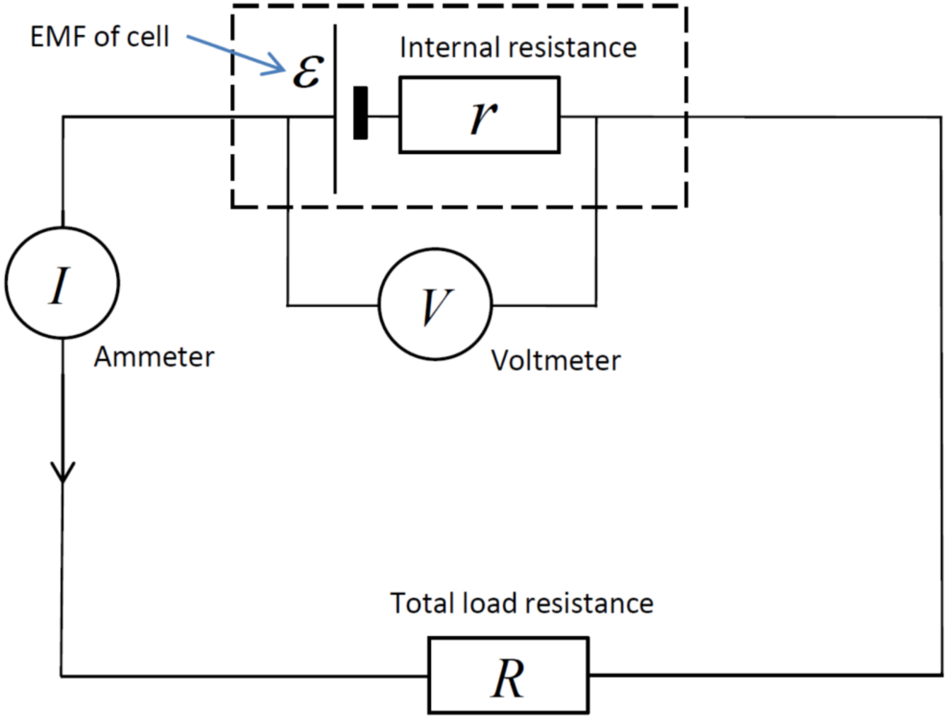
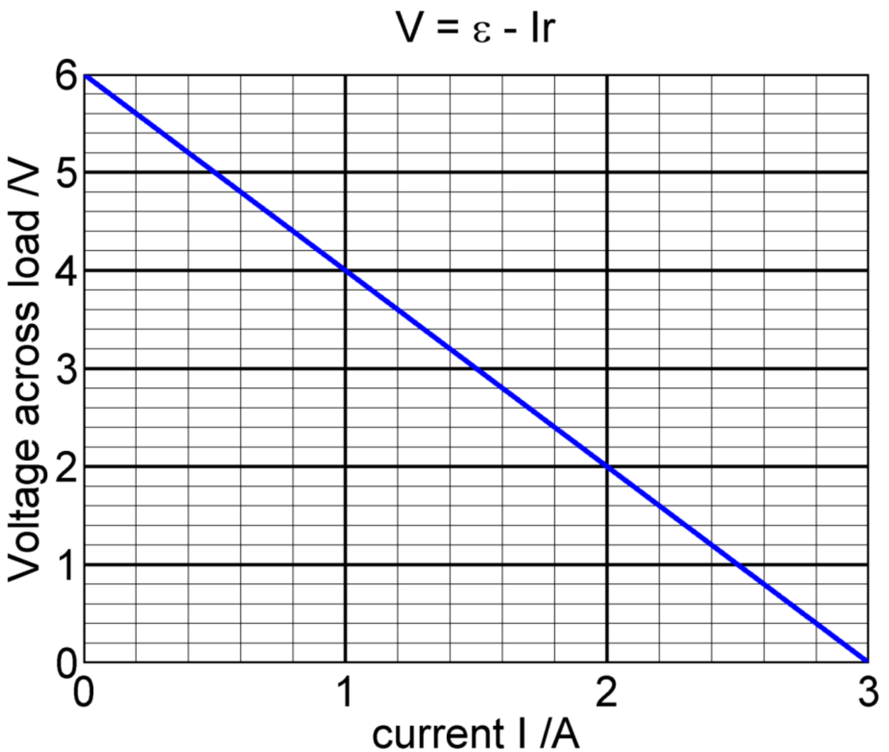
BPhO Computational Challenge

Electromagnetism

Dr Andrew French.
December 2023.

BPhO British Physics Olympiad





$$\varepsilon = Ir + IR$$

$$I = \frac{\varepsilon}{r + R}$$

$$P = I^2 R$$

Power dissipated in load R

$$\boxed{P = \frac{\varepsilon^2 R}{(r + R)^2}}$$

$$\varepsilon = Ir + V$$

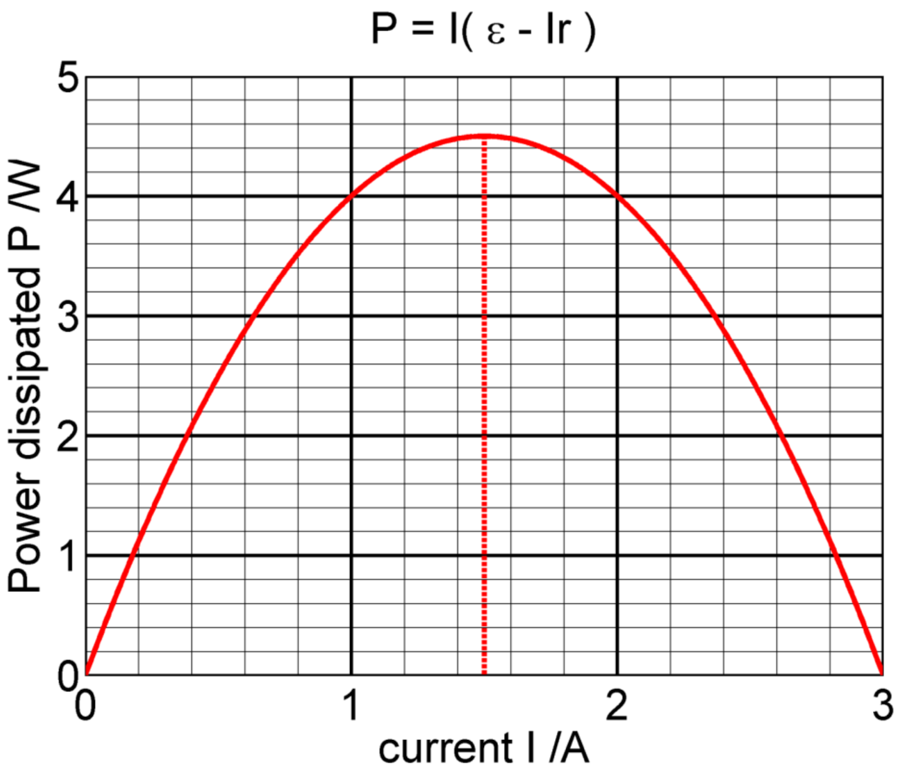
$$V = \varepsilon - Ir$$

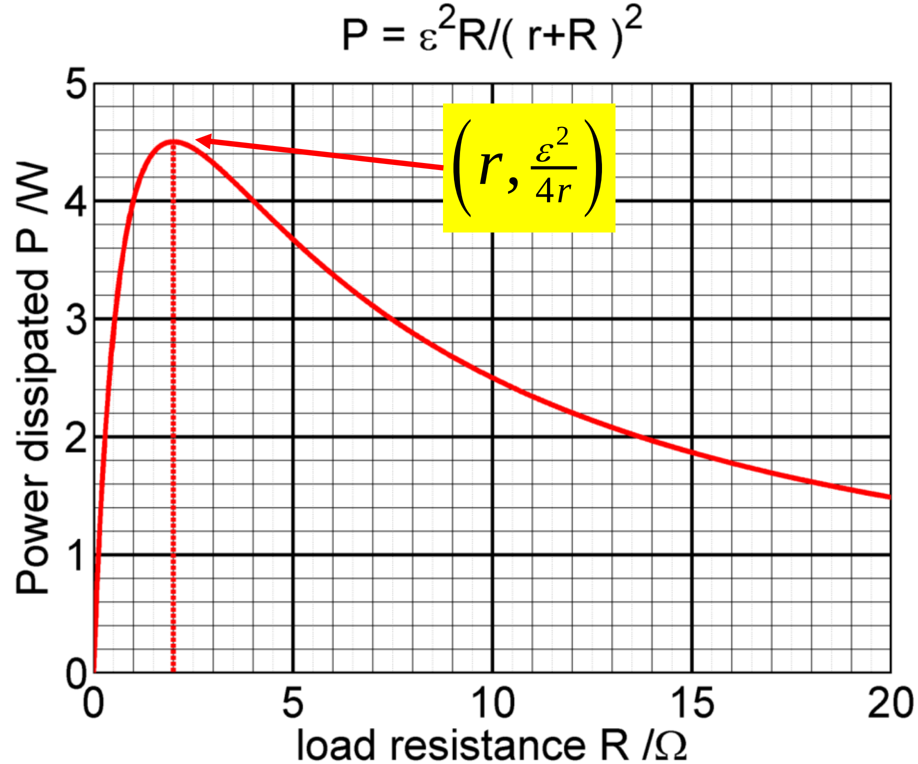
$$P = IV$$

$$P = I\varepsilon - I^2 r$$

$$P = -r\left\{I^2 - \frac{\varepsilon}{r} I\right\} = -r\left\{\left(I - \frac{\varepsilon}{2r}\right)^2 - \frac{\varepsilon^2}{4r^2}\right\}$$

$$\boxed{P = \frac{\varepsilon^2}{4r} - r\left(I - \frac{\varepsilon}{2r}\right)^2}$$





$$\varepsilon = Ir + V$$

$$V = \varepsilon - Ir$$

$$P = IV$$

$$P = I\varepsilon - I^2 r$$

$$P = -r\left\{I^2 - \frac{\varepsilon}{r}I\right\} = -r\left\{\left(I - \frac{\varepsilon}{2r}\right)^2 - \frac{\varepsilon^2}{4r^2}\right\}$$

$$P = \frac{\varepsilon^2}{4r} - r\left(I - \frac{\varepsilon}{2r}\right)^2$$

Maximum power dissipated in load R

$$P_{\max} = \frac{\varepsilon^2}{4r}$$

$$R = r$$

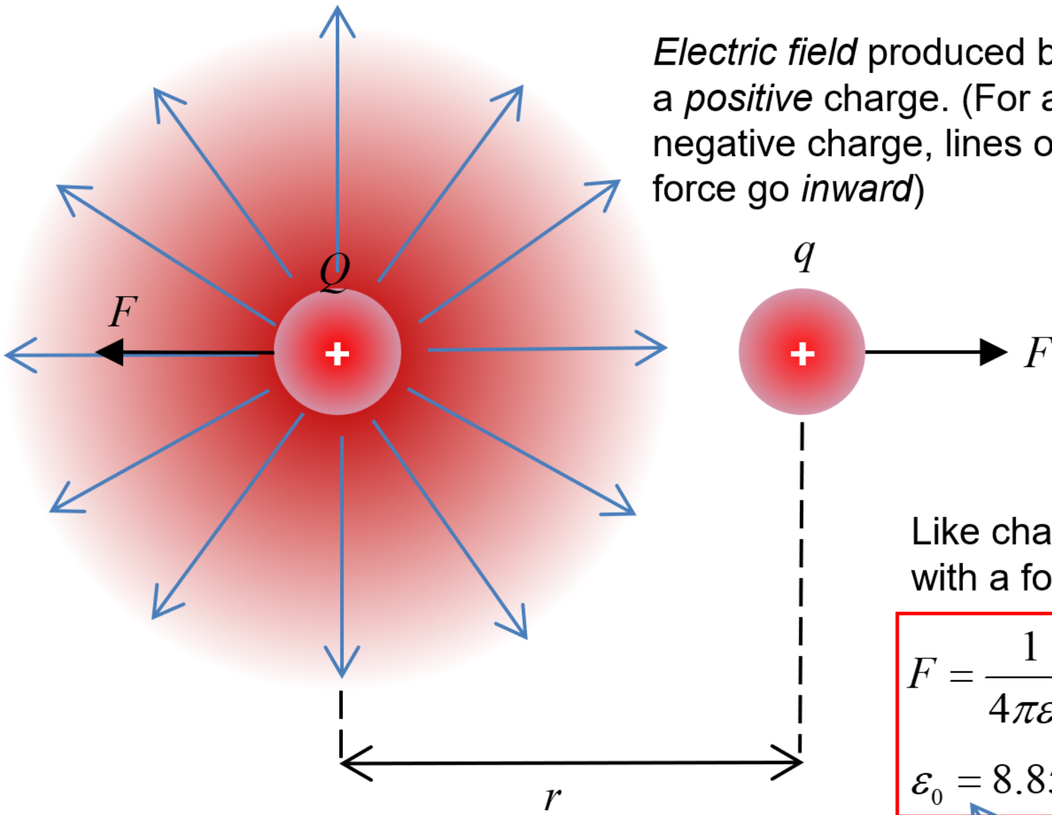
$$\varepsilon = Ir + IR$$

$$I = \frac{\varepsilon}{r + R}$$

$$P = I^2 R$$

$$P = \frac{\varepsilon^2 R}{(r + R)^2}$$

Electric field produced by a *positive* charge. (For a negative charge, lines of force go *inward*)



Like charges will *repel* with a force

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}$$

$$\varepsilon_0 = 8.8542\times10^{-12}\,\mathrm{Fm}^{-1}$$

This is Coulomb's Law of Electrostatics

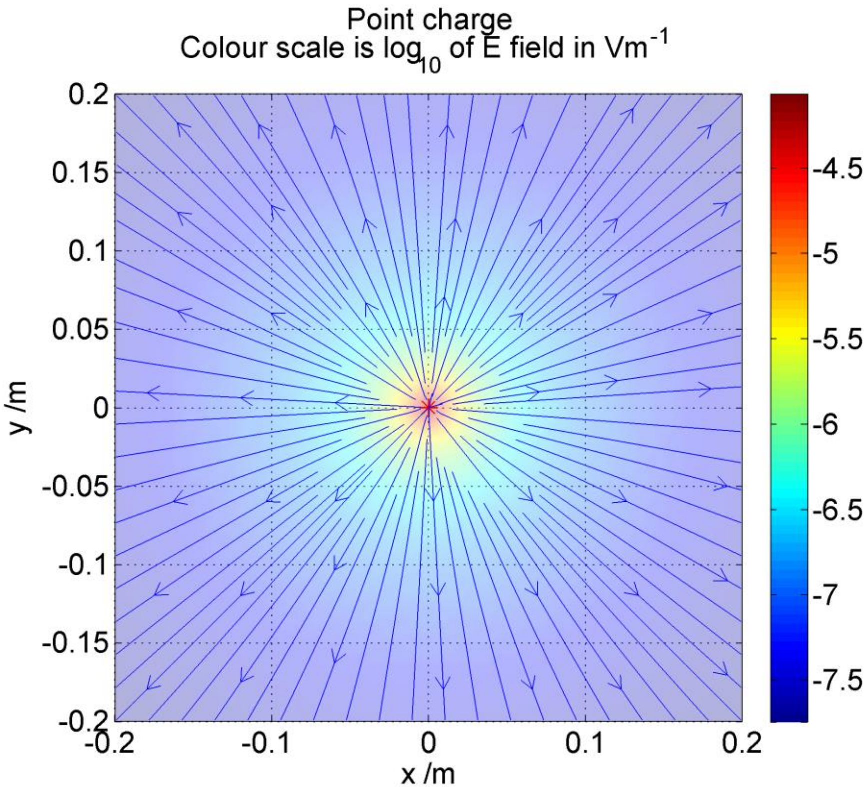
Permittivity of free space

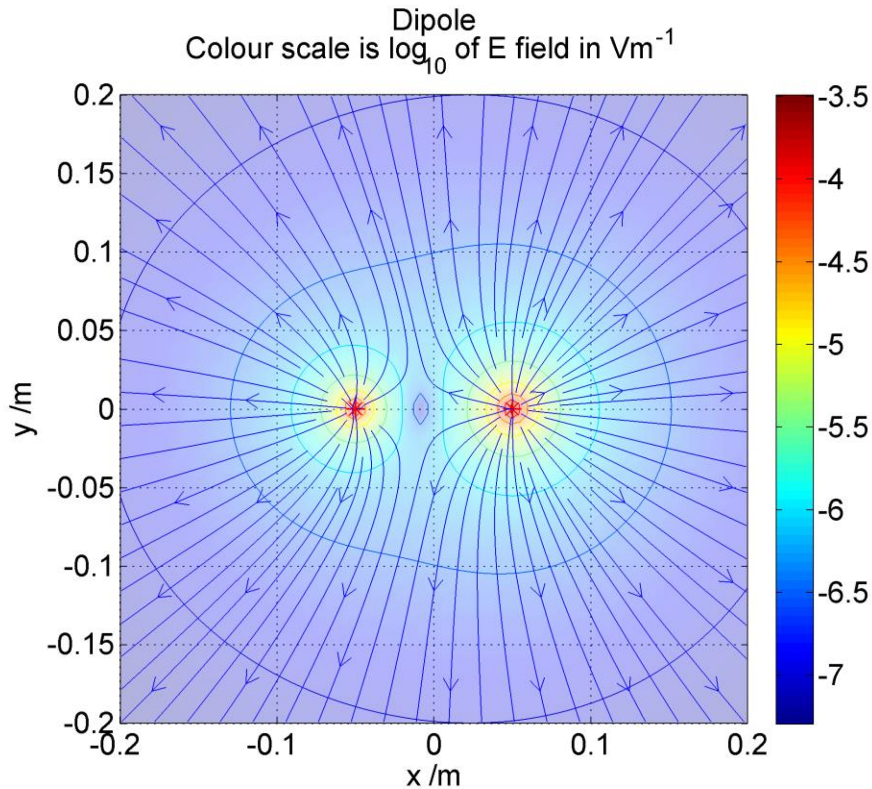
$$\mathbf{F} = q\mathbf{E}$$

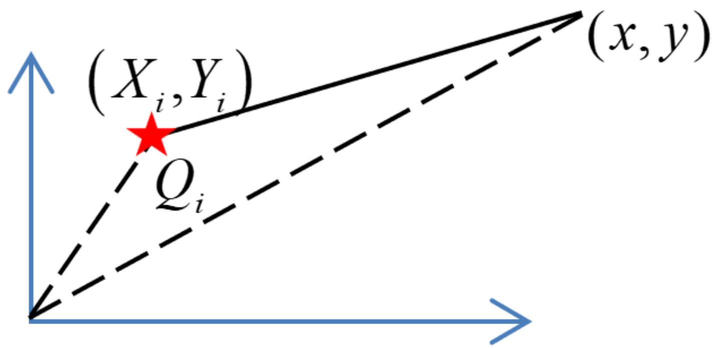
Force = charge x electric field strength

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

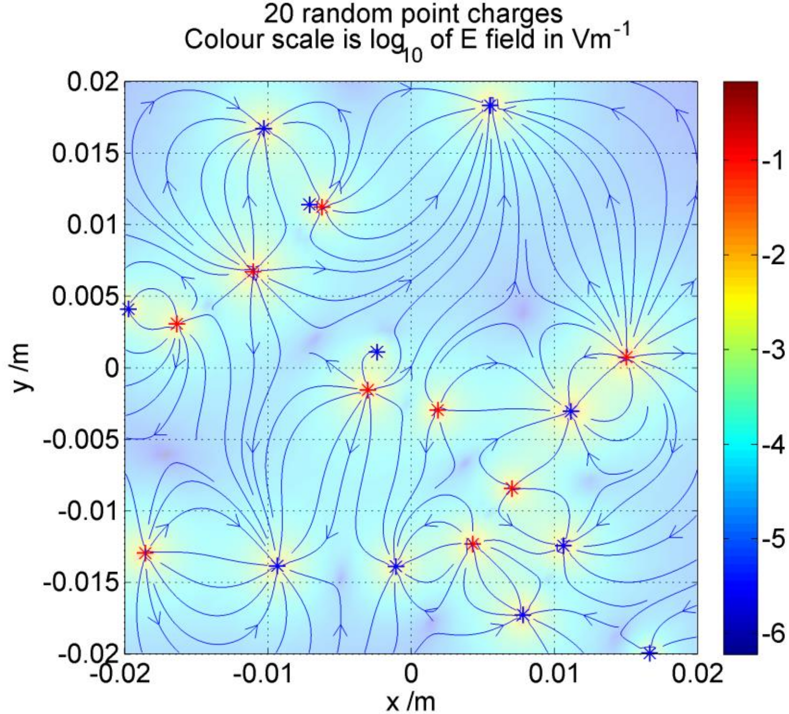
Charles-Augustin de Coulomb (1736-1806)

