 0.98^2\right)^{-\frac{1}{2}} \approx 5.03$$

$$\therefore \Delta t = \gamma \Delta t' = 5.03 \times 2.2\mu s = \boxed{11.1\mu s}$$

Moving frame → $\Delta t' = \dfrac{\Delta t}{\gamma}$ ← 'Lab' frame

So 2.2μs should in this case correspond to the 'moving' frame (of the muon)

The expected fraction of muons received should therefore be:

$$2^{-\frac{34}{11.1}} \approx 2^{-3.1} \approx \boxed{\frac{1}{8.4}}$$

It's not just my pupils which are dilated my dear Holmes!

But what about the Muon's perspective? Surely the muon will 'experience' a half life of 2.2μs? Don't we have a paradox?

The answer is that, from the Muon's perspective, the atmospheric distance it experiences 'coming towards it at 0.98c is **contracted**.
i.e. according to the Muon the detector is not 10km away, but **10km / γ = 1.99km**

Length of moving object → $$l = \frac{L}{\gamma}$$ ← Length at rest

We'll prove this result soon –
but first a poetic interlude....

Hendrik Lorentz
1853-1928

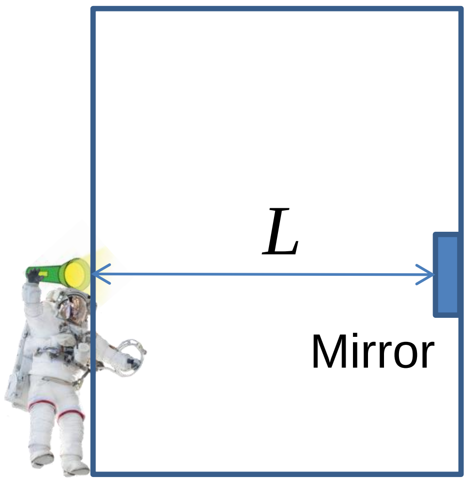
Observe that for muons created
The dilation of time is related
To Einstein’s insistence
Of shrunken-down distance
In the frame where decays aren’t
belated

Morin pp 522

Length contraction

Let's return to Mt Wonka's glass elevator

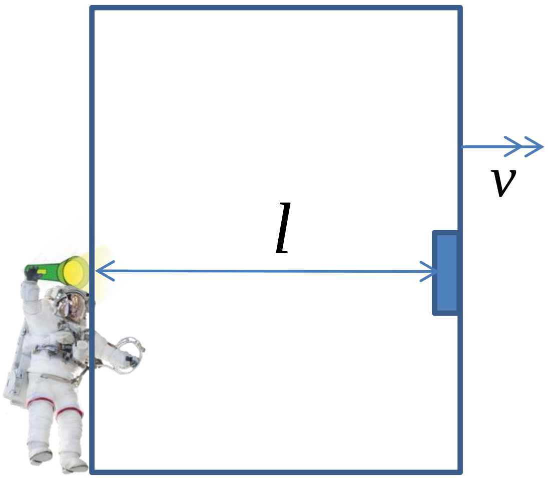
Length of moving object → $$l = \frac{L}{\gamma}$$ ← Length at rest



A **light pulse** is produced at one end, which is reflected off a **mirror**. The time difference $\Delta t'$ between the light pulse transmission and reception (the torch has an in-built lux meter data logger) enables the width L of the elevator to be measured.

'There and back' time → $$\Delta t' = \frac{2L}{c}$$

Now consider the situation as viewed by Prof. Feynman who observes the elevator moving with velocity v to the right. Let's assume *he* will measure the elevator width as l



total distance travelled by light

$$c\Delta t_{SM} = l + v\Delta t_{SM} \therefore \Delta t_{SM} = \frac{l}{c-v}$$

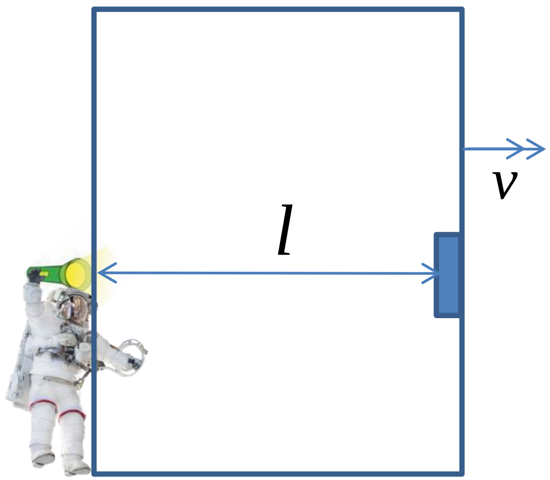
i.e. mirror has moved during transit!

Time from **S**ource to **M**irror

$$c\Delta t_{MS} = l - v\Delta t_{MS} \therefore \Delta t_{MS} = \frac{l}{c+v}$$

Closer in this case

Time from **M**irror to **S**ource



Total there-and back time is:

$$\Delta t = \Delta t_{SM} + \Delta t_{MS} = \frac{l}{c-v} + \frac{l}{c+v}$$

$$\Delta t = l\left(\frac{c+v+c-v}{(c-v)(c+v)}\right)$$

$$\Delta t = \frac{2lc}{c^2 - v^2} = \frac{2l}{c}\frac{1}{1-\frac{v^2}{c^2}} = \frac{2l}{c}\gamma^2$$

$$\gamma = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Now from the **time-dilation** result