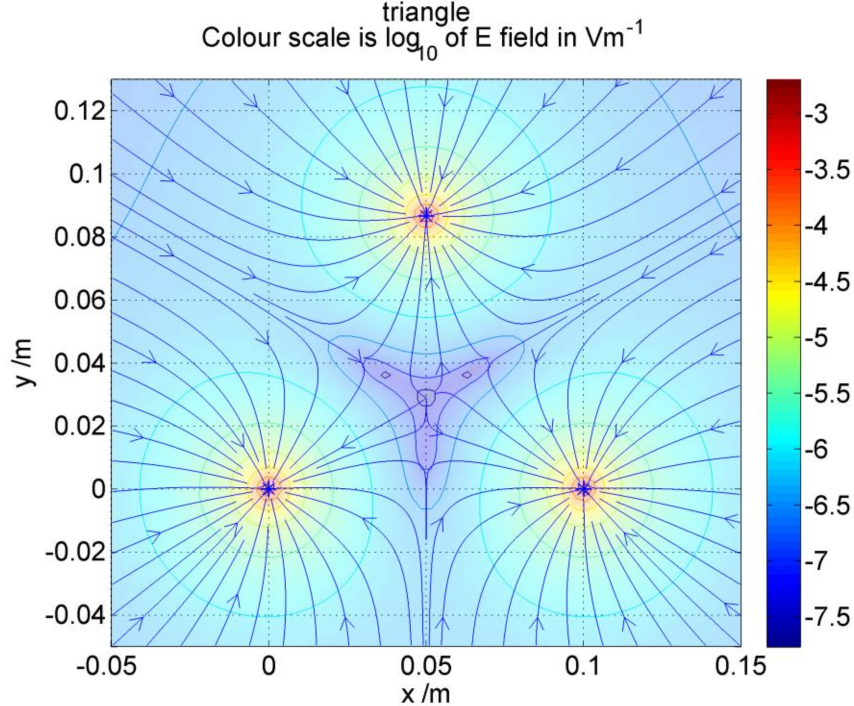


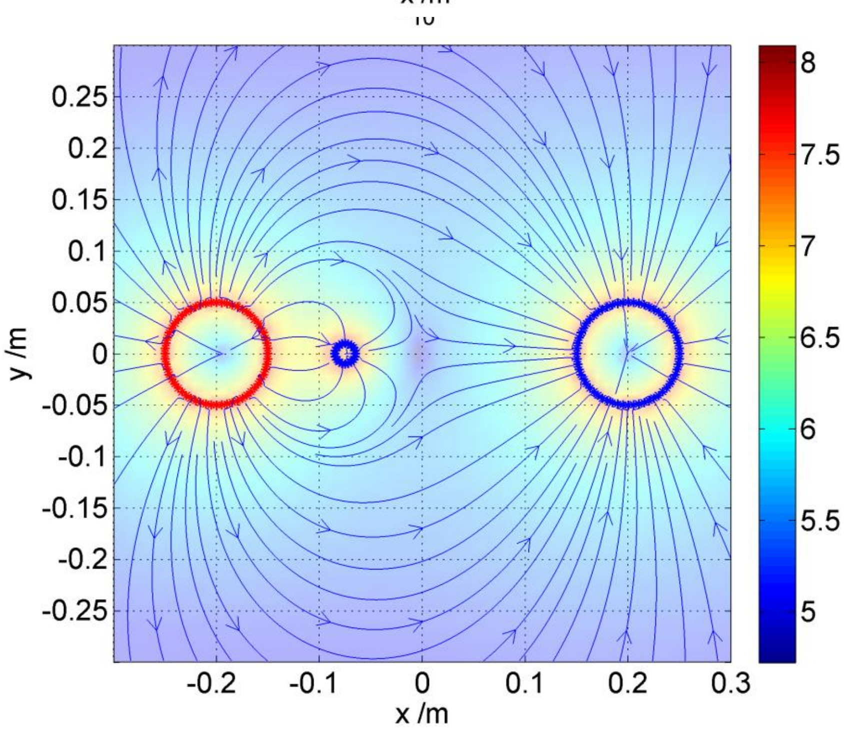


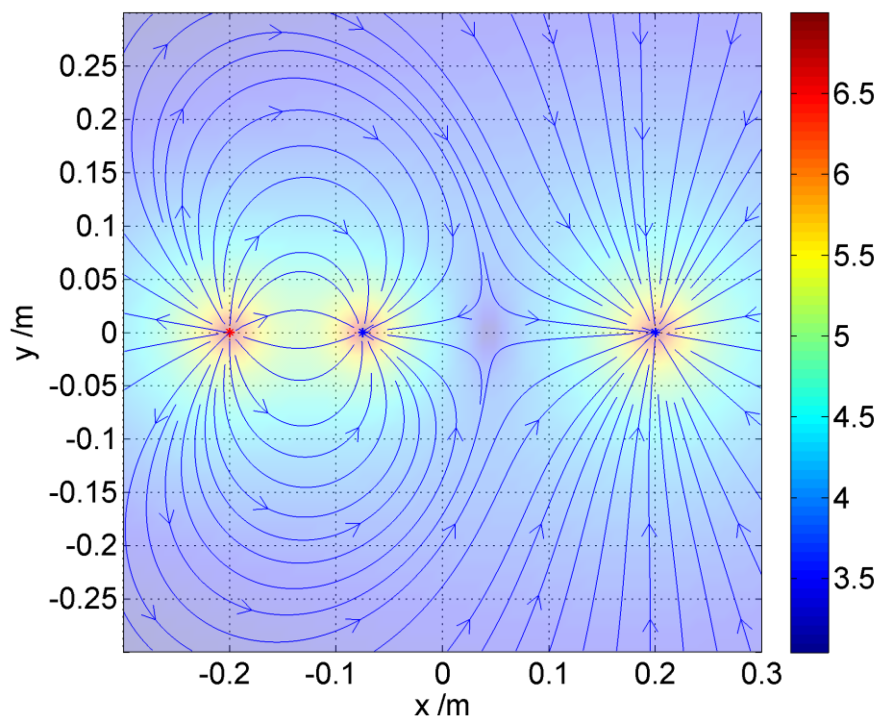


$$\mathbf{E}(x,y) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{Q_i}{(x-X_i)^2+(y-Y_i)^2}\,\frac{\hat{\mathbf{x}}(x-X_i)+\hat{\mathbf{y}}(y-Y_i)}{\sqrt{(x-X_i)^2+(y-Y_i)^2}}$$

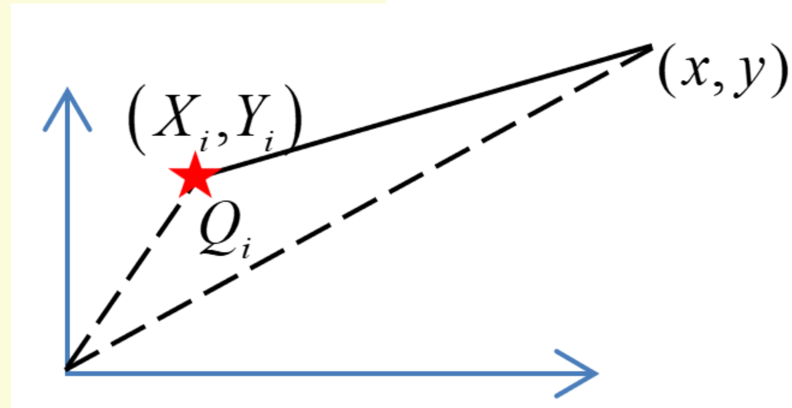

20 random point charges
Colour scale is $\log_{10}$ of E field in Vm^{-1}
y /m
x /m


triangle
Colour scale is $\log_{10}$ of E field in Vm^{-1}
y /m
x /m

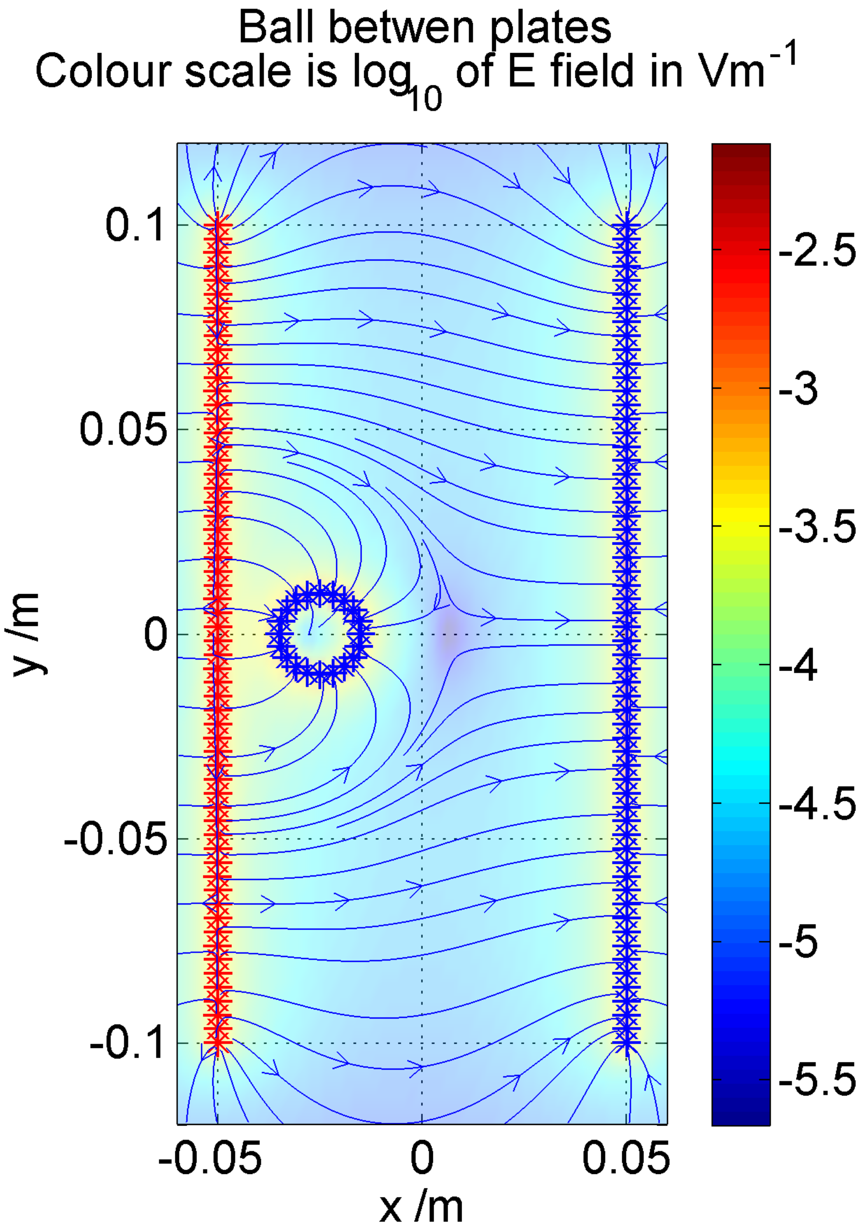

y /m
x /m


y /m
x /m

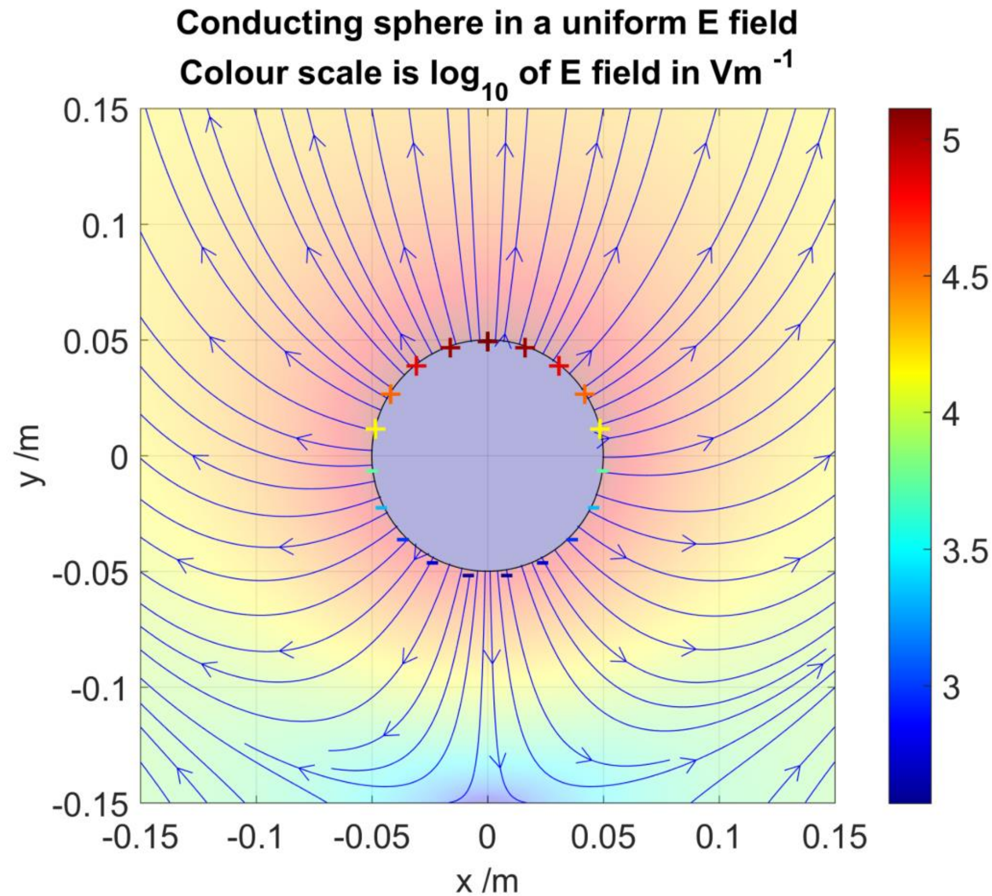
```matlab
% Compute electric potential and electric field vectors
function [Ex,Ey,V] = field_calc( x,y, xq,yq,zq, q )

%Permittivity of free space / m^-3 kg^-1 s^4 A^2 (or Fm^-1)
e0 = 8.854e-12;

%Calculate electric potential and electric field in (x,y,z=0) plane
dim = size(x);
V = zeros(dim);
Ex = zeros(dim);
Ey = zeros(dim);
Xq = zeros( dim(1),dim(2),numel(q) );
Yq = zeros( dim(1),dim(2),numel(q) );
Zq = zeros( dim(1),dim(2),numel(q) );
Q = zeros( dim(1),dim(2),numel(q) );
for k=1:numel(q)
    Xq(:,:,k) = xq(k);
    Yq(:,:,k) = yq(k);
    Zq(:,:,k) = zq(k);
    Q(:,:,k) = q(k);
end
x = repmat(x,[1,1,numel(q)] );
y = repmat(y,[1,1,numel(q)] );
z = zeros( dim(1),dim(2),numel(q) );
r = sqrt( ( x - Xq ).^2 + ( y - Yq ).^2 + ( z - Zq ).^2 );
V  = sum(( Q./(4*pi*e0) )./r,3);
Ex = sum(( x - Xq ).*( Q./(4*pi*e0) )./(r.^3),3);
Ey = sum(( y - Yq ).*( Q./(4*pi*e0) )./(r.^3),3);

%%
```



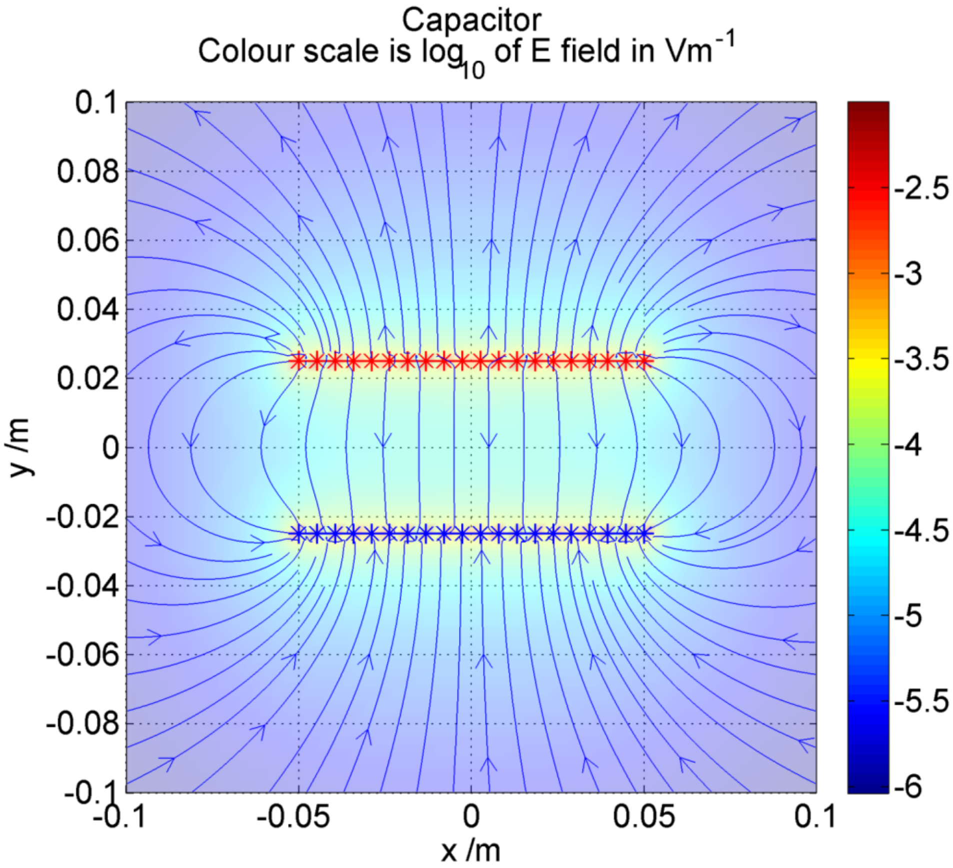
$$\mathbf{E}(x,y) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{Q_i}{(x-X_i)^2+(y-Y_i)^2}\,\frac{\hat{\mathbf{x}}(x-X_i)+\hat{\mathbf{y}}(y-Y_i)}{\sqrt{(x-X_i)^2+(y-Y_i)^2}}$$

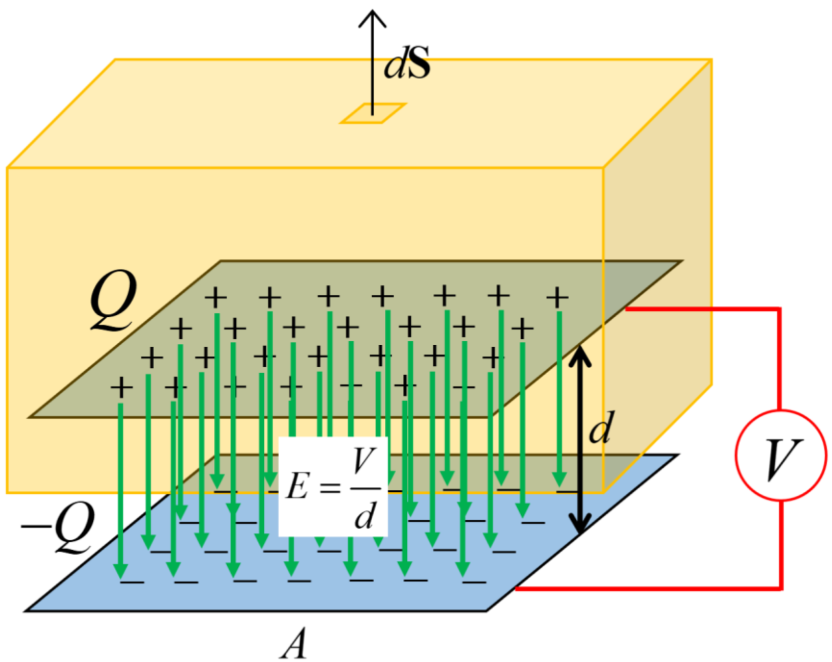


Not quite!