$$\Delta t' = \frac{\Delta t}{\gamma}$$

and $\frac{2L}{c} = \Delta t'$

There-and-back time measured by the astronaut

$$\therefore \frac{2L}{c} = \frac{2l}{c}\gamma$$

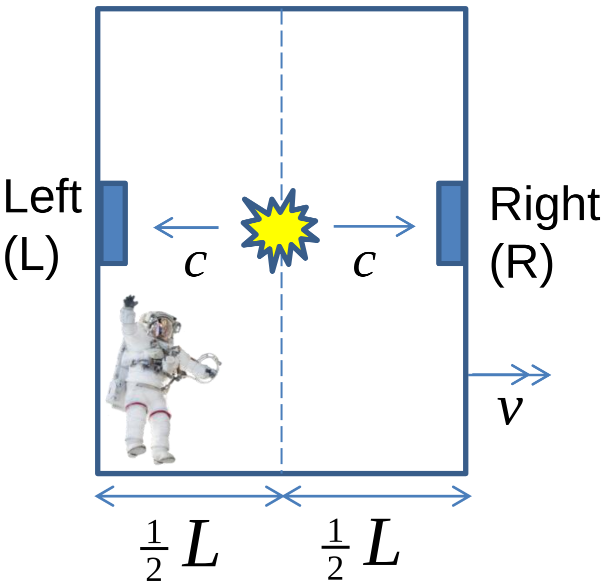
$$\therefore l = \frac{L}{\gamma}$$

So Prof. Feynman will observe the elevator width to be **contracted**, relative to the measurement of the astronaut

Loss of simultaneity

A light source is placed in the centre of Mr Wonka's elevator. Detectors placed at opposite side will clearly receive a light pulse simultaneously, since light travels at speed c and travels the same distance.

Note this is *not* the "Rear Clock Ahead" example in Morin, but the *previous* scenario on p512. I think this gets the point across more clearly – and gives a result consistent with the Lorentz transform!



Let us define two **events**:
R means right detector receives the pulse
L means left detector receives the pulse

What does Prof Feynman see? Let us work out the time elapsed since the light source was created for events R and L

$$c\Delta t_R = \tfrac{1}{2}l + v\Delta t_R \;\therefore\; \Delta t_R = \frac{\frac{1}{2}l}{c-v}$$

$$c\Delta t_L = \tfrac{1}{2}l - v\Delta t_L \;\therefore\; \Delta t_L = \frac{\frac{1}{2}l}{c+v}$$

i.e. work out the distance travelled as in the length contraction example

$$\Delta t = \Delta t_R - \Delta t_L$$

$$\Delta t = \frac{\frac{1}{2}l}{c-v} - \frac{\frac{1}{2}l}{c+v} = \tfrac{1}{2}l\,\frac{c+v-(c-v)}{c^2-v^2} = \frac{vl}{c^2}\frac{1}{1-\frac{v^2}{c^2}}$$

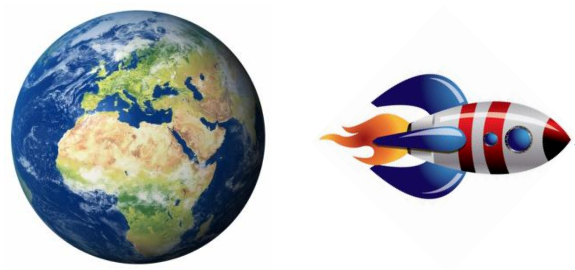
Note I measure the width to be $l = \dfrac{L}{\gamma}$

$$\Delta t = \frac{vl}{c^2}\gamma^2 = \boxed{\frac{\gamma v L}{c^2}}$$

So events R and L are ***not simultaneous*** Prof. Feynman's frame

The Twins 'Paradox'

Consider a pair of twins. One journeys to Alpha Centauri and stays for a year and the other stays on Earth. According to the analysis of time dilation:



$v = \frac{4}{5}c$

Earth twin:

$$\Delta t_{\oplus} = 2 \times \frac{4cT_{year}}{\frac{4}{5}c} + 1 = \boxed{11\,\text{years}}$$

4 light-years $= 4cT_{year}$

Astronaut twin:

$$\Delta t' = \frac{\Delta t_{\oplus} - 1}{\gamma} + 1$$

$$\boxed{\Delta t' = \frac{\Delta t}{\gamma}}$$

$$\Delta t' = \frac{10}{\frac{5}{3}} + 1 = \boxed{7\,\text{years}}$$

So the Earth twin should be **older** by 4 years

But from the perspective of the Astronaut, the brother has receded away (and then returned) at the same speed. So shouldn't the Earth-bound brother be four years older, rather than the other way round?

Youtube video solution!

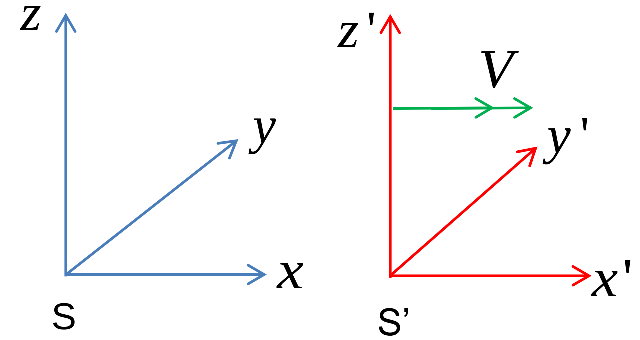
Resolution of the Twins paradox. The Earth-bound twin *is* older by four years. Morin (Appendix H) gives a number of reasons, but I think the correct thing to do is to properly consider the **Lorentz transforms.** The issue is that the problem involves the astronaut going there *and back*. The change of direction is the problem. We can't simply apply time dilation because the reference frames **change**

$$x = \gamma\left(x' + Vt'\right) \qquad x' = \gamma\left(x - Vt\right)$$

$$y = y' \qquad y = y'$$

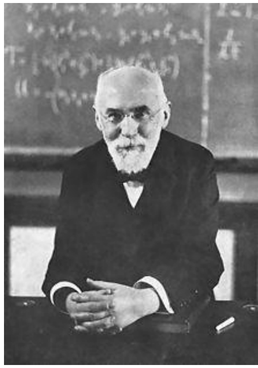
$$z = z' \qquad z = z'$$

$$t = \gamma\left(t' + \frac{Vx'}{c^2}\right) \qquad t' = \gamma\left(t - \frac{Vx}{c^2}\right)$$



The Lorentz Transforms

Hendrik Lorentz
1853-1928



$$\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\gamma t' + \frac{\gamma V x'}{c^2}$$

time dilation

loss of simultaneity

Earth is a rest, spaceship moves

T = time in years

$$x = \gamma\left(x' + Vt'\right)$$

$$t = \gamma\left(t' + \frac{Vx'}{c^2}\right)$$

Journey to Alpha Centauri

$$x = 4cT, \quad x' = 0$$

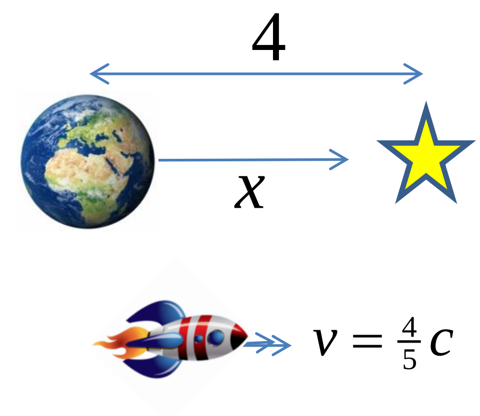
there is essentially no motion of the astronaut in the spaceship frame

This is why time dilation alone works in this scenario

$$t = 5T, \ V = \tfrac{4}{5}c$$

$$\gamma = \tfrac{5}{3}$$

$$t = \gamma t' \quad \therefore t' = \frac{5T}{\frac{5}{3}} = 3T$$



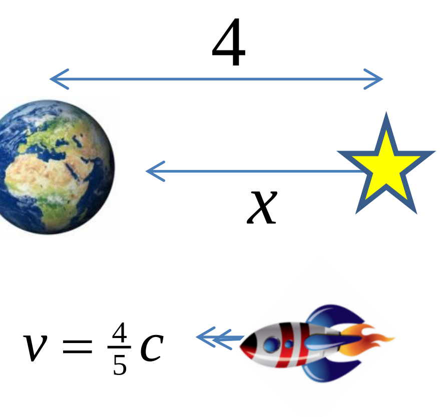
Journey home from Alpha Centauri

$$x = 4cT, \quad x' = 0$$

$$t = 5T, \ V = \tfrac{4}{5}c$$

$$\gamma = \tfrac{5}{3}$$

$$t = \gamma t' \quad \therefore t' = \frac{5T}{\frac{5}{3}} = 3T$$



So total time elapsed on **Earth** is 5+5 = **10 years** which is equivalent to 3+3 = **6 years** for the Astronaut twin

Spaceship is at rest, Earth moves t' is now time on Earth

$$x' = \gamma\left(x - Vt'\right)$$

$$t' = \gamma\left(t - \frac{Vx}{c^2}\right)$$

Journey to Alpha Centauri

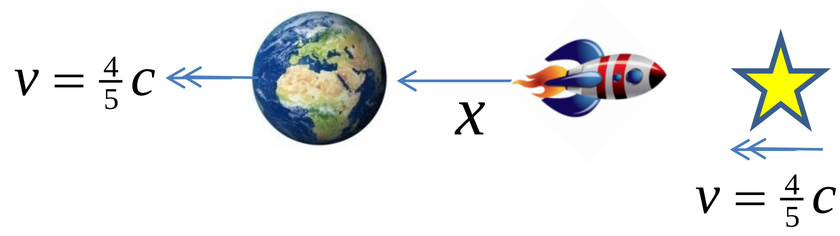
$V = \frac{4}{5}c, \quad \gamma = \frac{5}{3}$

$$t = 3T, \quad x = \frac{4cT}{\frac{5}{3}} = \frac{12}{5}cT$$

Lorentz contracted distance to Alpha Centauri, as observed by the spacecraft

$$t' = \frac{5}{3}\left(3T - \frac{\frac{4}{5}c \times \frac{12}{5}cT}{c^2}\right)$$

$$t' = \frac{9}{5}T$$



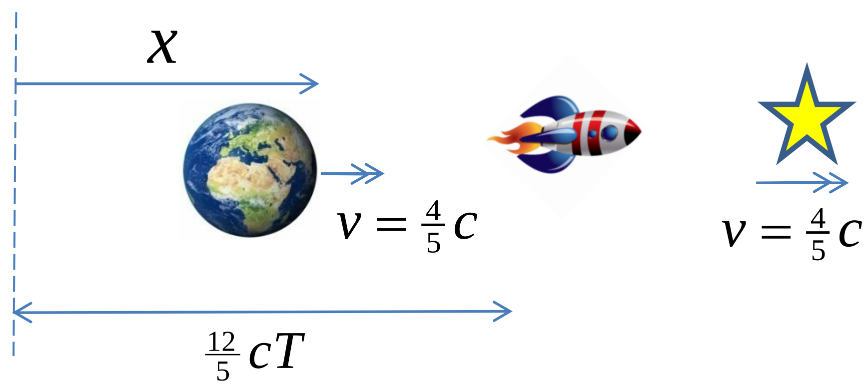
Journey home from Alpha Centauri

$V = \frac{4}{5}c, \quad \gamma = \frac{5}{3}$

$$t = 3T, \quad x = \frac{4}{\frac{5}{3}}cT = \frac{12}{5}cT$$

$$t' = \frac{5}{3}\left(3T - \frac{\frac{4}{5}c \times \frac{12}{5}cT}{c^2}\right)$$

$$t' = \frac{9}{5}T$$



So total time elapsed on Earth, during relative motion, is $\frac{18}{5}T = 3\frac{3}{5}T$
What has gone wrong? *What has happened to the missing* $6\frac{2}{5}$ years ?