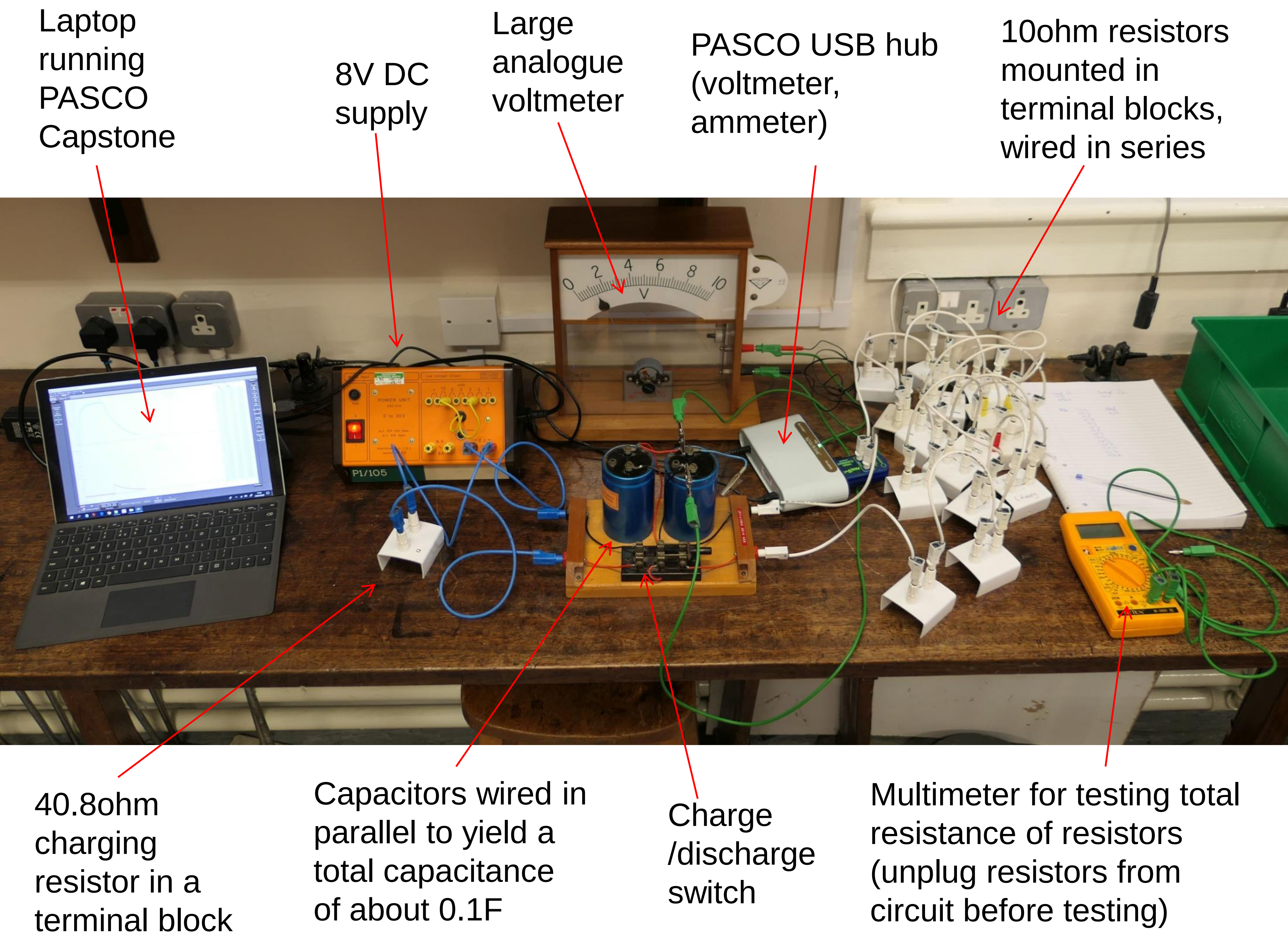


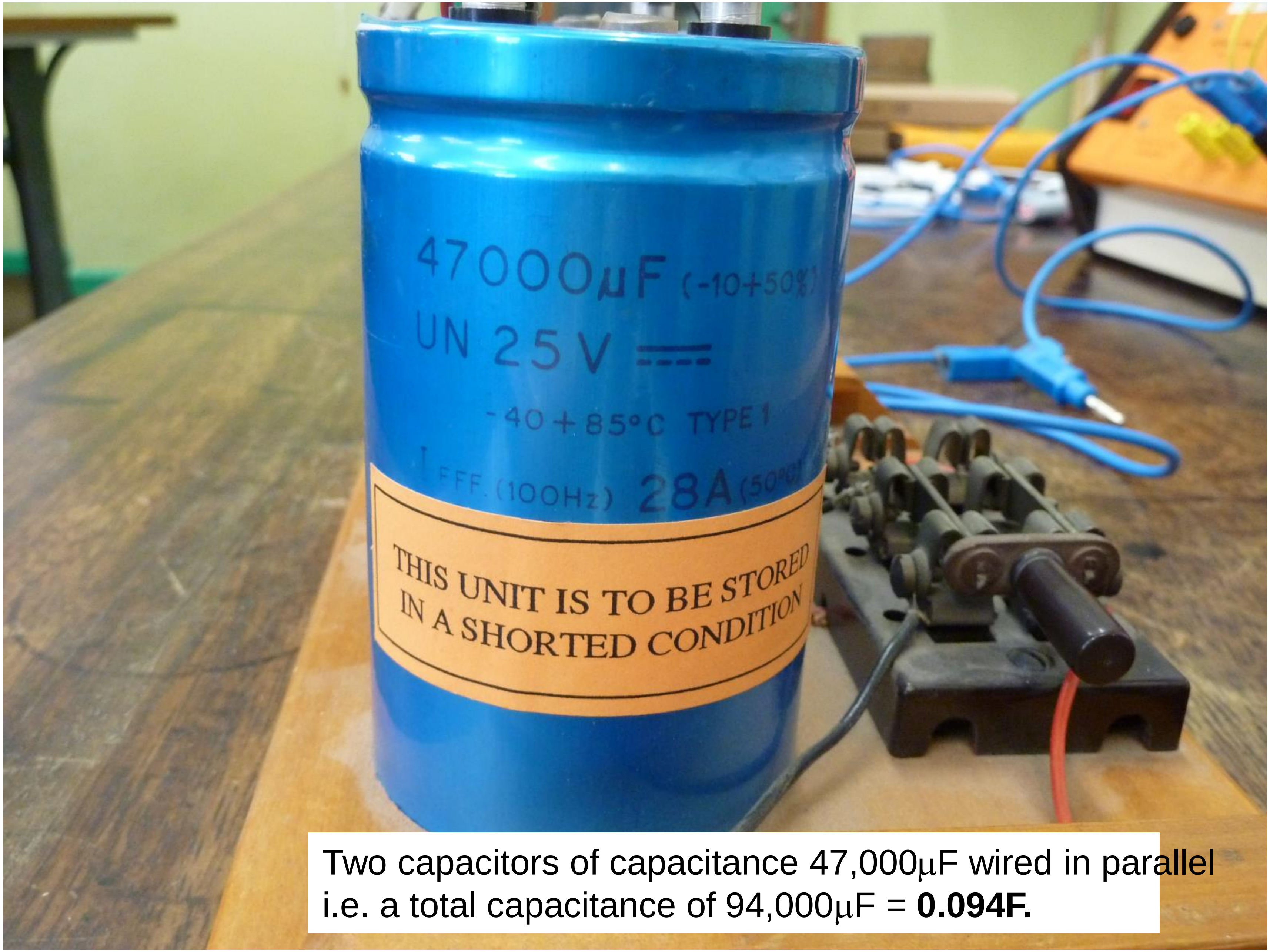
Actually the charge distribution on a conducting sphere will be *polarized* by the electric field between the plates





Capacitor model


Laptop running PASCO Capstone
8V DC supply
Large analogue voltmeter
PASCO USB hub (voltmeter, ammeter)
10ohm resistors mounted in terminal blocks, wired in series
40.8ohm charging resistor in a terminal block
Capacitors wired in parallel to yield a total capacitance of about 0.1F
Charge /discharge switch
Multimeter for testing total resistance of resistors (unplug resistors from circuit before testing)



Two capacitors of capacitance 47,000µF wired in parallel
i.e. a total capacitance of 94,000µF = **0.094F.**

Discharging a capacitor

$$Q = CV \qquad V = IR \qquad I = -\frac{dQ}{dt}$$

capacitor charge, voltage relationship

Ohm's law

Definition of current, and negative since charge is discharged from plates

$$\therefore I = \frac{V}{R} = -C\frac{dV}{dt}$$

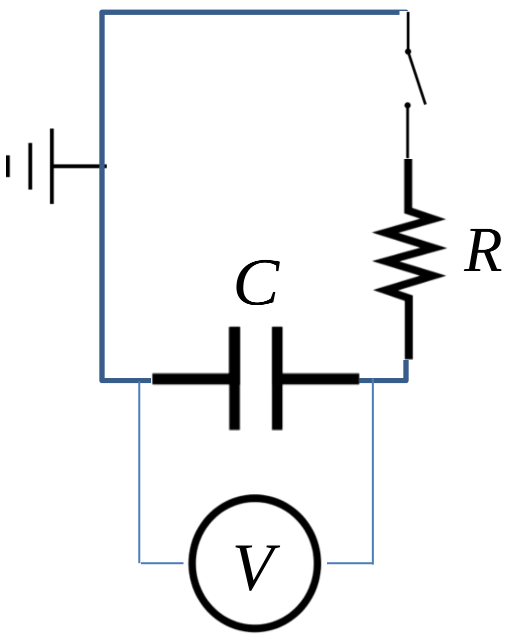
$$\frac{1}{RC}\int_0^t dt = -\int_{V_0}^{V}\frac{dV}{V}$$

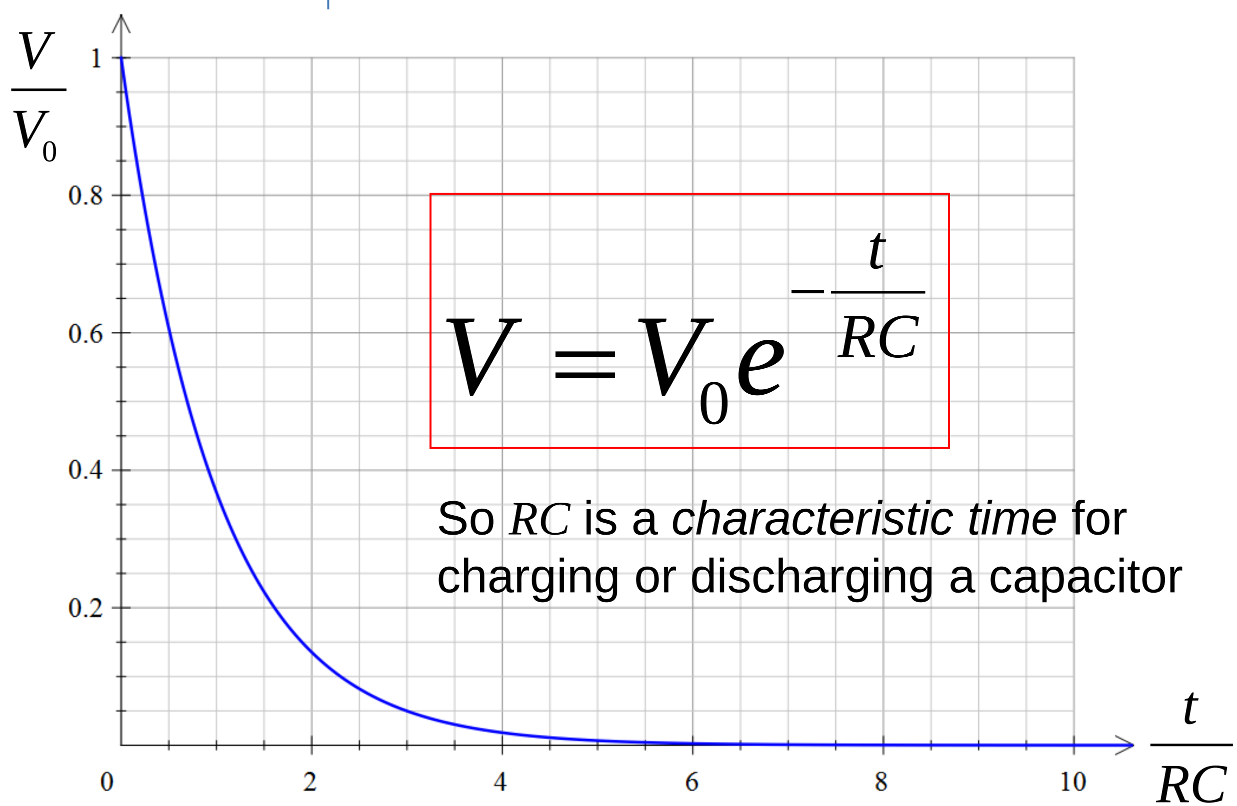
$$\frac{t}{RC} = -\left[\ln|V|\right]_{V_0}^{V}$$

$$\frac{t}{RC} = -\ln\left(\frac{V}{V_0}\right)$$

$$V = V_0 e^{-\frac{t}{RC}}$$

Note $V=V_0$ when $t = 0$





$$V = V_0 e^{-\frac{t}{RC}}$$

So RC is a *characteristic time* for charging or discharging a capacitor

Charging a capacitor using a DC source

$$Q = CV$$

capacitor charge, voltage relationship

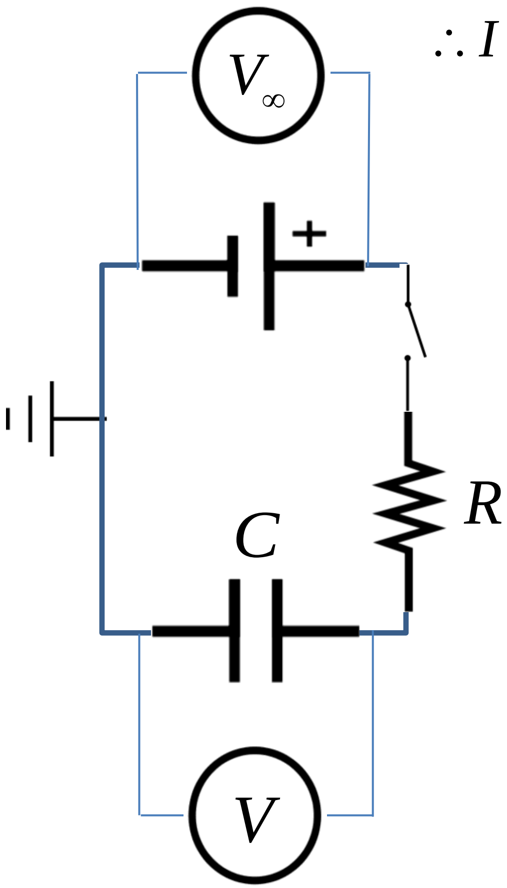
$$V_\infty - V = IR$$

Ohm's law

$$I = \frac{dQ}{dt}$$

Definition of current

$$\therefore I = \frac{V_\infty - V}{R} = C\frac{dV}{dt}$$

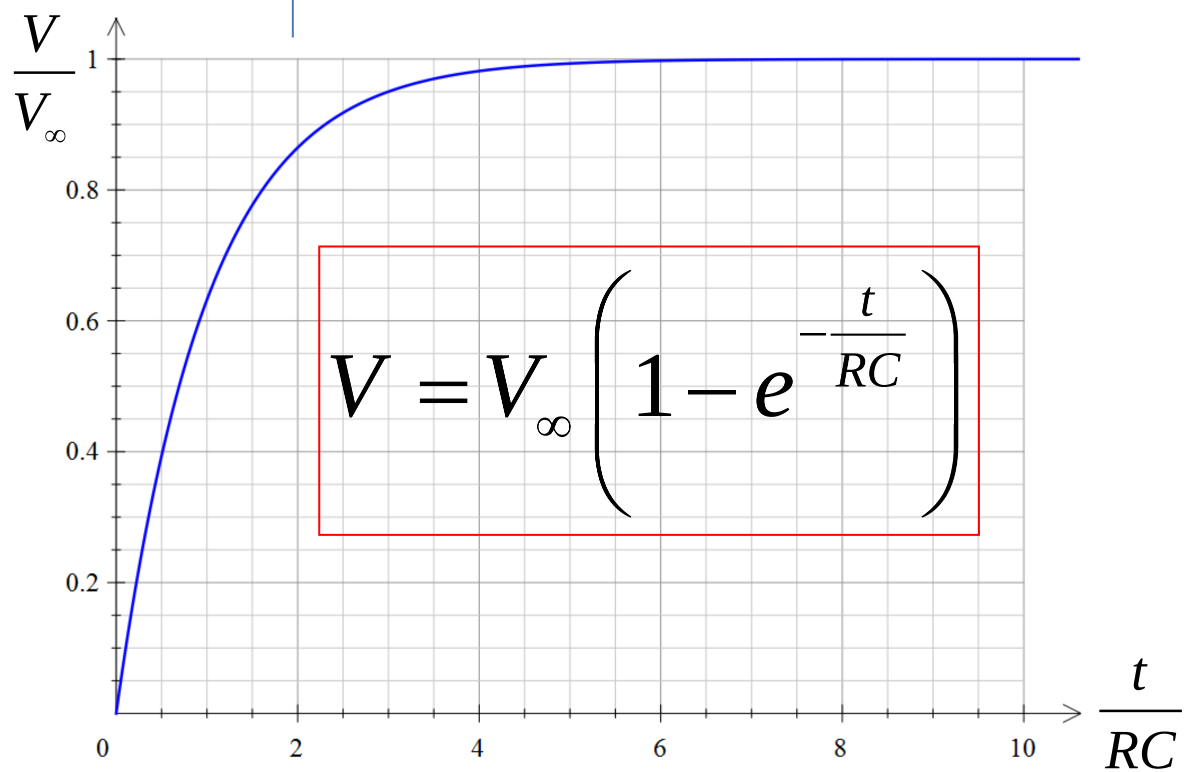


$$\frac{1}{RC}\int_0^t dt = \int_0^V \frac{dV}{V_\infty - V} = -\int_0^V \frac{-dV}{V_\infty - V}$$

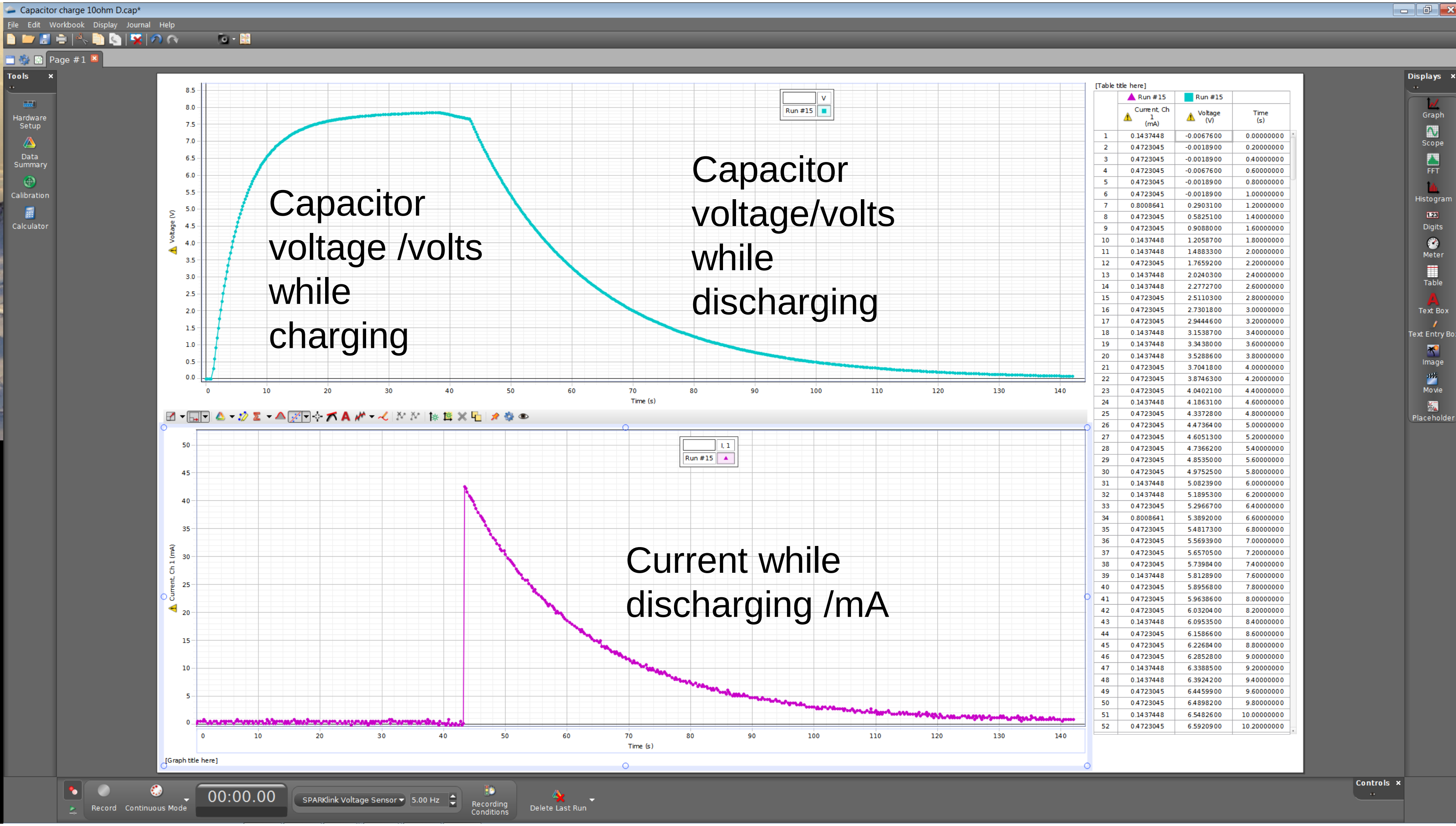
$$\frac{t}{RC} = -\Big[\ln\left|V_\infty - V\right|\Big]_0^V$$

$$-\frac{t}{RC} = \ln\left(V_\infty - V\right) - \ln\left(V_\infty\right) = \ln\left(\frac{V_\infty - V}{V_\infty}\right)$$

$$\frac{V_\infty - V}{V_\infty} = e^{-\frac{t}{RC}}$$

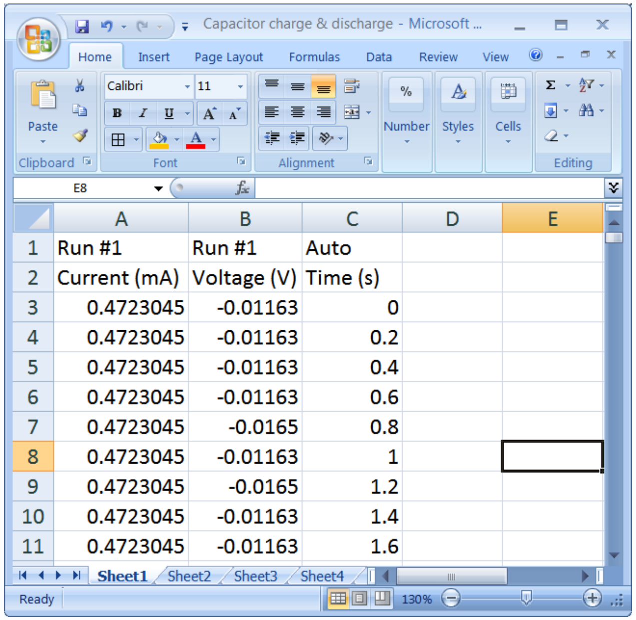


$$V = V_\infty\left(1 - e^{-\frac{t}{RC}}\right)$$



Charge and discharge recorded using Capstone software, interfacing via USB to the PASCO datalogger hub. Note Ammeter is in series with discharge loop, so no current recorded during charging.

Capstone → Copy and paste data to text files (one per discharge resistance)

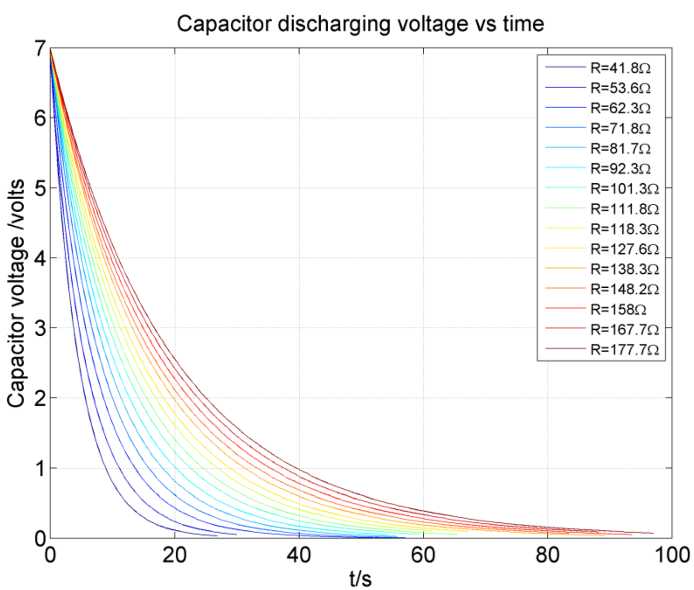


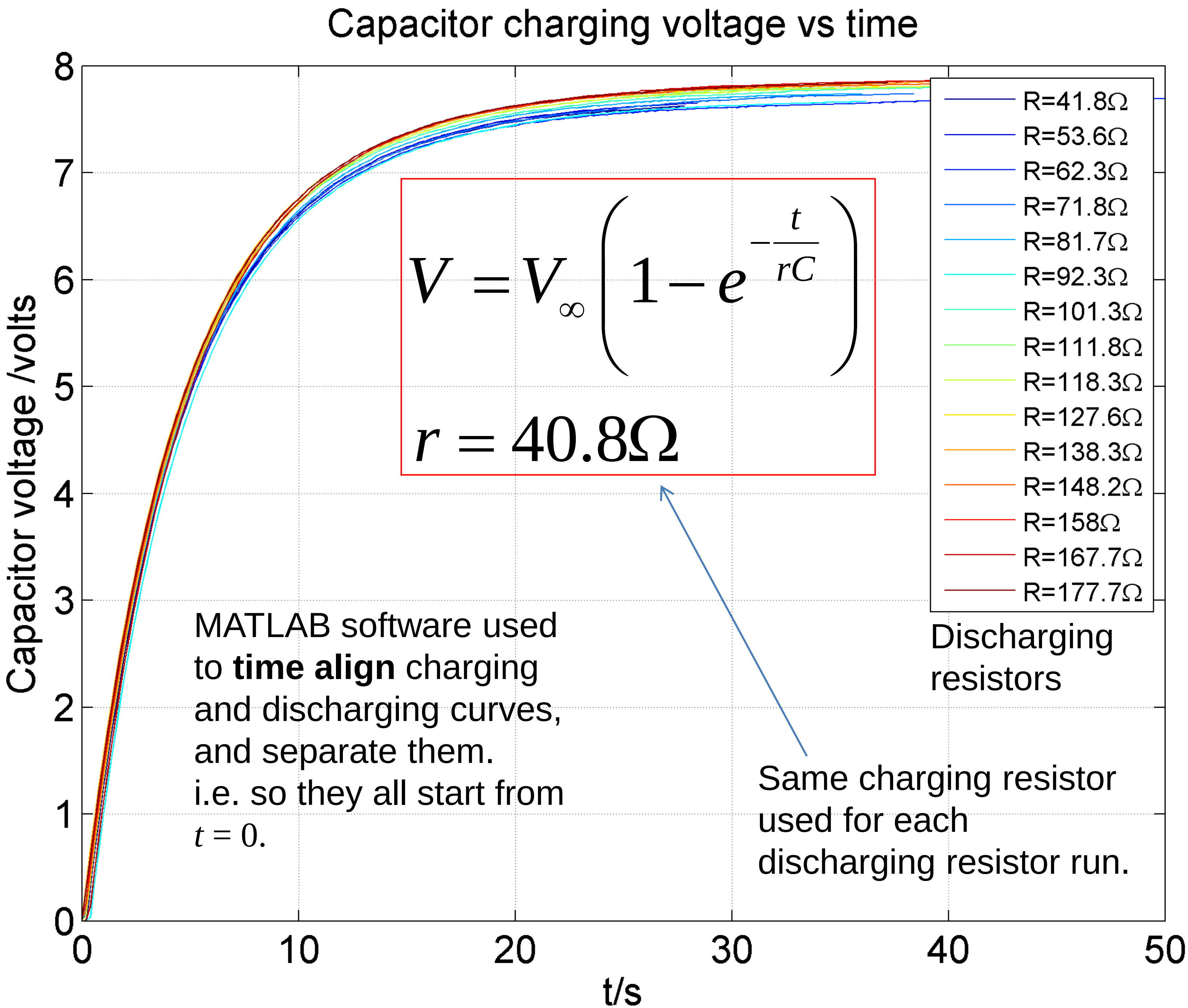
	A	B	C	D	E
1	Run #1	Run #1	Auto		
2	Current (mA)	Voltage (V)	Time (s)		
3	0.4723045	-0.01163	0		
4	0.4723045	-0.01163	0.2		
5	0.4723045	-0.01163	0.4		
6	0.4723045	-0.01163	0.6		
7	0.4723045	-0.0165	0.8		
8	0.4723045	-0.01163	1		
9	0.4723045	-0.0165	1.2		
10	0.4723045	-0.01163	1.4		
11	0.4723045	-0.01163	1.6		

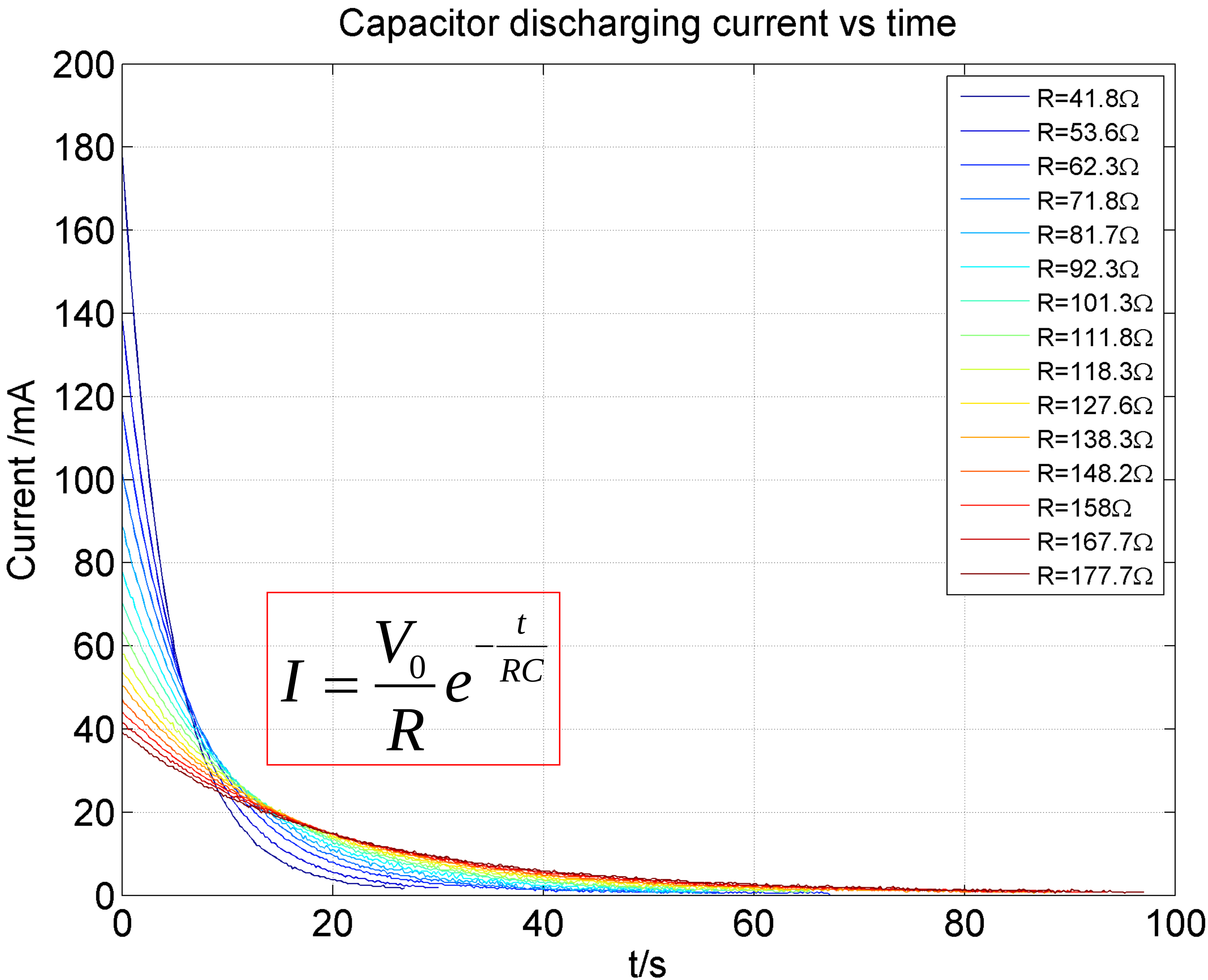
```
%Import Capacitor charge & discharge data
% LAST UPDATED by Andy French Mar 2020

function import_data
disp(' '); disp(' Importing data from Excel...' )

%Import resistances /ohms
[num,txt,raw] = xlsread( 'Capacitor charge & discharge.xlsx',...
    'Resistances' );
R = num(:,2).';

%Import data from Excel
num_runs = 15;
for n=1:15
    [num,txt,raw] = xlsread( 'Capacitor charge & discharge.xlsx',...
        ['Sheet',num2str(n)] );
    data(n).I_mA = num(:,1);
    data(n).V_volts = num(:,2);
    data(n).t_s = num(:,3);
    data(n).R_ohms = R(n);
end

%Save data to a .mat file
save( 'capacitor data','data','R' );
disp(' Data saved to file capacitor_data.mat. ']);

%End of code
```

MATLAB

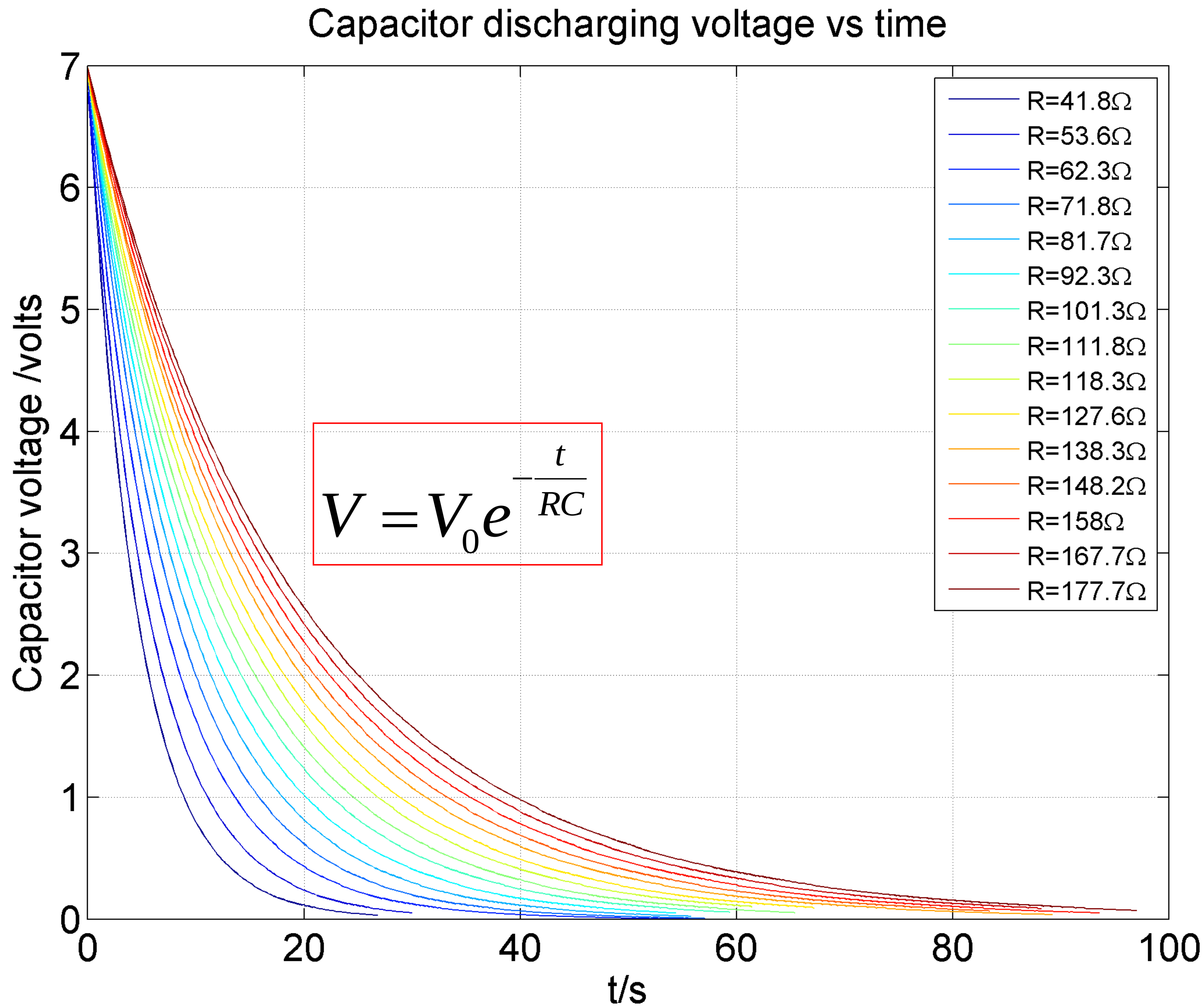
```
%Process Capacitor charge & discharge data
% LAST UPDATED by Andy French Mar 2020
function process_data

%Fontsize for graphs
fsize = 16;

%Load imported data
load('capacitor data')

%Initialize charge, discharge and line of best fit curves
fig_Vcharge = figure; ax_V_charge = axes('nextplot','add'); grid on; set(gca,'f
xlabel('t/s','fontsize',fsize); ylabel('Capacitor voltage /volts','fontsize',fs
title('Capacitor charging voltage vs time','fontsize',fsize)
fig_Vdischarge = figure; ax_V_discharge = axes('nextplot','add'); grid on; set(
xlabel('t/s','fontsize',fsize); ylabel('Capacitor voltage /volts','fontsize',fs
title('Capacitor discharging voltage vs time','fontsize',fsize)
fig_Idischarge = figure; ax_I_discharge = axes('nextplot','add'); grid on; set(
xlabel('t/s','fontsize',fsize); ylabel('Current /mA','fontsize',fsize);
title('Capacitor discharging current vs time','fontsize',fsize)
fig_best_fit = figure; ax_best_fit = axes('nextplot','add'); grid on; set(gca,'
xlabel('t/s','fontsize',fsize); ylabel('ln(V/volts)','fontsize',fsize);
title('ln(V) vs t line of best fit to find RC time','fontsize',fsize)
fig_V_over_I = figure; ax_V_over_I = axes('nextplot','add'); grid on; set(gca,'
xlabel('t/s','fontsize',fsize); ylabel('R = V/I /\Omega','fontsize',fsize);
title('R = V/I for capacitor discharge','fontsize',fsize)

%Determine legend string
lgnd_str = {};
```



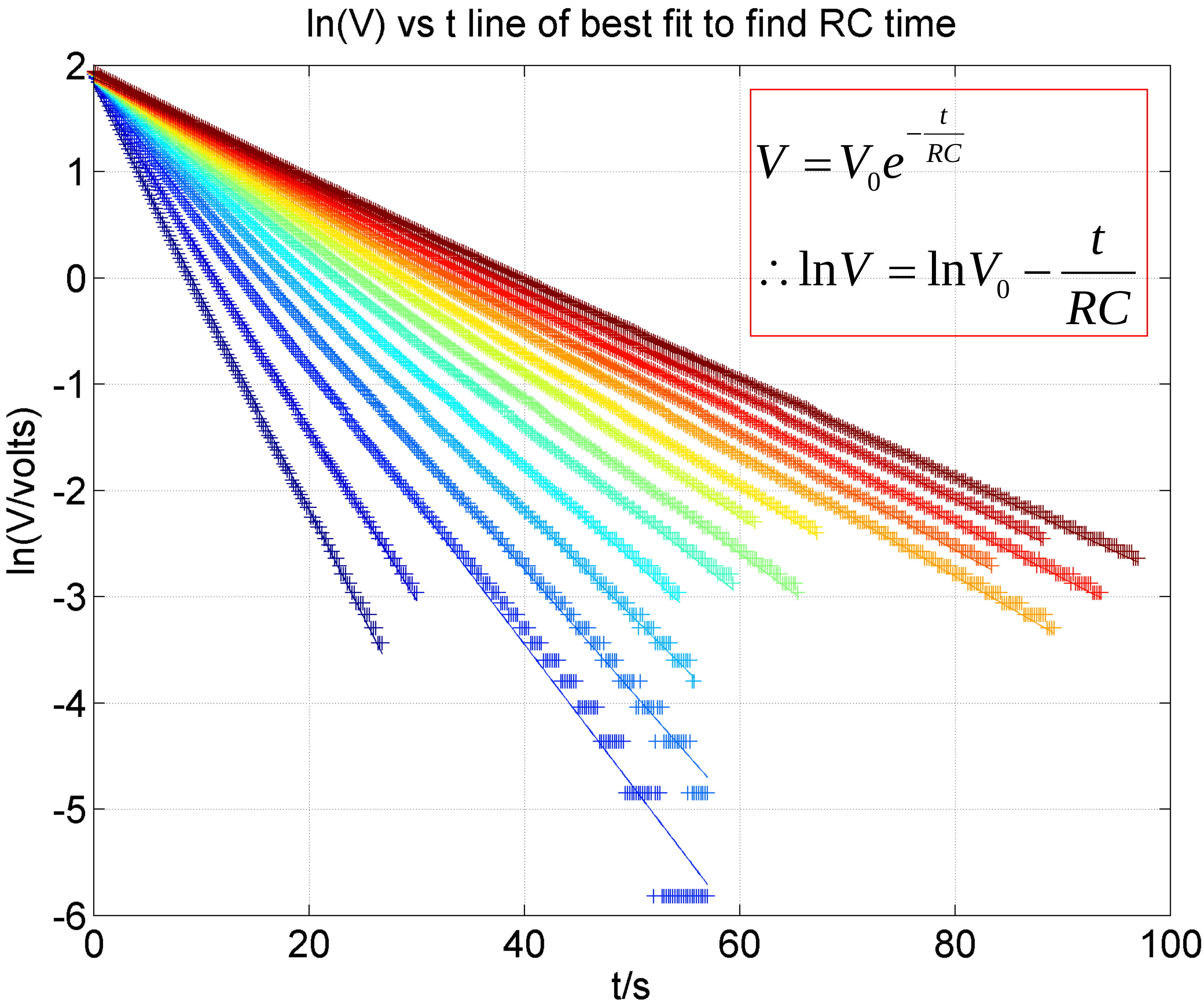

Capacitor charging voltage vs time
Capacitor voltage /volts
t/s
0
1
2
3
4
5
6
7
8
10
20
30
40
50
R=41.8Ω
R=53.6Ω
R=62.3Ω
R=71.8Ω
R=81.7Ω
R=92.3Ω
R=101.3Ω
R=111.8Ω
R=118.3Ω
R=127.6Ω
R=138.3Ω
R=148.2Ω
R=158Ω
R=167.7Ω
R=177.7Ω
Discharging resistors
$V = V_\infty \left(1 - e^{-\frac{t}{rC}}\right)$
$r = 40.8\Omega$
MATLAB software used to **time align** charging and discharging curves, and separate them. i.e. so they all start from $t = 0$.
Same charging resistor used for each discharging resistor run.