The reason is that we haven't taken into account a **time offset** resulting from the fact that, from the Spaceship's perspective, the Earth has **changed reference frame.** In other words, simply adding the t' contributions will not cover the entire time elapsed on Earth.

i.e. sticking with the *original* frame of reference just before departure

$$t_1' = \tfrac{5}{3}\left(3T - \frac{\tfrac{4}{5}c\,\tfrac{12}{5}cT}{c^2}\right) = \tfrac{9}{5}T$$

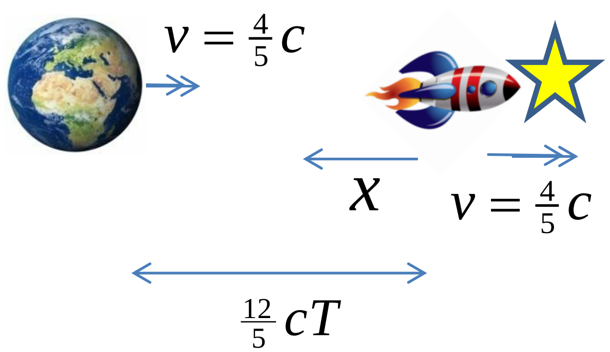
Earth time when spacecraft *arrives* at Alpha Centauri

$$t_2' = \tfrac{5}{3}\left(3T - \frac{\left(-\tfrac{4}{5}c\right)\tfrac{12}{5}cT}{c^2}\right) + 1$$

Earth time when spacecraft *leaves* Alpha Centauri (after the one year visit)

So **extra** Earth time is:

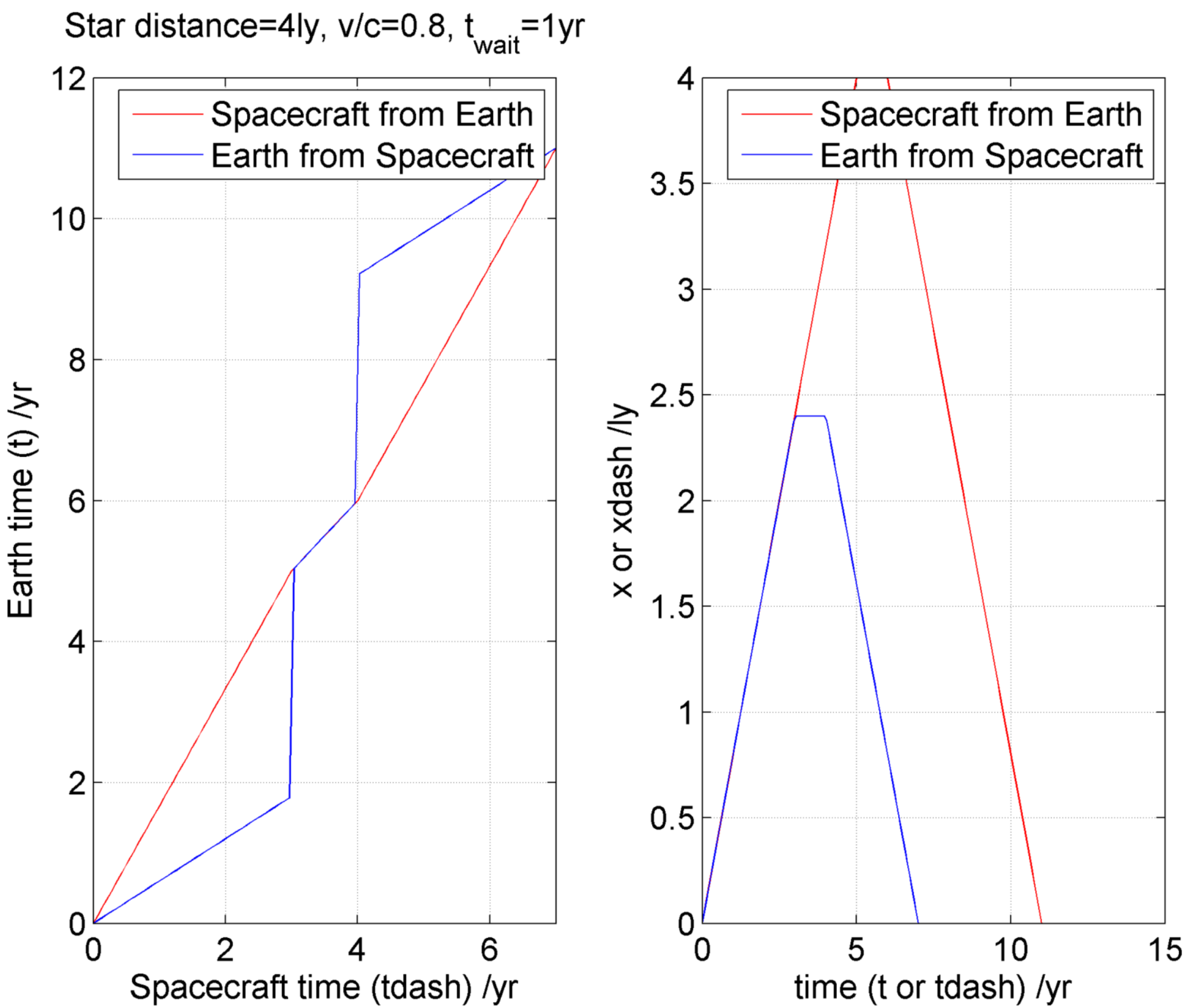
$$\Delta t' = t_2' - 1 - t_1' = \tfrac{5}{3}\frac{2\times\left(\tfrac{4}{5}c\right)\tfrac{12}{5}cT}{c^2}$$



$$\Delta t' = 6\tfrac{2}{5}T$$

which is **exactly** what was missing

This offset should exactly equate to the effects of **acceleration** as the rocket slows down to Alpha Centauri and then speeds up as it leaves.