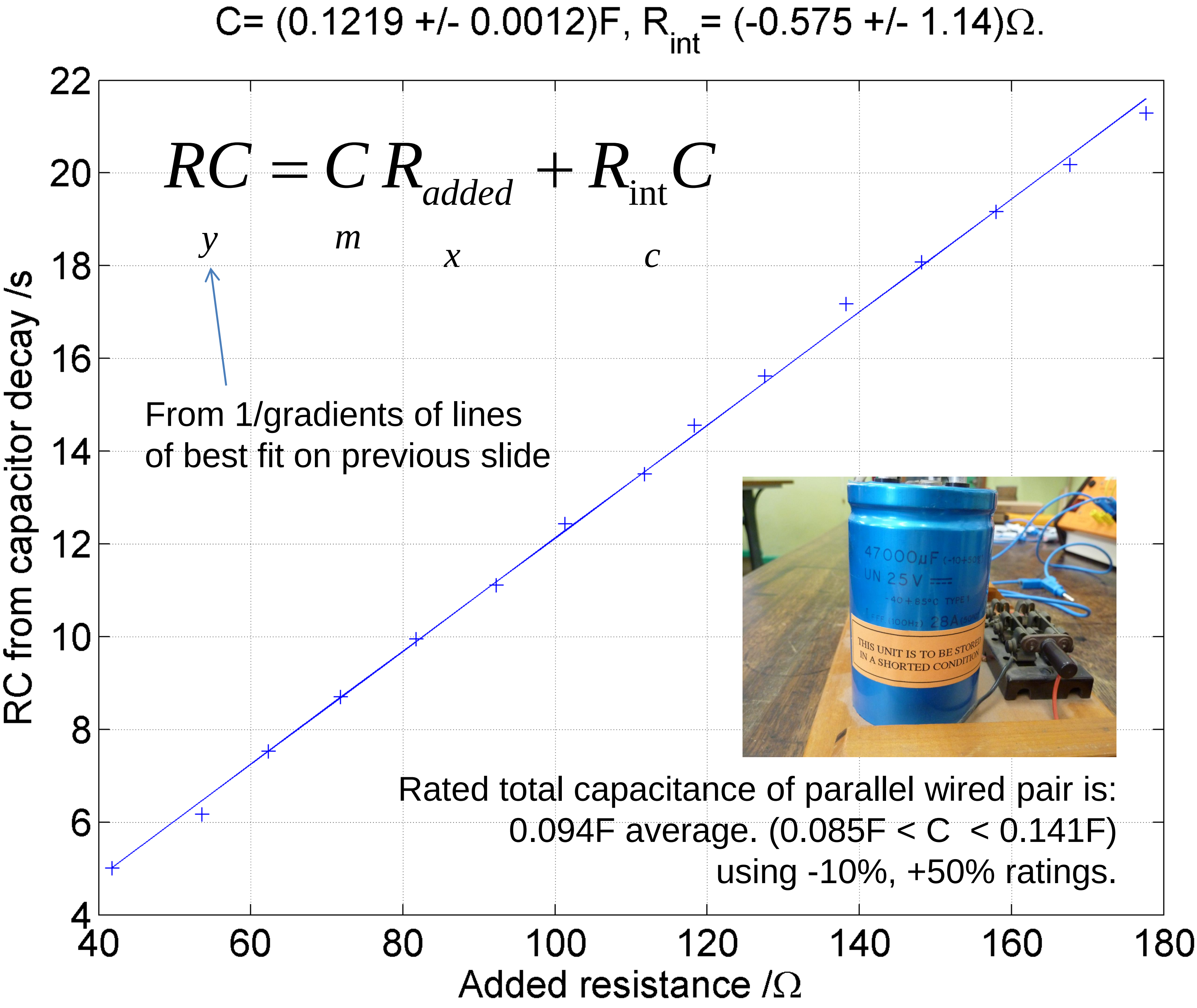
Capacitor discharging current vs time
Current /mA
t/s
0
20
40
60
80
100
120
140
160
180
200
0
20
40
60
80
100
R=41.8Ω
R=53.6Ω
R=62.3Ω
R=71.8Ω
R=81.7Ω
R=92.3Ω
R=101.3Ω
R=111.8Ω
R=118.3Ω
R=127.6Ω
R=138.3Ω
R=148.2Ω
R=158Ω
R=167.7Ω
R=177.7Ω
$I = \frac{V_0}{R} e^{-\frac{t}{RC}}$



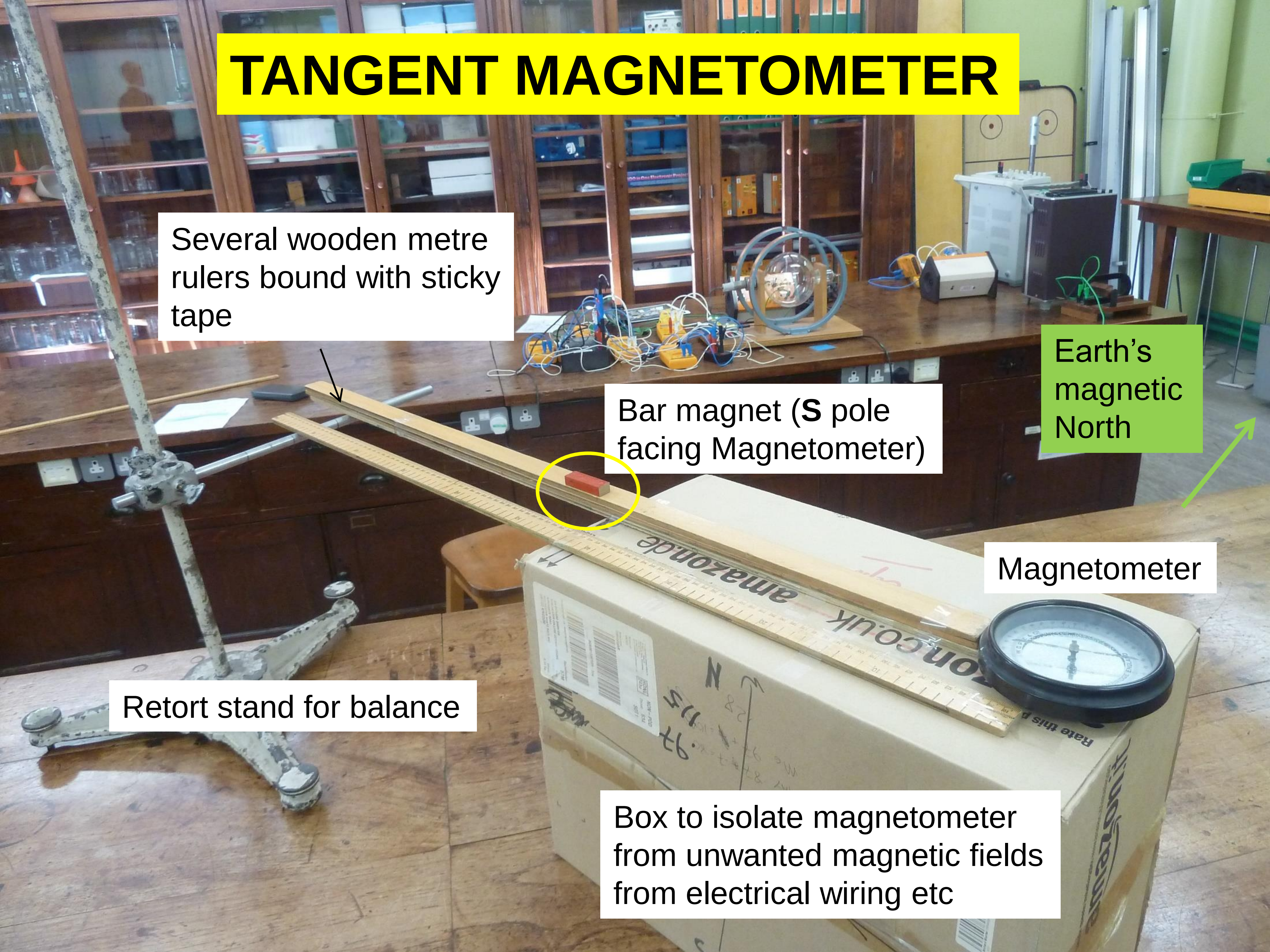
Capacitor discharging voltage vs time
Capacitor voltage /volts
t/s
0
1
2
3
4
5
6
7
0
20
40
60
80
100
R=41.8Ω
R=53.6Ω
R=62.3Ω
R=71.8Ω
R=81.7Ω
R=92.3Ω
R=101.3Ω
R=111.8Ω
R=118.3Ω
R=127.6Ω
R=138.3Ω
R=148.2Ω
R=158Ω
R=167.7Ω
R=177.7Ω
$V = V_0 e^{-\frac{t}{RC}}$



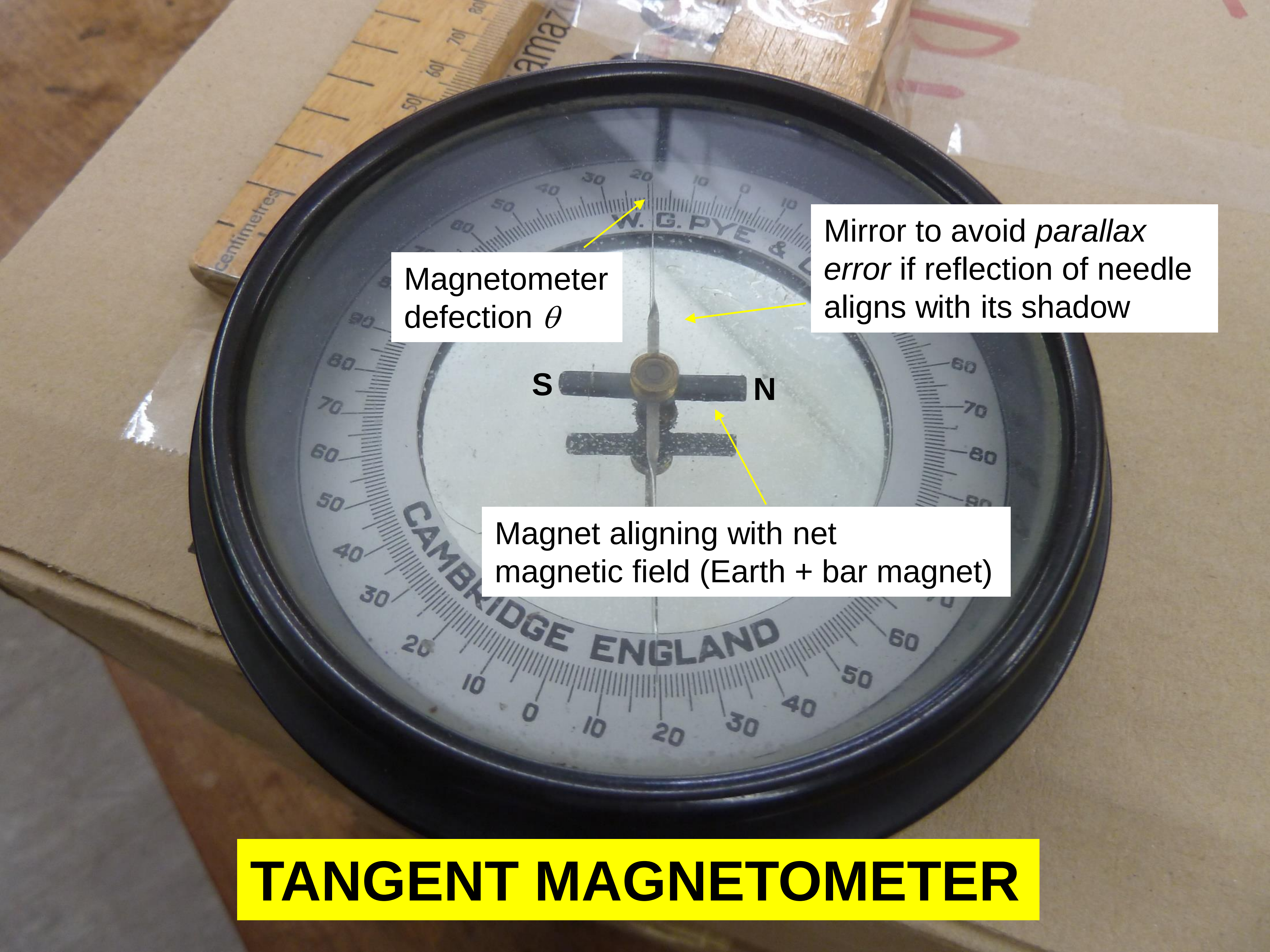
ln(V) vs t line of best fit to find RC time
$V = V_0 e^{-\frac{t}{RC}}$
$\therefore \ln V = \ln V_0 - \frac{t}{RC}$
ln(V/volts)
t/s
2
1
0
-1
-2
-3
-4
-5
-6
0
20
40
60
80
100

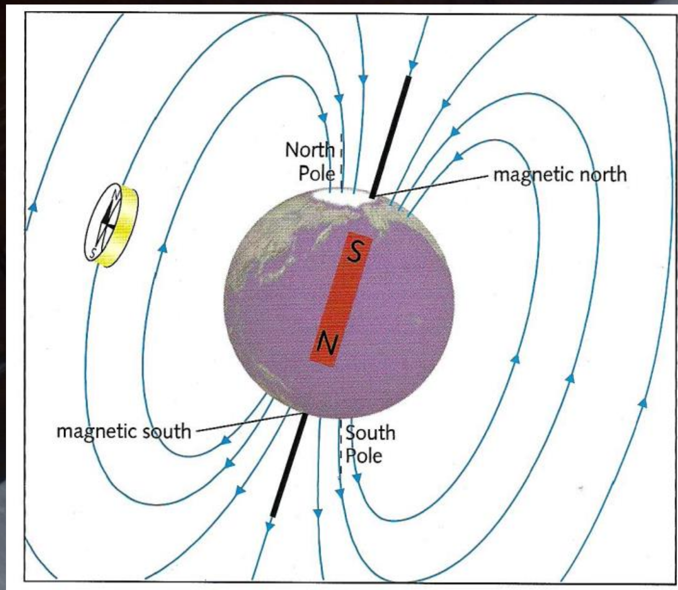

C= (0.1219 +/- 0.0012)F, R_{int}= (-0.575 +/- 1.14)Ω.
$RC = C\,R_{added} + R_{int}C$
y m x c
From 1/gradients of lines
of best fit on previous slide
47000µF
UN 25 V
THIS UNIT IS TO BE STORED
IN A SHORTED CONDITION
Rated total capacitance of parallel wired pair is:
0.094F average. (0.085F < C < 0.141F)
using -10%, +50% ratings.
RC from capacitor decay /s
Added resistance /Ω
4
6
8
10
12
14
16
18
20
22
40
60
80
100
120
140
160
180

TANGENT MAGNETOMETER



TANGENT MAGNETOMETER

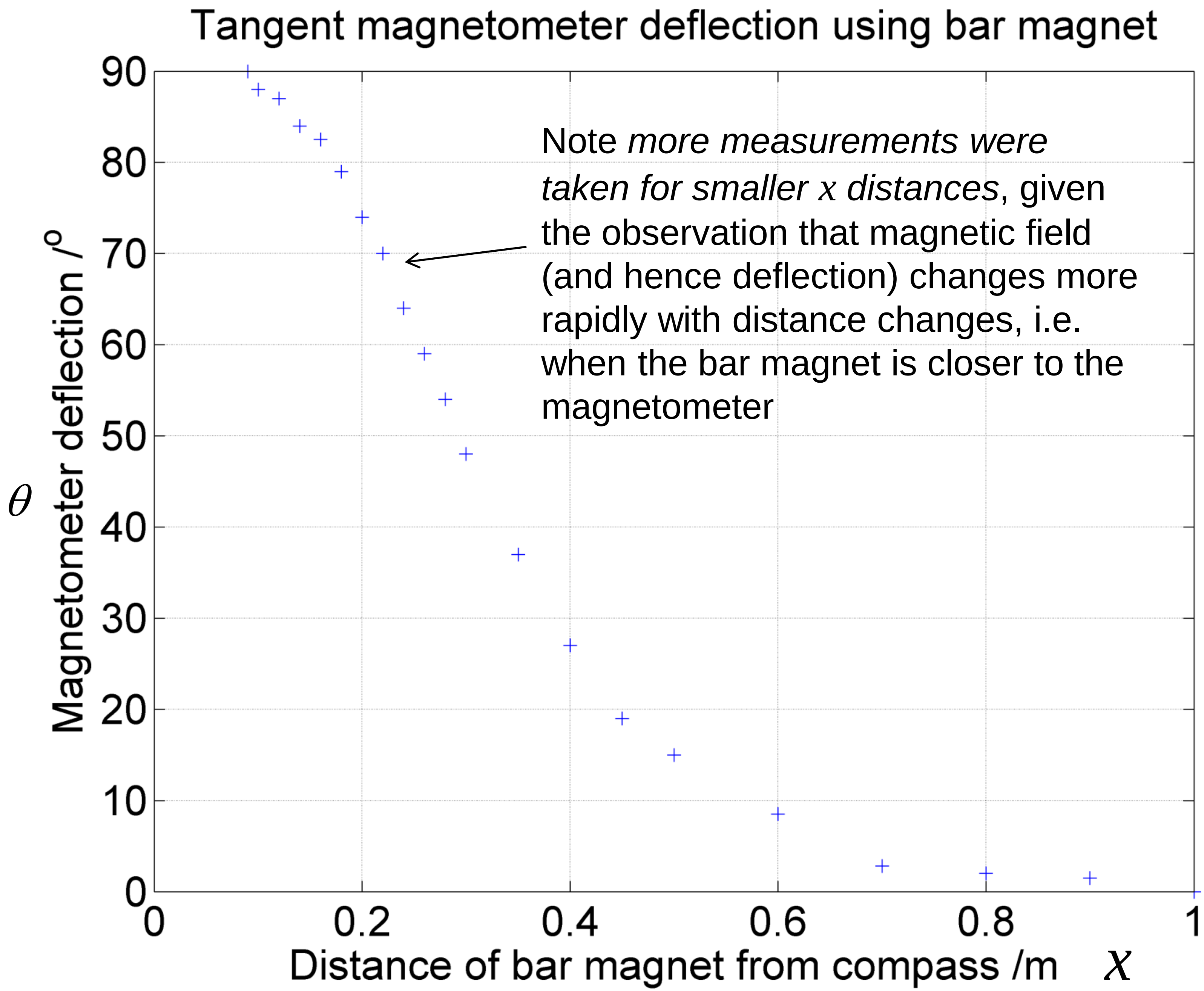




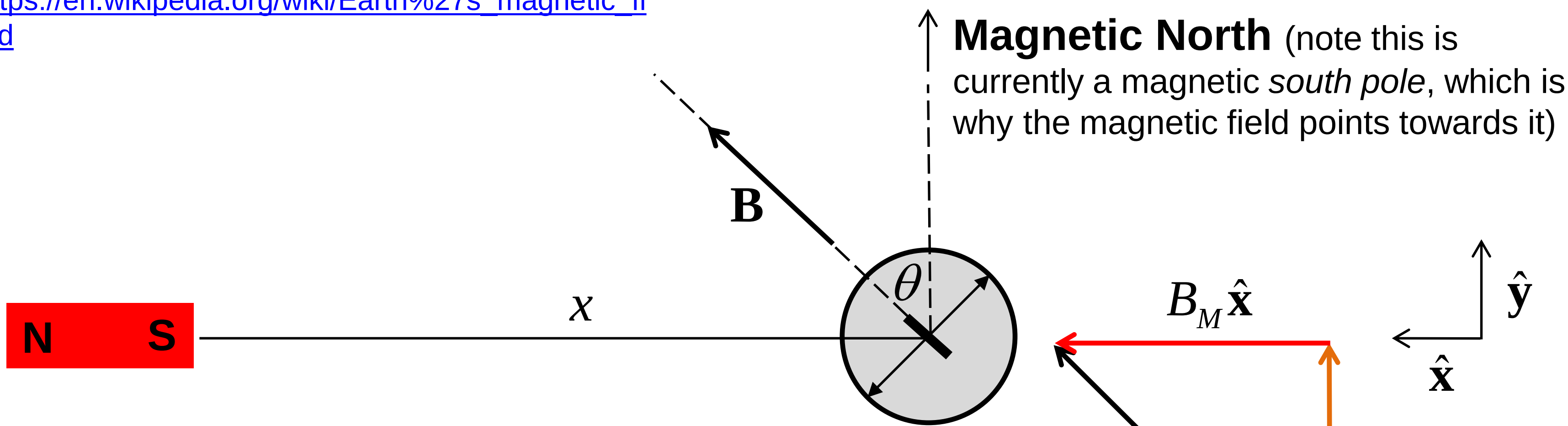
Note by convention magnetic field lines point **towards the south pole** and **emerge from the north pole**.

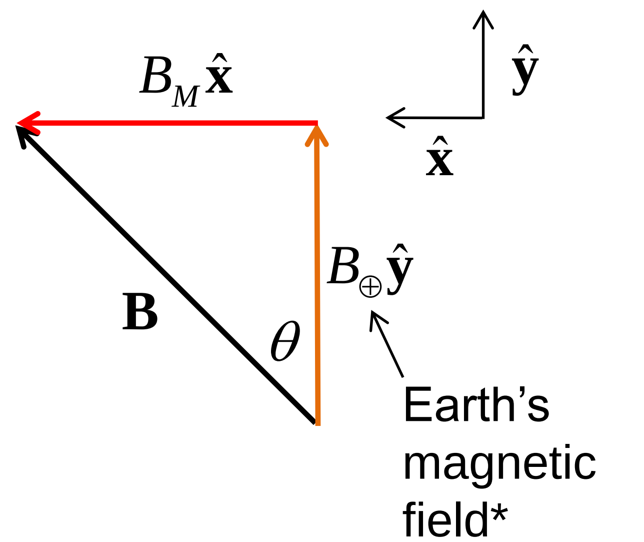
Note also that, as of 11 Nov 2017, geomagnetic north is actually a south pole! (i.e. *field lines point north*, not south).

TANGENT MAGNETOMETER


Tangent magnetometer deflection using bar magnet
Note *more measurements were taken for smaller x distances*, given the observation that magnetic field (and hence deflection) changes more rapidly with distance changes, i.e. when the bar magnet is closer to the magnetometer
θ
Magnetometer deflection /°
90
80
70
60
50
40
30
20
10
0
0
0.2
0.4
0.6
0.8
1
Distance of bar magnet from compass /m
x

* https://en.wikipedia.org/wiki/Earth%27s_magnetic_field

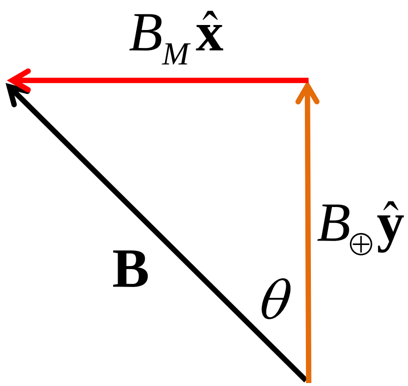




Net magnetic field acting on magnetometer magnet (which it aligns with) is:

$$\mathbf{B} = B_M\hat{\mathbf{x}} + B_\oplus\hat{\mathbf{y}}$$

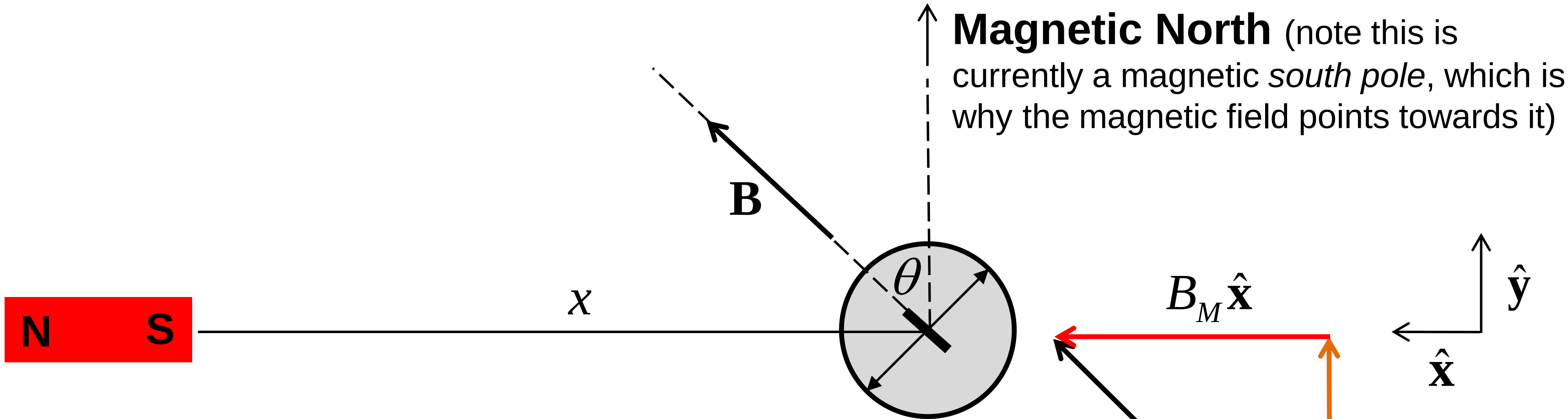
$$25\mu\mathrm{T} < B_\oplus < 65\mu\mathrm{T}$$

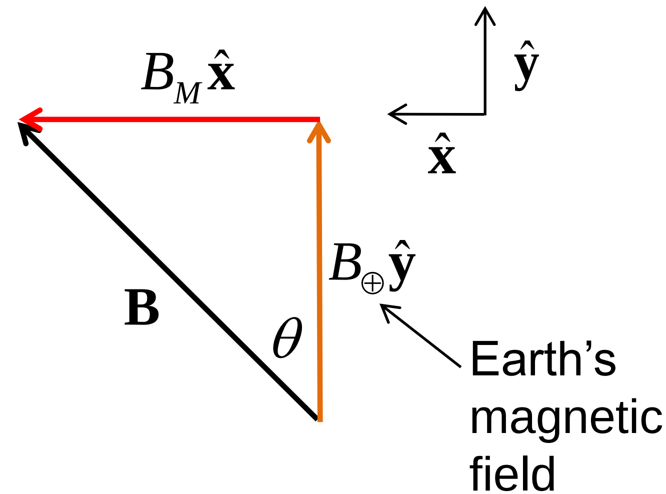


Hence:

$$\frac{B_M}{B_\oplus} = \tan\theta$$

$B_M\hat{\mathbf{x}}$ is the magnetic field due to the bar magnet





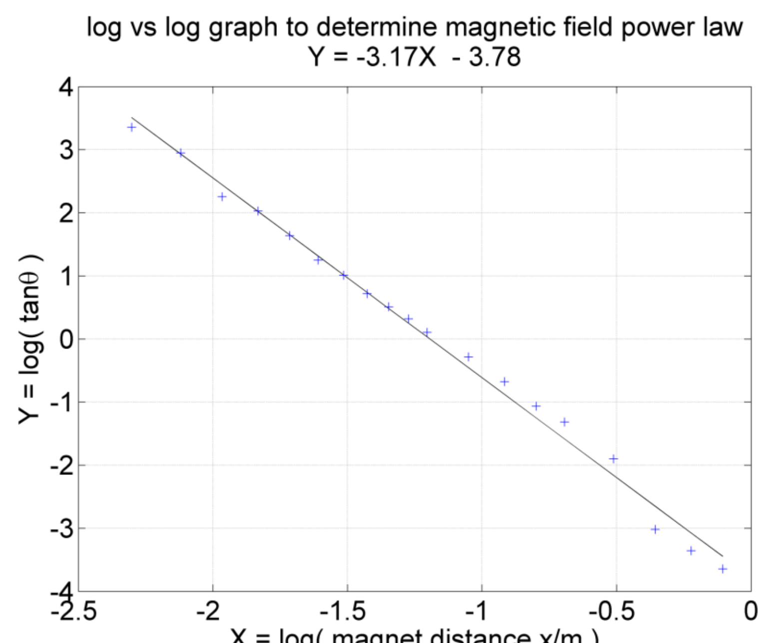
$$\frac{B_M}{B_\oplus} = \frac{k}{x^n}$$

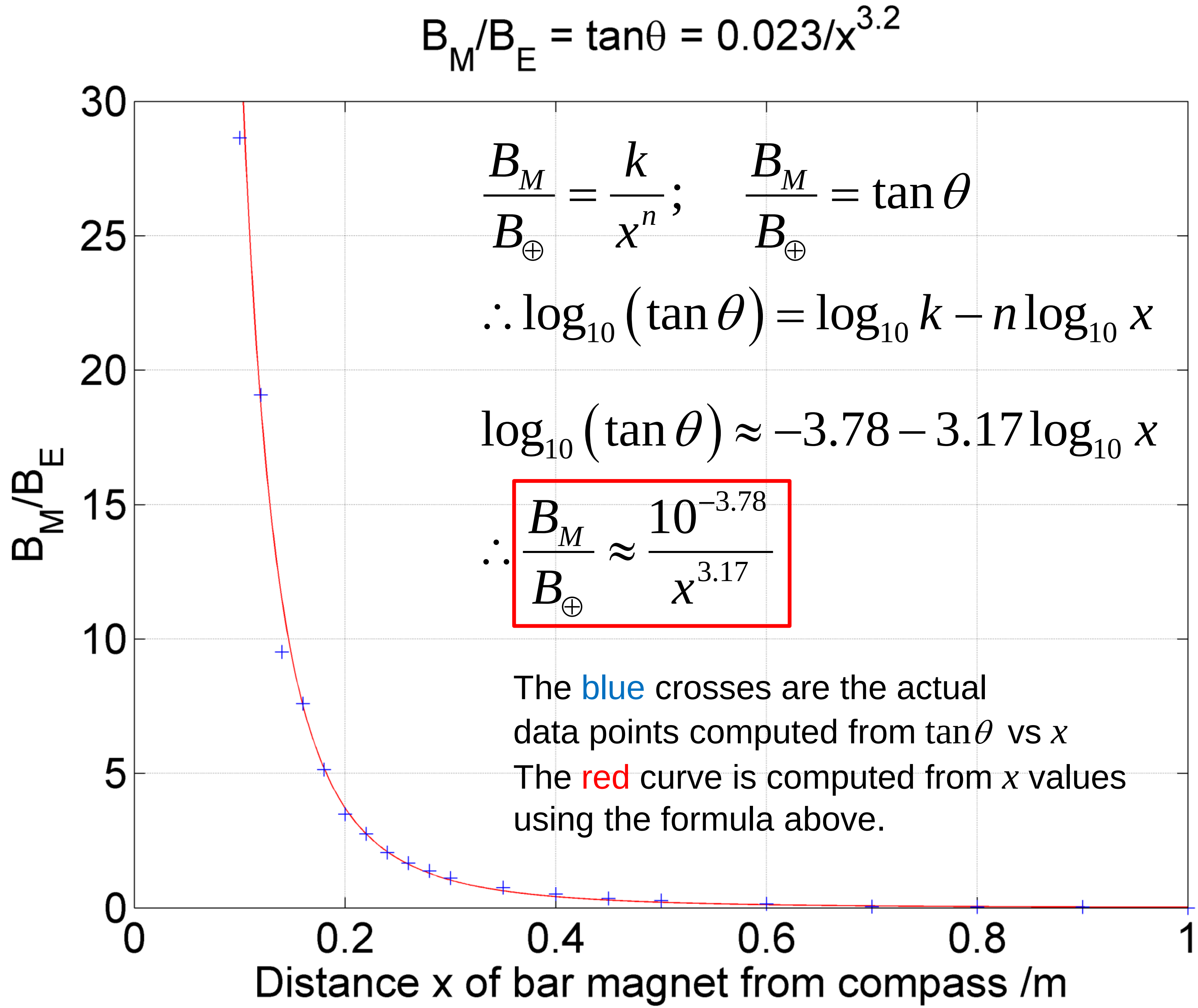
Postulate a **power-law** variation of bar magnet field strength with distance. *k* is a constant.

$$\frac{B_M}{B_\oplus} = \tan\theta$$

$$\therefore \log_{10}(\tan\theta) = \log_{10} k - n \log_{10} x$$

So plotting $Y = \log_{10}(\tan\theta)$ vs $X = \log_{10} x$ should yield a **straight line** with gradient -*n*

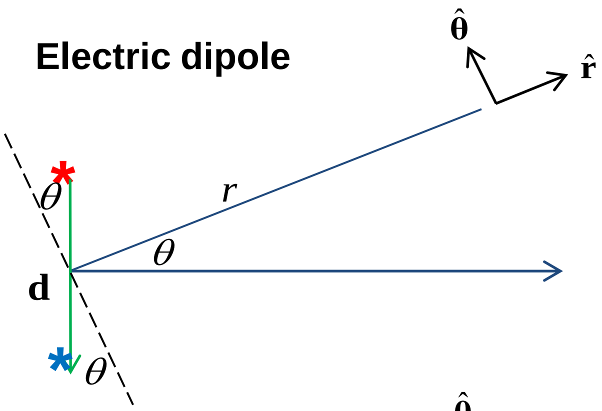




$B_M/B_E = \tan\theta = 0.023/x^{3.2}$
$\frac{B_M}{B_\oplus} = \frac{k}{x^n}; \quad \frac{B_M}{B_\oplus} = \tan\theta$
$\therefore \log_{10}(\tan\theta) = \log_{10} k - n \log_{10} x$
$\log_{10}(\tan\theta) \approx -3.78 - 3.17 \log_{10} x$
$\therefore \frac{B_M}{B_\oplus} \approx \frac{10^{-3.78}}{x^{3.17}}$
The blue crosses are the actual data points computed from $\tan\theta$ vs x
The red curve is computed from x values using the formula above.
B_M/B_E
Distance x of bar magnet from compass /m
0
5
10
15
20
25
30
0.2
0.4
0.6
0.8
1

The field of a **Magnetic dipole** is mathematically very similar to that of an electric dipole (see Electric dipole notes).

Electric dipole



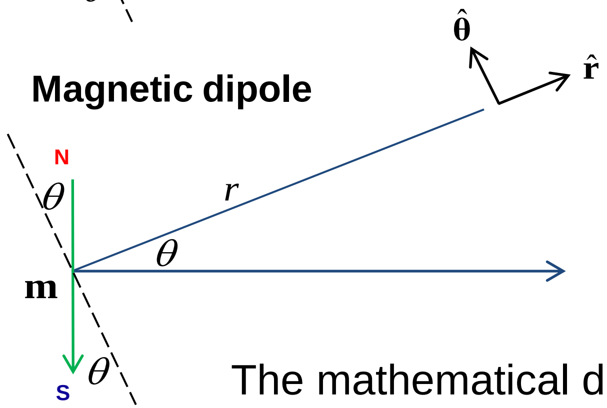
$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 r^3}\left(2\hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta\right)$$

In both cases assume r is much greater than the dimensions associated with the dipole

This explains why the variation of magnetic field strength vs distance is $B_M \propto r^{-3}$

We measured -3.2 as the power.

Magnetic dipole



$$\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\hat{\mathbf{r}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta\right)$$

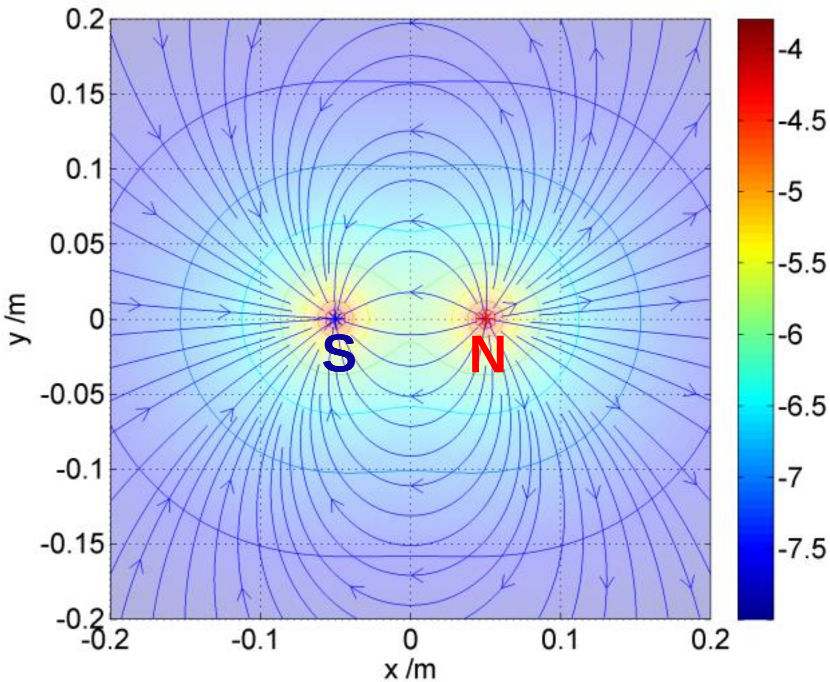
Permittivity of free space $\varepsilon_0 = 8.854\times10^{-12}\,\mathrm{Fm}^{-1}$

Permeability of free space $\mu_0 = 4\pi\times10^{-7}\,\mathrm{Hm}^{-1}$

The mathematical difference between the electric and magnetic dipoles is the quantity

$$\frac{qd}{\varepsilon_0} \to \mu_0 m$$

m is the **magnetic dipole moment**



For a magnetic dipole formed from a small current loop (or indeed solenoid) of radius a

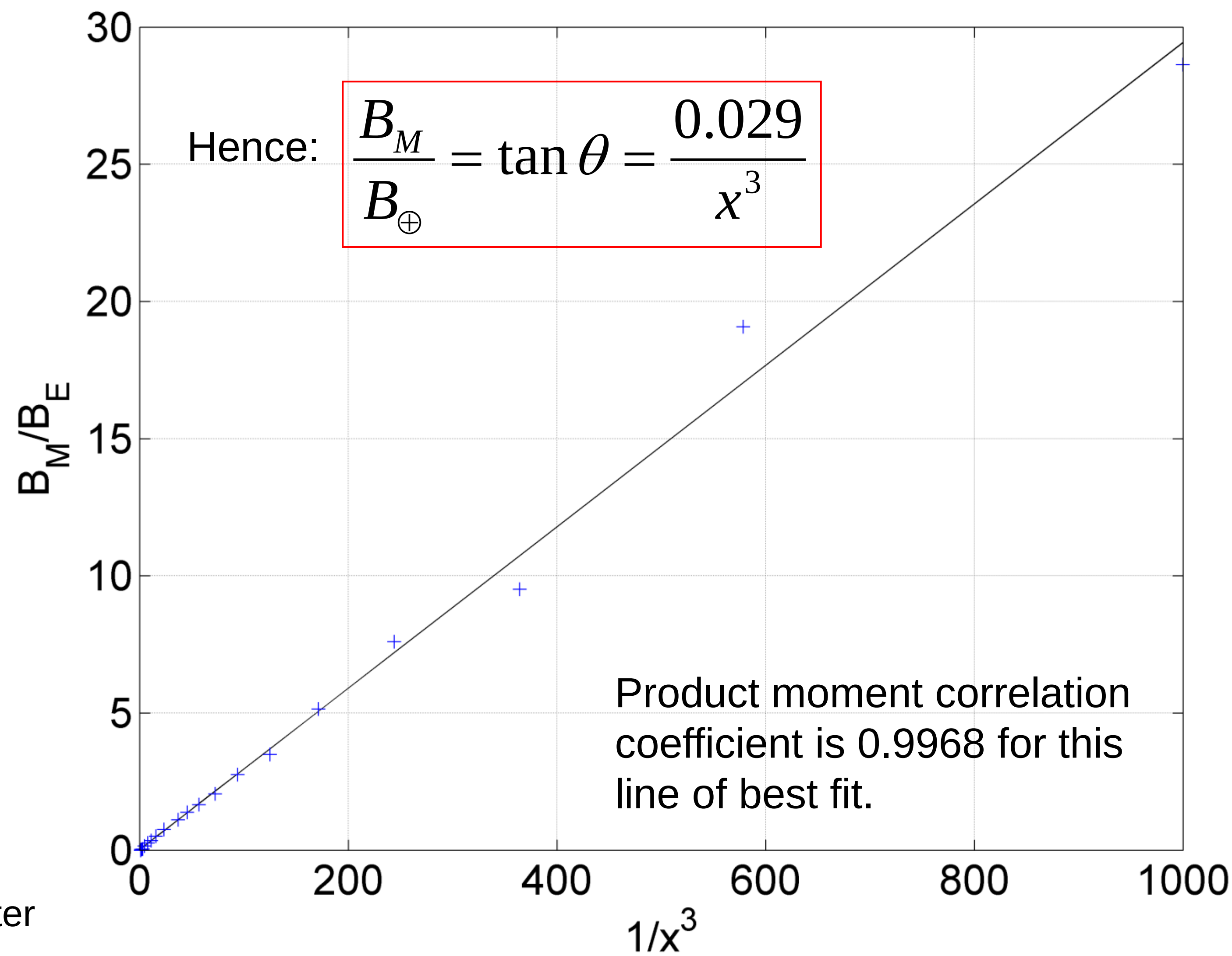
$$m = I\pi a^2$$

Rather than a curve fit using $B_M \propto r^{-3.17}$ we can construct an alternative linearization

$$\frac{B_M}{B_\oplus} = \tan\theta = \frac{k}{x^3}$$

So plotting $\tan\theta$ vs $1/x^3$ should yield a **straight line**

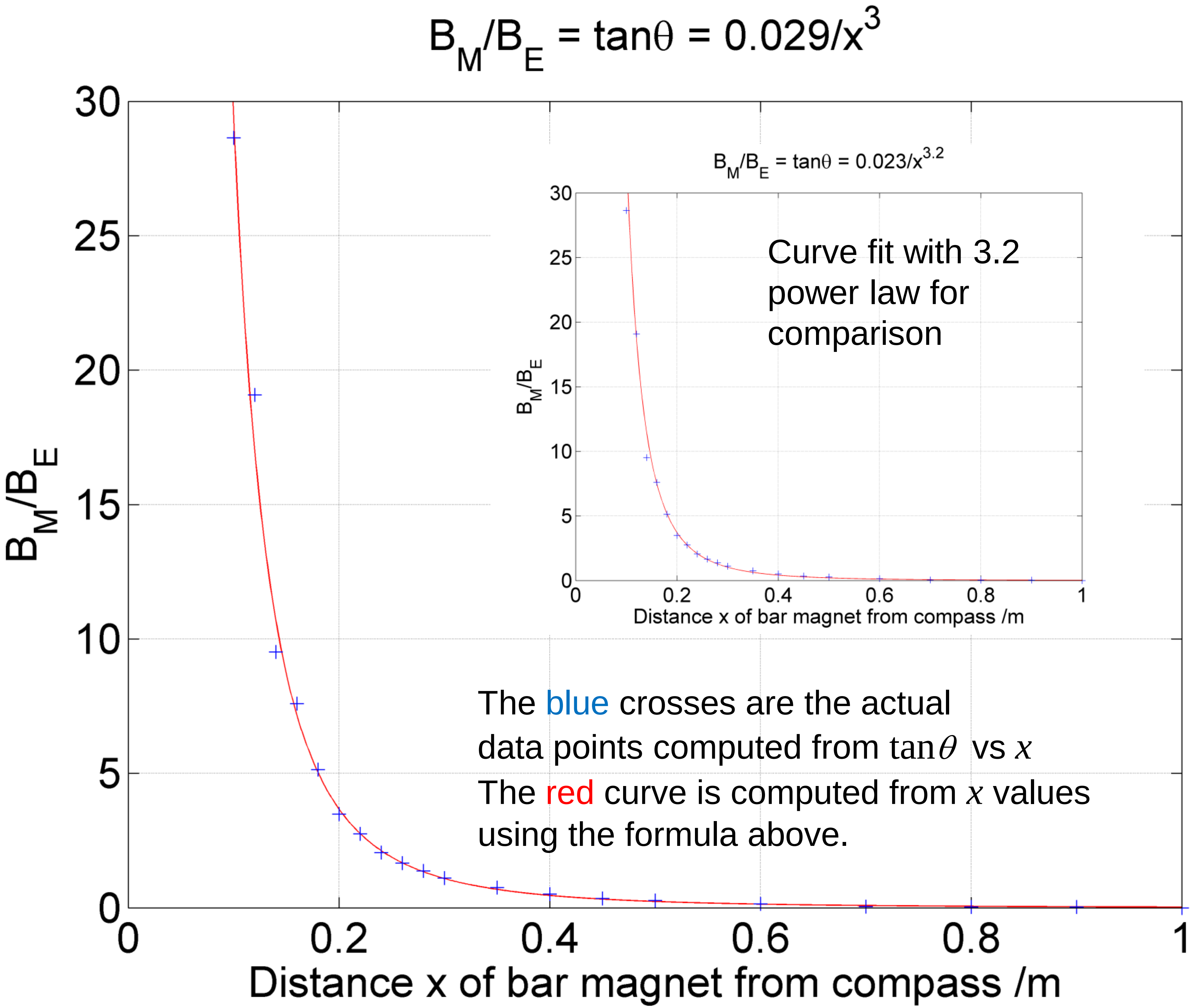
i.e. using our theoretical model of the magnetic dipole