Star distance=4ly, v/c=0.8, t_wait=1yr
Spacecraft from Earth
Earth from Spacecraft
Earth time (t) /yr
Spacecraft time (tdash) /yr
x or xdash /ly
time (t or tdash) /yr
0
2
4
6
8
10
12
0
0.5
1
1.5
2
2.5
3
3.5
4
5
15

$$E^2 - |\mathbf{p}|^2 c^2 = m^2 c^4$$

$$\mathbf{f} = m\gamma\mathbf{a} + m\gamma^3\left(\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\right)\mathbf{u}$$

EXTRAS

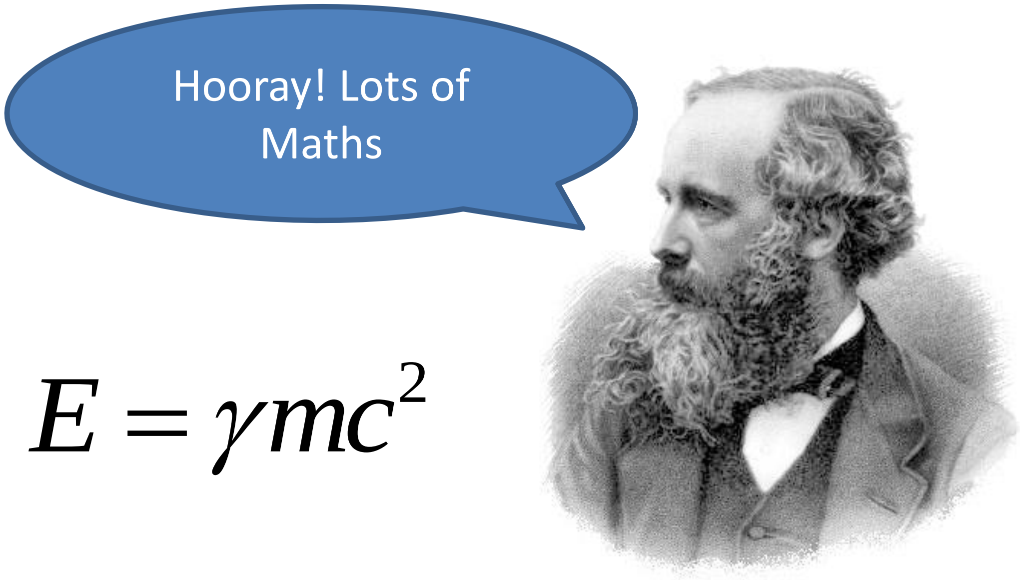
$$x = \gamma\left(x' + Vt'\right)$$

$$y = y'$$

$$z = z'$$

$$t = \gamma\left(t' + \frac{Vx'}{c^2}\right)$$

$$E = \gamma mc^2$$



See www.eclecticon.info Maths Mechanics page (Special Relativity)
for more detailed explanations of the results presented here

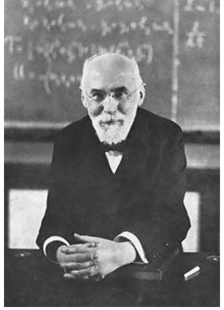
Extras

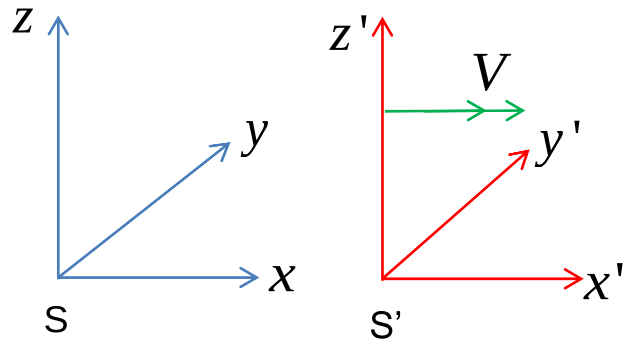
Summary of Special Relativity Results

- The **Lorentz Transform**
- Relativistic transformation of velocities
- Relativistic Doppler Shift
- Relativistic Momentum
- Relativistic Newton's Second Law
- Work done and $E = mc^2$
- Energy, momentum invariant
- Momentum of a photon

The Lorentz transform

Hendrik Lorentz
1853-1928





Time dilation, length contraction and loss of simultaneity can be incorporated into a general transformation of **spacetime** coordinates!

$$x = \gamma\left(x' + Vt'\right) \qquad x' = \gamma\left(x - Vt\right)$$

$$y = y' \qquad y = y'$$

$$z = z' \qquad z = z'$$

$$t = \gamma\left(t' + \frac{Vx'}{c^2}\right) \qquad t' = \gamma\left(t - \frac{Vx}{c^2}\right)$$

$$\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}}$$

We can generalize to an *S'* velocity which is not parallel to the x axis of the *S* frame

$$\mathbf{r} = (x, y, z), \quad \mathbf{r}' = (x', y', z')$$

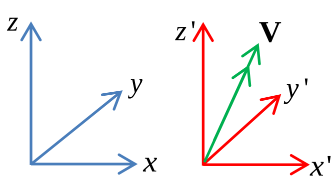
$$\mathbf{r} = \mathbf{r}' + \left(\frac{\gamma - 1}{V^2}\left(\mathbf{V}\cdot\mathbf{r}'\right) + \gamma t'\right)\mathbf{V}$$

$$t = \gamma\left(t' + \frac{\mathbf{V}\cdot\mathbf{r}'}{c^2}\right)$$

$$\mathbf{r}' = \mathbf{r} + \left(\frac{\gamma - 1}{V^2}\left(\mathbf{V}\cdot\mathbf{r}\right) - \gamma t'\right)\mathbf{V}$$

$$t' = \gamma\left(t - \frac{\mathbf{V}\cdot\mathbf{r}}{c^2}\right)$$

$$V = |\mathbf{V}|$$



Relativistic transformation of velocities

$$x = \gamma\left(x' + Vt'\right)$$

$$y = y'$$

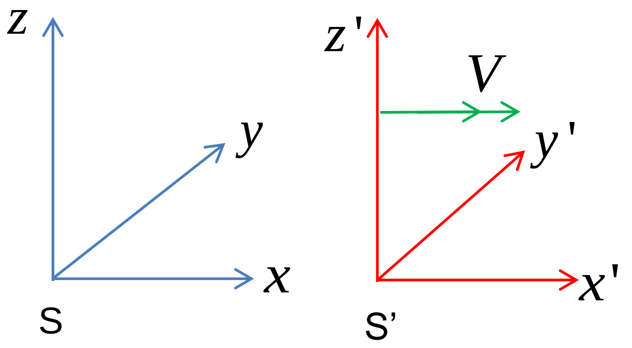
$$z = z'$$

$$t = \gamma\left(t' + \frac{Vx'}{c^2}\right)$$

$$v_x = \frac{dx}{dt} = \frac{\gamma\left(dx' + Vdt'\right)}{\gamma\left(dt' + \frac{V}{c^2}dx'\right)}$$

$$v_x = \frac{\frac{dx'}{dt'} + V}{1 + \frac{V}{c^2}\frac{dx'}{dt'}}$$

$$v_x = \frac{v_x' + V}{1 + \frac{v_x' V}{c^2}}$$



$$v_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{V}{c^2}dx'\right)}$$

$$v_z = \frac{dz}{dt} = \frac{dz'}{\gamma\left(dt' + \frac{V}{c^2}dx'\right)}$$

$$v_y = \frac{\frac{dy'}{dt'}}{\gamma\left(1 + \frac{V}{c^2}\frac{dx'}{dt'}\right)}$$

$$v_z = \frac{\frac{dz'}{dt'}}{\gamma\left(1 + \frac{V}{c^2}\frac{dx'}{dt'}\right)}$$

$$\boxed{v_y = \frac{v'_y}{\gamma\left(1 + \frac{v'_x V}{c^2}\right)}}$$

$$\boxed{v_z = \frac{v'_z}{\gamma\left(1 + \frac{v'_z V}{c^2}\right)}}$$

Hence:

$$v'_x = \frac{v_x - V}{1 - \frac{v_x V}{c^2}}$$

$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{v_x V}{c^2}\right)}$$

$$v'_z = \frac{v_z}{\gamma\left(1 - \frac{v_x V}{c^2}\right)}$$

If the velocity was the speed of light ....

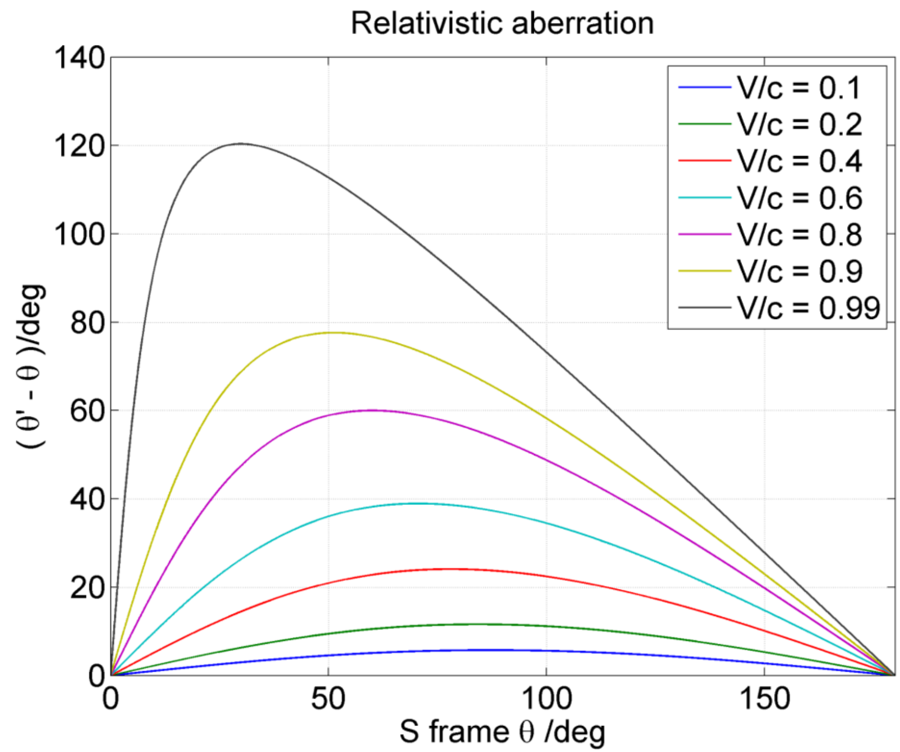
$$v_x = c\cos\theta$$

$$v'_x = c\cos\theta'$$

$$\cos\theta = \frac{\cos\theta' + \frac{V}{c}}{1 + \frac{V}{c}\cos\theta'}$$

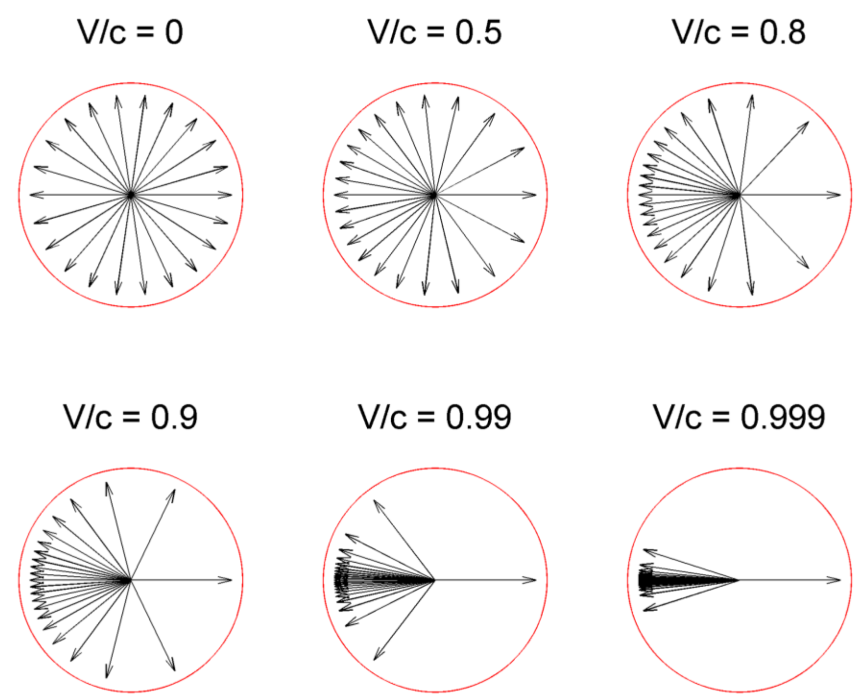
$$\cos\theta' = \frac{\cos\theta - \frac{V}{c}}{1 - \frac{V}{c}\cos\theta}$$