Hence: $$\frac{B_M}{B_\oplus} = \tan\theta = \frac{0.029}{x^3}$$

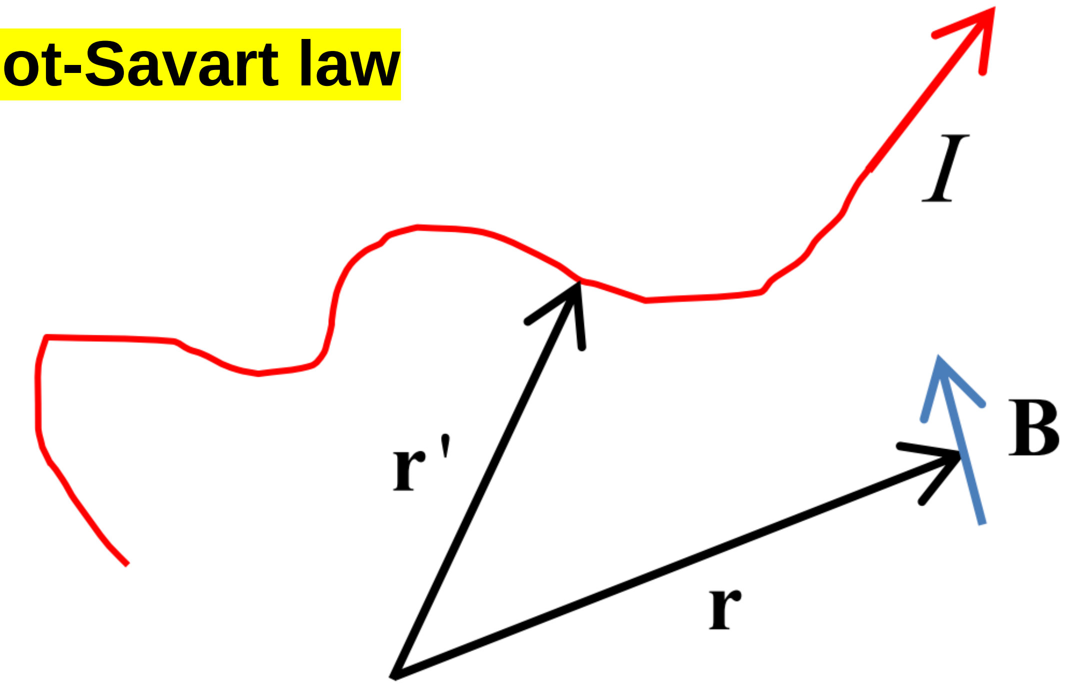
Product moment correlation coefficient is 0.9968 for this line of best fit.

Note distance x is in metres from centre of magnetometer



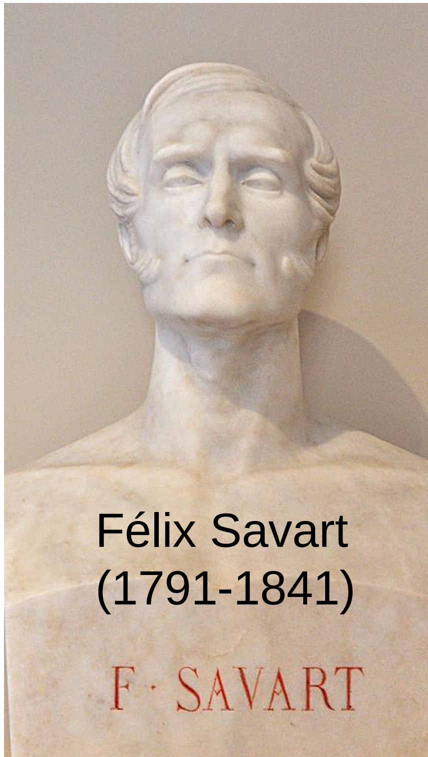
B_M/B_E = tanθ = 0.029/x^3
B_M/B_E
Distance x of bar magnet from compass /m
0
5
10
15
20
25
30
0
0.2
0.4
0.6
0.8
1
B_M/B_E = tanθ = 0.023/x^3.2
B_M/B_E
Distance x of bar magnet from compass /m
Curve fit with 3.2 power law for comparison
The blue crosses are the actual data points computed from tan θ vs x
The red curve is computed from x values using the formula above.

Biot-Savart law

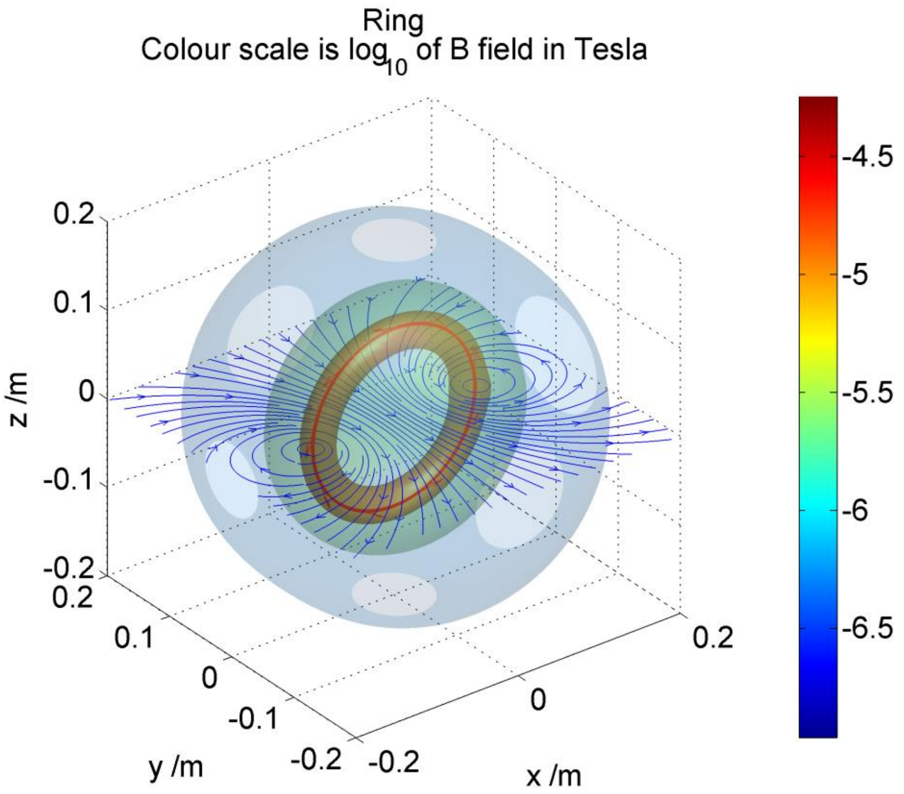


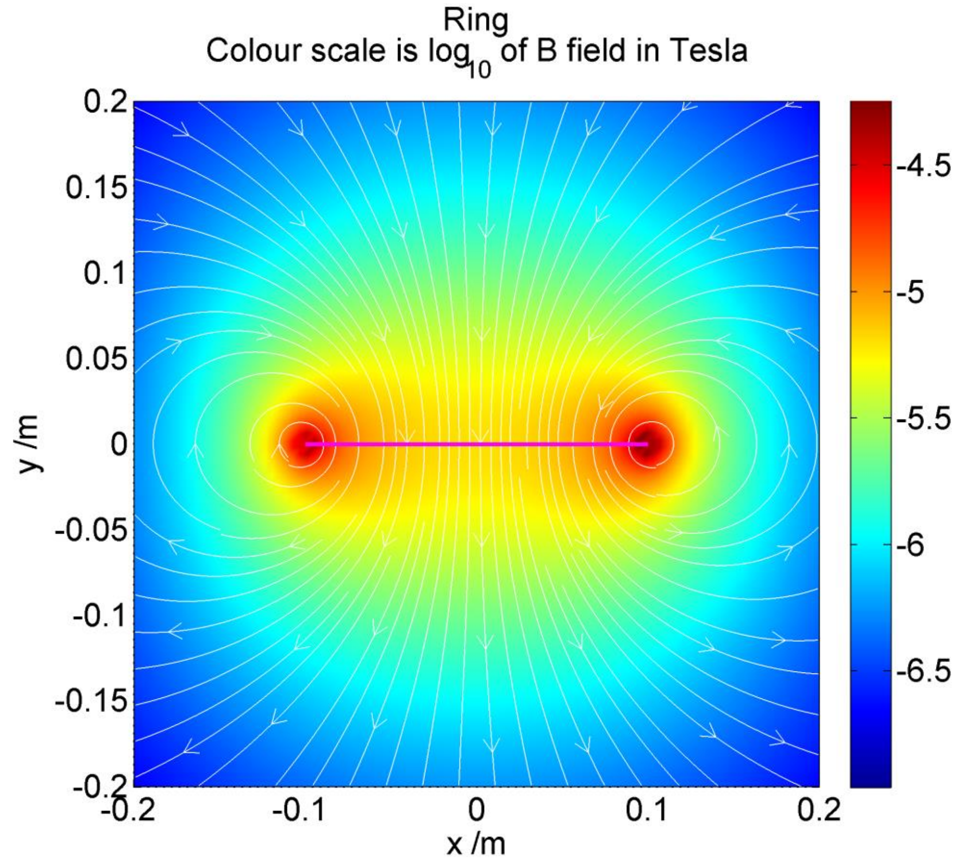
$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

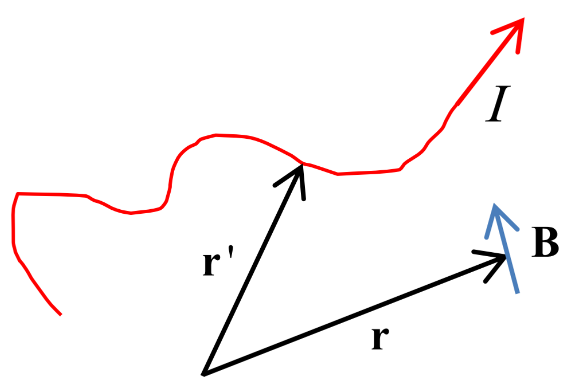
Jean-Baptiste Biot
(1774-1862)



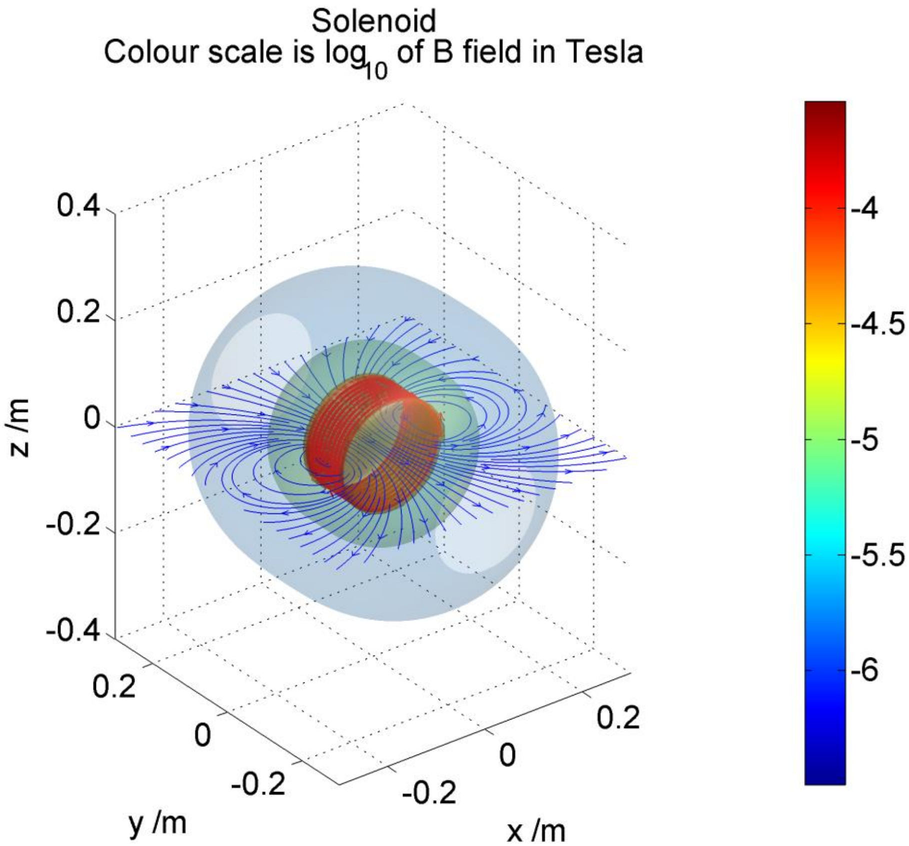
Félix Savart
(1791-1841)

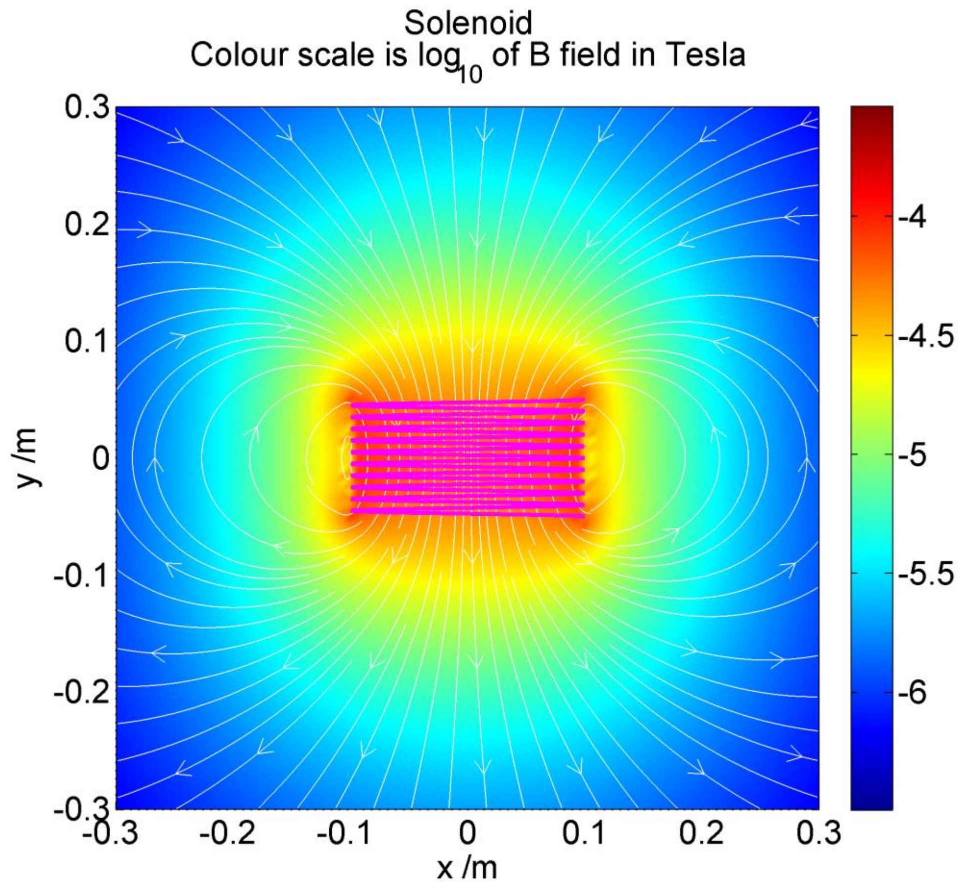


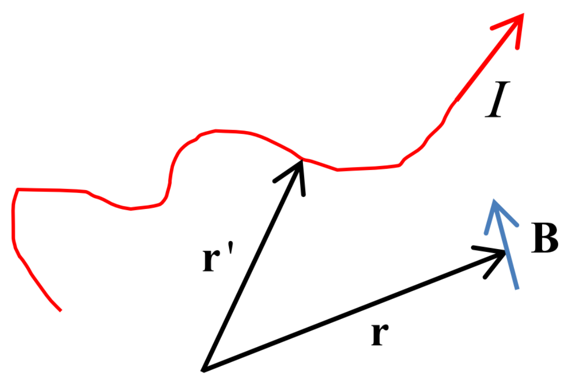




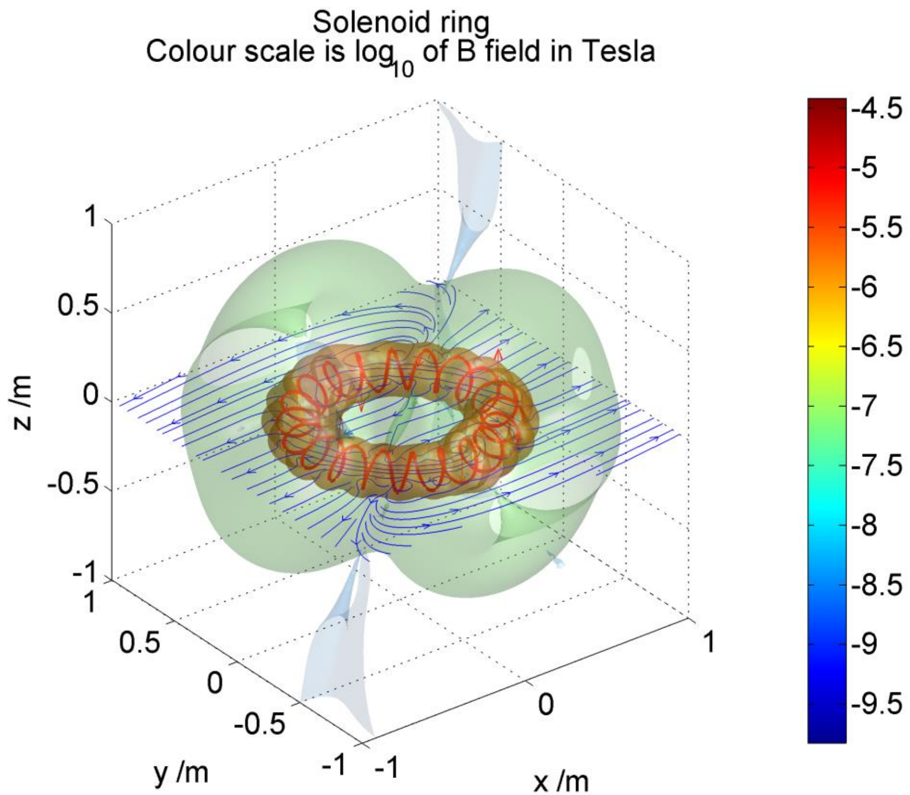
$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