This is called '**relativistic aberration**'



$$\cos\theta = \frac{\cos\theta' + \frac{V}{c}}{1 + \frac{V}{c}\cos\theta'}$$

$$\cos\theta' = \frac{\cos\theta - \frac{V}{c}}{1 - \frac{V}{c}\cos\theta}$$



$$v_x' = \frac{v_x - V}{1 - \frac{v_x V}{c^2}}$$

$$v_y' = \frac{v_y}{\gamma\left(1 - \frac{v_x V}{c^2}\right)}$$

$$v_z' = \frac{v_z}{\gamma\left(1 - \frac{v_x V}{c^2}\right)}$$

$$v_x = c\cos\theta$$

$$v_x' = c\cos\theta'$$

Relativistic Doppler shift

Consider a receding wave source of frequency f' in the S' frame. It crosses the x axis of the S frame at angle θ. and speed u. The velocity of waves emitted is w, in S.

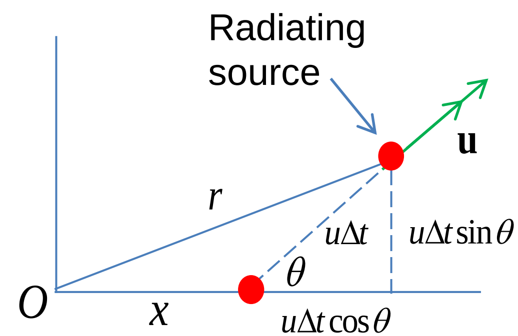
The period T of waves received by an observer (in the x direction) at the origin O of the S frame is:

$$T = \Delta t + \frac{r - x}{w}$$

extra distance travelled by source between wave crests

time between wave crests *at source*

wave speed



Note if waves are *electromagnetic*

$$w = c$$

$$f = \frac{f'}{\gamma\left(1 + \frac{u\cos\theta}{w}\right)}$$

See Eclecticon note for proof

Define **Doppler frequency shift**

$$\Delta f = f - f'$$

$$\frac{\Delta f}{f'} = \frac{1}{\gamma\left(1 + \frac{u\cos\theta}{w}\right)} - 1$$

Relativistic Momentum

We might expect ‘force = rate of change of momentum’ to be true in a relativistic sense as well as in the classical. However, the speed limit of c would imply an *upper limit on the amount of momentum a given mass could* have, if we use the classical momentum formula

$$\mathbf{p} = m\mathbf{u}$$

This would be *counter to reality* – we could easily devise a theoretical system which applies a finite amount of power, indefinitely, to a fixed mass system. e.g. a ball rolling down a infinitely long slope!

To get around this problem, let us *redefine* momentum such that it *can* become infinite as velocity tends towards c. i.e. multiply by γ....

$$\mathbf{p} = \gamma m\mathbf{u}$$

$$\gamma = \left(1 - \frac{\mathbf{u}\cdot\mathbf{u}}{c^2}\right)^{-\frac{1}{2}} = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

Some useful derivatives involving γ

$$\mathbf{a} = \frac{d\mathbf{u}}{dt}$$

$$\frac{d\gamma}{dt} = -\frac{1}{2}\left(1 - \frac{\mathbf{u}\cdot\mathbf{u}}{c^2}\right)^{-\frac{3}{2}}\left(-2\frac{\mathbf{u}}{c^2}\cdot\frac{d\mathbf{u}}{dt}\right)$$

$$\boxed{\frac{d\gamma}{dt} = \gamma^3 \frac{\mathbf{a}\cdot\mathbf{u}}{c^2}}$$

$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

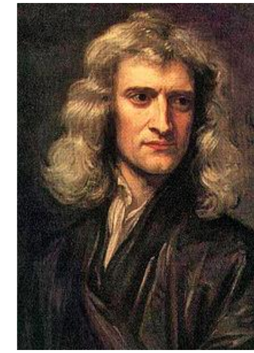
$$\frac{d\gamma}{du} = -\frac{1}{2}\left(1 - \frac{u^2}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{2u}{c^2}\right)$$

$$u^2 = \mathbf{u}\cdot\mathbf{u}$$

$$\boxed{\frac{d\gamma}{du} = \gamma^3 \frac{u}{c^2}}$$

Force, work & energy

$$\mathbf{f} = \frac{d}{dt}(\gamma m \mathbf{u})$$ 'Relativistic Newton's Second Law'



$$\mathbf{f} = ma$$

$$\mathbf{f} = m\gamma \frac{d\mathbf{u}}{dt} + m\mathbf{u}\frac{d\gamma}{dt}$$

$$\mathbf{f} = m\gamma\mathbf{a} + m\gamma^3\left(\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\right)\mathbf{u}$$

$$\mathbf{f} = m\gamma\mathbf{a} + m\gamma^3\left(\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\right)\mathbf{u}$$

$$W = \int \mathbf{f}\cdot d\mathbf{r} = \int \mathbf{f}\cdot\mathbf{u}\,dt$$ Work done

$$W = m\int\left(\gamma\mathbf{a}\cdot\mathbf{u} + \gamma^3\left(\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\right)u^2\right)dt$$

$$W = m\int \gamma(\mathbf{a}\cdot\mathbf{u})\left(1+\frac{\gamma^2u^2}{c^2}\right)dt$$

$$1+\frac{\gamma^2u^2}{c^2} = \gamma^2 \Rightarrow \gamma^2\left(1-\frac{u^2}{c^2}\right) = 1$$

$$\Rightarrow \gamma = \left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

$$W = m\int \gamma^3(\mathbf{a}\cdot\mathbf{u})dt$$

$$W = mc^2\int \gamma^3\frac{(\mathbf{a}\cdot\mathbf{u})}{c^2}dt$$

$$\frac{d\gamma}{dt} = \gamma^3\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}$$

$$W = mc^2\int \frac{d\gamma}{dt}dt$$

$$W = mc^2\int_{\gamma_0}^{\gamma_1} d\gamma$$

$$W = (\gamma_1-\gamma_0)mc^2$$

So the **total energy** of a mass m is:

$$E = \gamma mc^2$$

When the *mass is at rest*

$$\gamma = 1$$

$$E_0 = mc^2$$

Hence **kinetic energy** is

$$E_k = (\gamma-1)mc^2$$

Now in the **classical limit**

$$u \ll u \qquad \therefore \gamma \approx 1+\tfrac{1}{2}\frac{u^2}{c^2}$$

$$\therefore (\gamma-1)mc^2 = \tfrac{1}{2}mu^2$$

Energy, momentum invariant

Consider the following quantity:

$$E^2 - |\mathbf{p}|^2 c^2 = m^2 c^4$$

$$k = E^2 - |\mathbf{p}|^2 c^2$$

$$k = \left(\gamma m c^2\right)^2 - \left(\gamma m \mathbf{u}\right)\cdot\left(\gamma m \mathbf{u}\right)c^2$$

$$k = \gamma^2 m^2 c^4 - \gamma^2 m^2 u^2 c^2$$

$$k = m^2 c^4 \gamma^2 \left(1 - \frac{u^2}{c^2}\right)$$

$$k = m^2 c^4 \left(1 - \frac{u^2}{c^2}\right)^{-1} \left(1 - \frac{u^2}{c^2}\right)$$

$$k = m^2 c^4$$

This is clearly an invariant, *regardless* of the frame of reference.

This is *very* useful in **Particle Physics**, and collision problems at relativistic speeds. e.g. **Compton Scattering**.

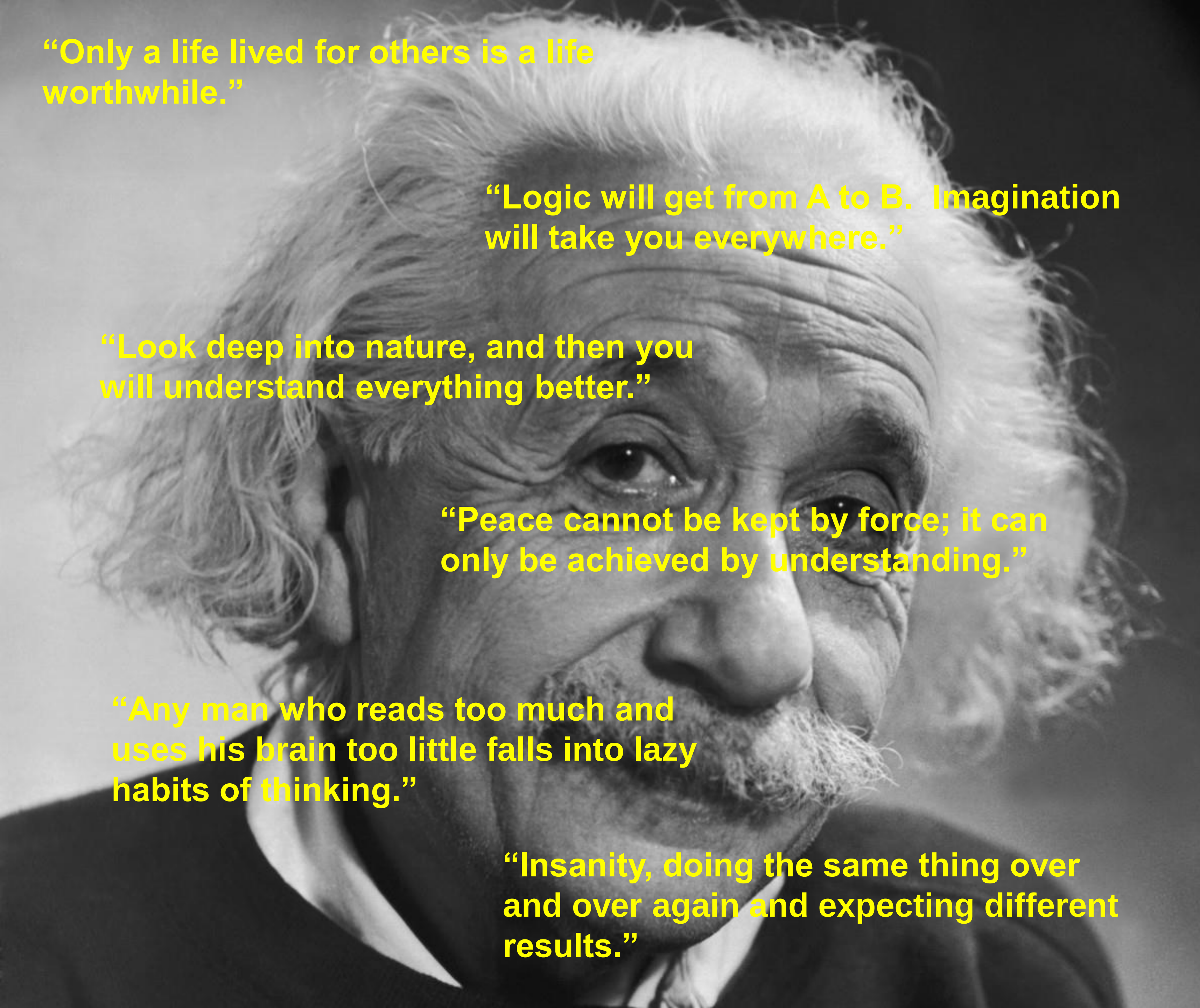
So for a **photon** $m = 0$

$$E = pc$$

$$E = hf$$

h is **Planck's constant**

$$\therefore p = \frac{hf}{c}$$



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