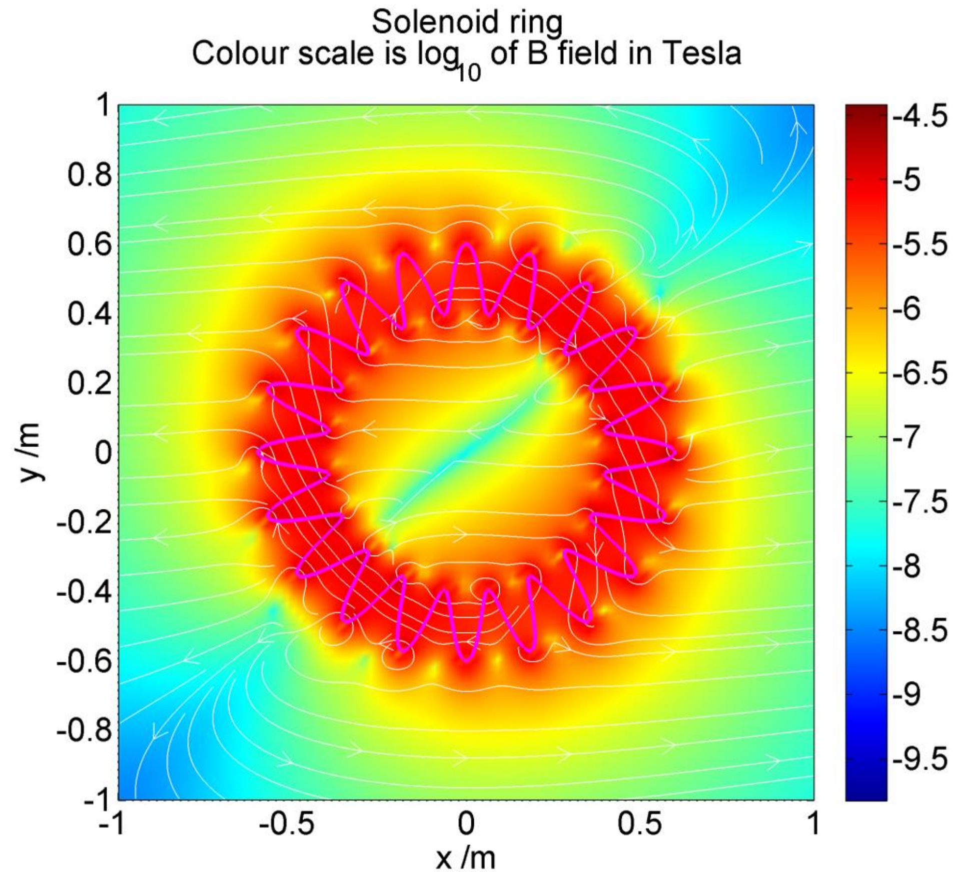


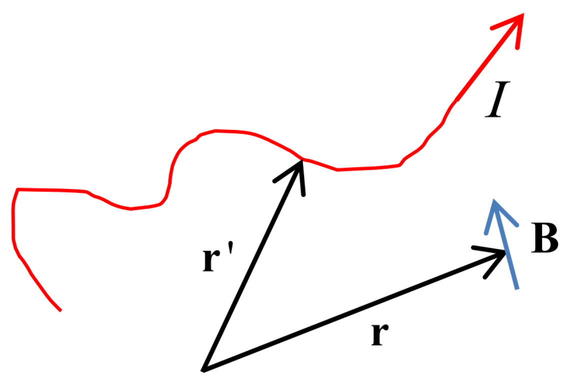




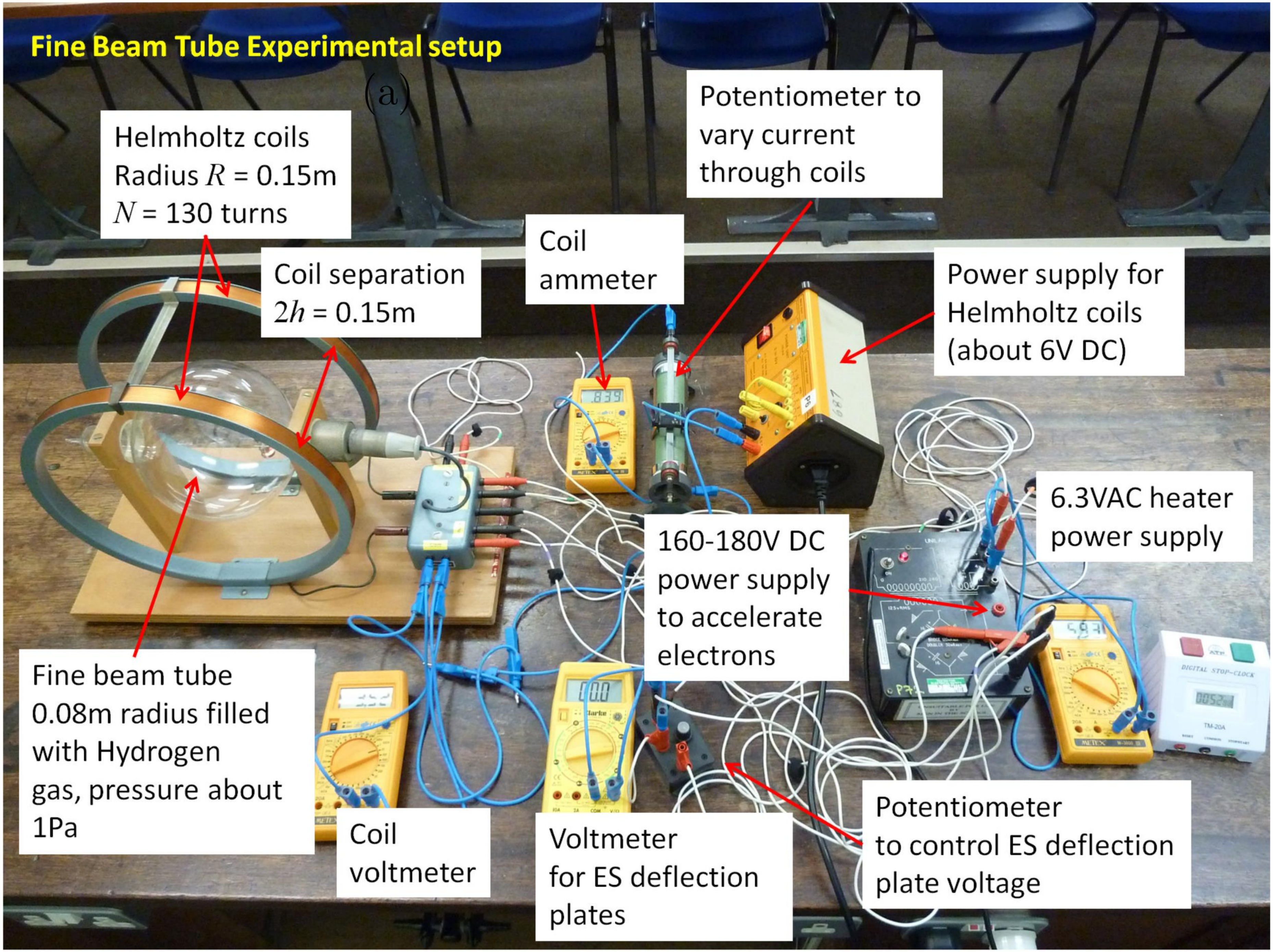
$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$







$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$


Fine Beam Tube Experimental setup
(a)
Helmholtz coils
Radius R = 0.15m
N = 130 turns
Coil separation
$2h$ = 0.15m
Coil ammeter
Potentiometer to vary current through coils
Power supply for Helmholtz coils (about 6V DC)
6.3VAC heater power supply
160-180V DC power supply to accelerate electrons
Fine beam tube 0.08m radius filled with Hydrogen gas, pressure about 1Pa
Coil voltmeter
Voltmeter for ES deflection plates
Potentiometer to control ES deflection plate voltage

Calculating the electron charge to mass ratio using a Fine Beam Tube

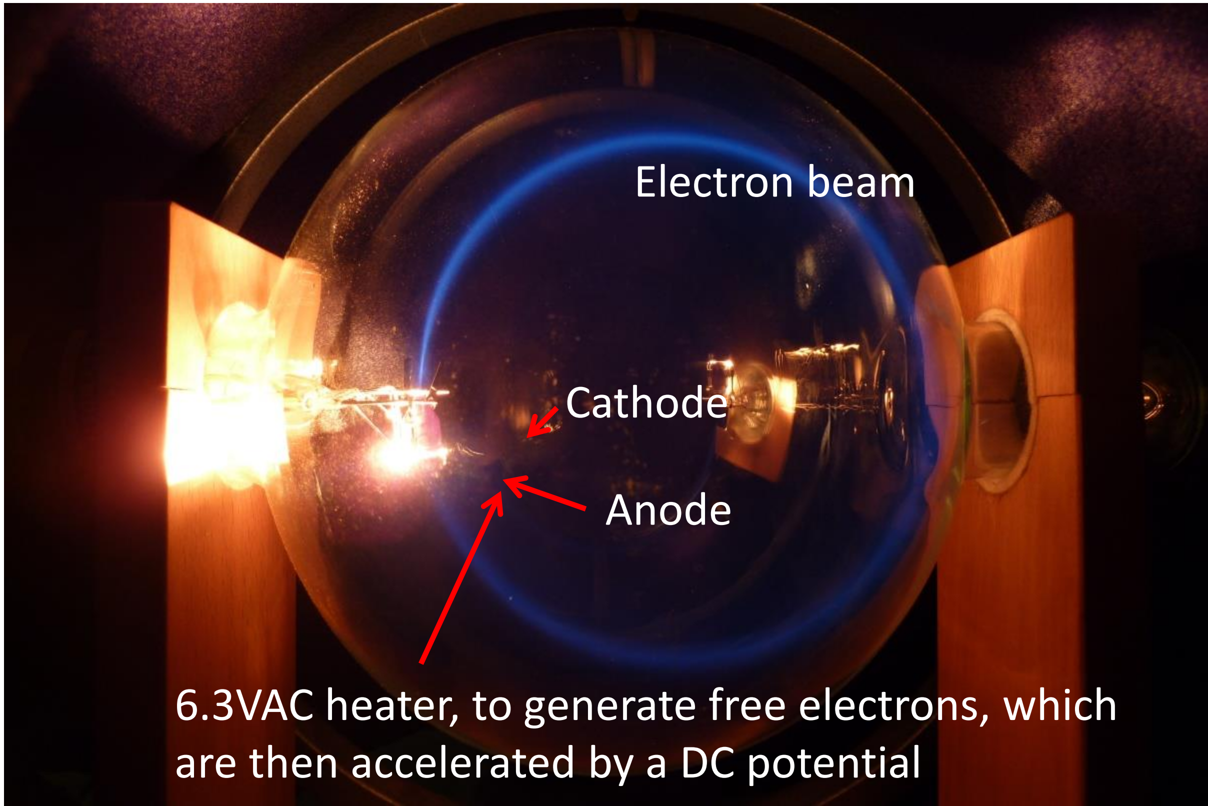
Low pressure hydrogen gas inside a spherical tube is ionized by a beam of electrons, which are accelerated via a voltage of approximately 100V. A pair of **Helmholtz coils** (solenoids) produce a highly uniform magnetic field which bends the beam into a circle.

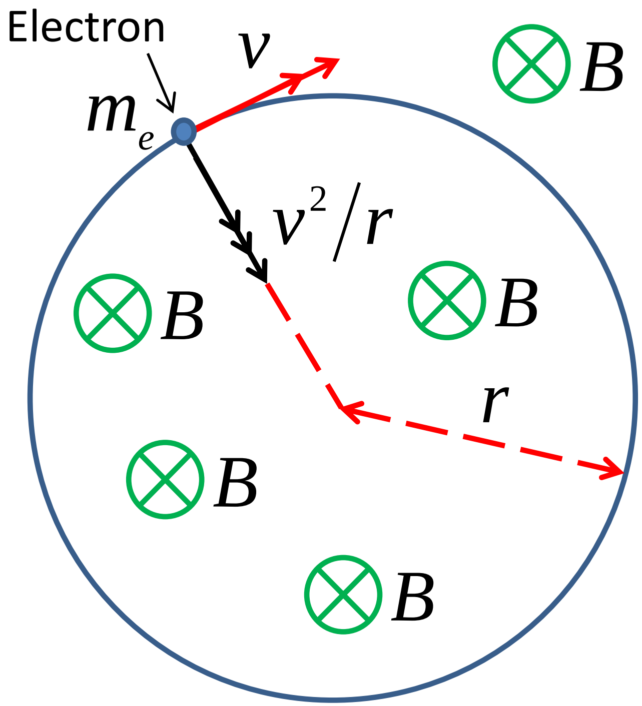
If the accelerating voltage, the coil current and the beam radius are measured, it is possible to calculate from these parameters the **electron charge to mass ratio** e/m_e

$$m_e = 9.109 \times 10^{-31}\,\text{kg}$$

$$e = 1.602 \times 10^{-19}\,\text{C}$$

$$\frac{e}{m_e} = 1.76 \times 10^{11}\,\text{Ckg}^{-1}$$





Assume *uniform* magnetic field of strength B between the Helmholtz coils.

The **force** on an electron (beyond cathode and deflection plates) is $\mathbf{F} = -e\mathbf{v} \times \mathbf{B}$

i.e. a purely *centripetal* force if the beam is initially vertical and perpendicular to the uniform magnetic field.

Newton II (+ve in radially inward direction):

$$\frac{m_e v^2}{r} = Bev \Rightarrow v = \frac{Ber}{m_e}$$

Assume electron **kinetic energy** is solely from the **accelerating potential**, and velocities are low enough such that relativistic effects can be ignored.

$$\tfrac{1}{2} m_e v^2 = eV \therefore v = \sqrt{\frac{2eV}{m_e}}$$

Hence:

$$\sqrt{\frac{2eV}{m_e}} = \frac{Ber}{m_e} \therefore \frac{2eV}{m_e} = \frac{B^2 e^2 r^2}{m_e^2}$$

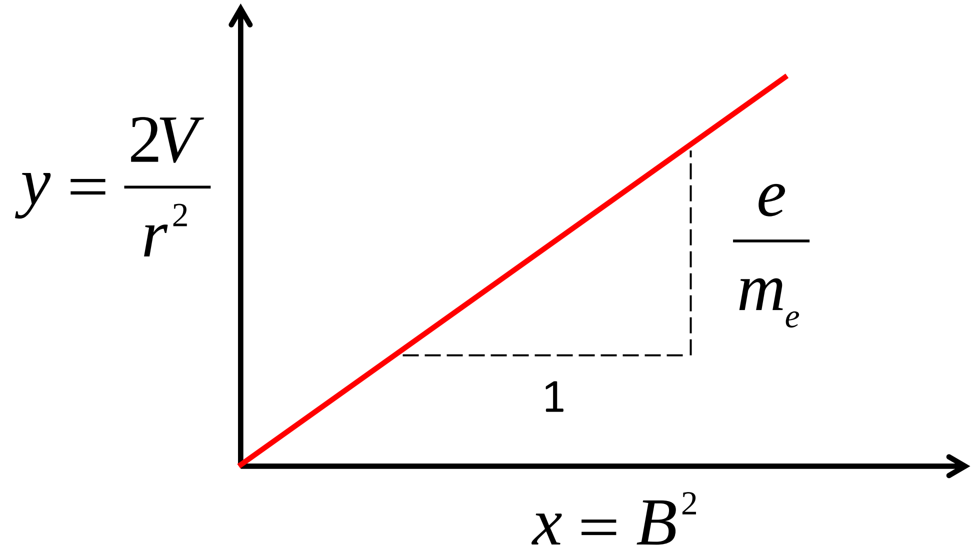
The charge to mass ratio for an electron can therefore be determined in terms of readily measurable quantities via the Fine Beam Tube!

$$\therefore \boxed{\frac{e}{m_e} = \frac{2V}{B^2 r^2}}$$

Classical result: $\dfrac{e}{m_e} = \dfrac{2V}{B^2 r^2}$

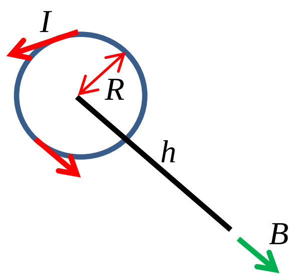
So the Fine Beam tube can be used to measure the **electron charge to mass ratio** by plotting a graph of y vs x and finding the gradient.

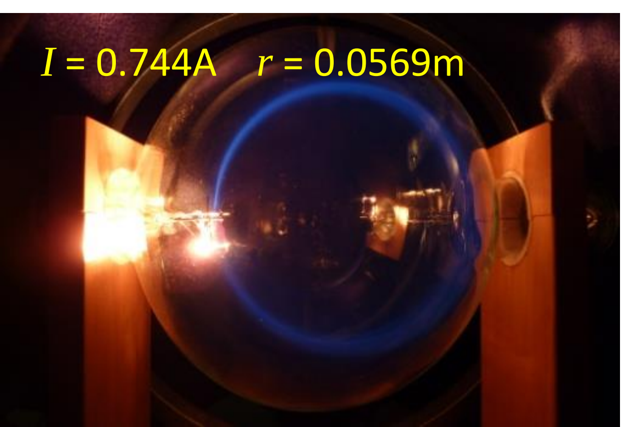
$$x = B^2, \quad y = \frac{2V}{r^2}$$



$$B = \frac{\frac{1}{2}\mu_0 N I R^2}{\left(R^2 + h^2\right)^{\frac{3}{2}}}$$

Magnetic field on axis from a current loop of N turns





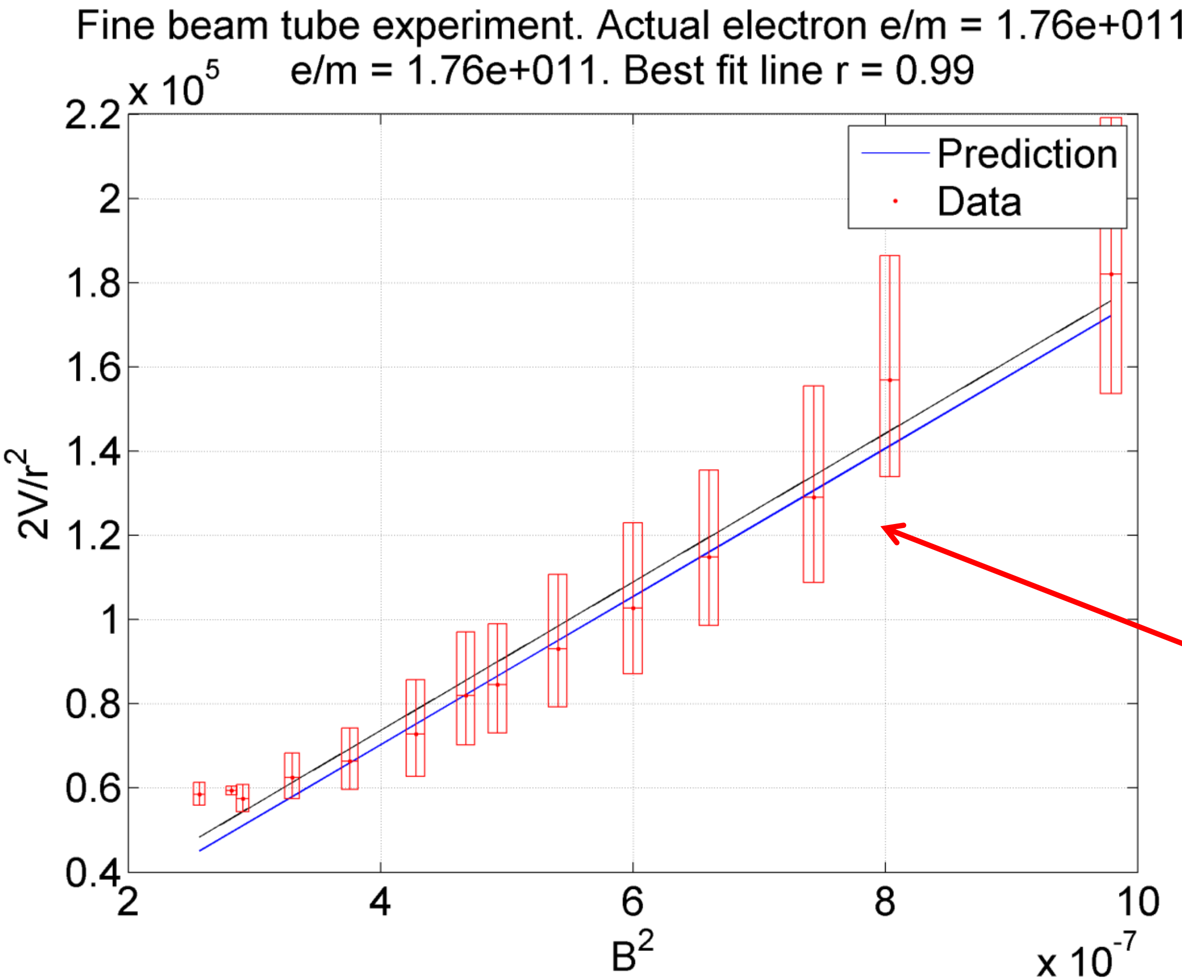
For a *pair* of Helmholtz coils with N turns and radius R separated by distance $2h$, the magnetic field strength along the coil centre line, half way between the coils, is:

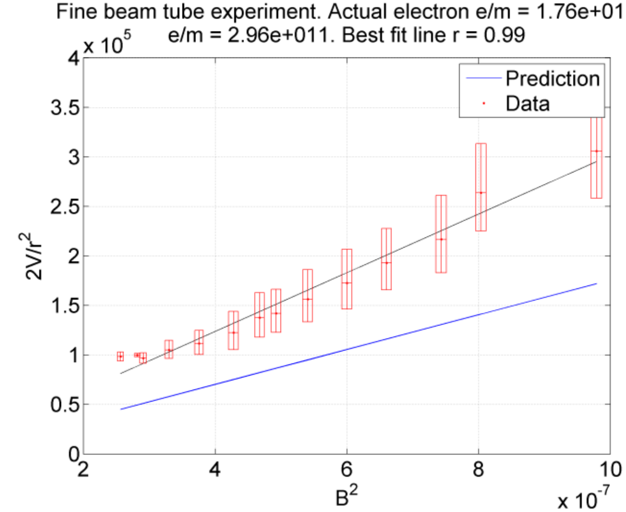
$$B = \frac{\mu_0 N I R^2}{\left(R^2 + h^2\right)^{\frac{3}{2}}} = \frac{\mu_0 N I}{R}\left(1 + \left(\frac{h}{R}\right)^2\right)^{-\frac{3}{2}}$$

$$R = 0.15\text{m}, \quad h = 0.075\text{m}$$

$$\therefore 1 + \left(\frac{h}{R}\right)^2 = \tfrac{5}{4} \Rightarrow B = \frac{\mu_0 N I}{R}\left(\tfrac{4}{5}\right)^{\frac{3}{2}}$$

Permeability of free space $\mu_0 = 4\pi \times 10^{-7}\,\text{Hm}^{-1}$





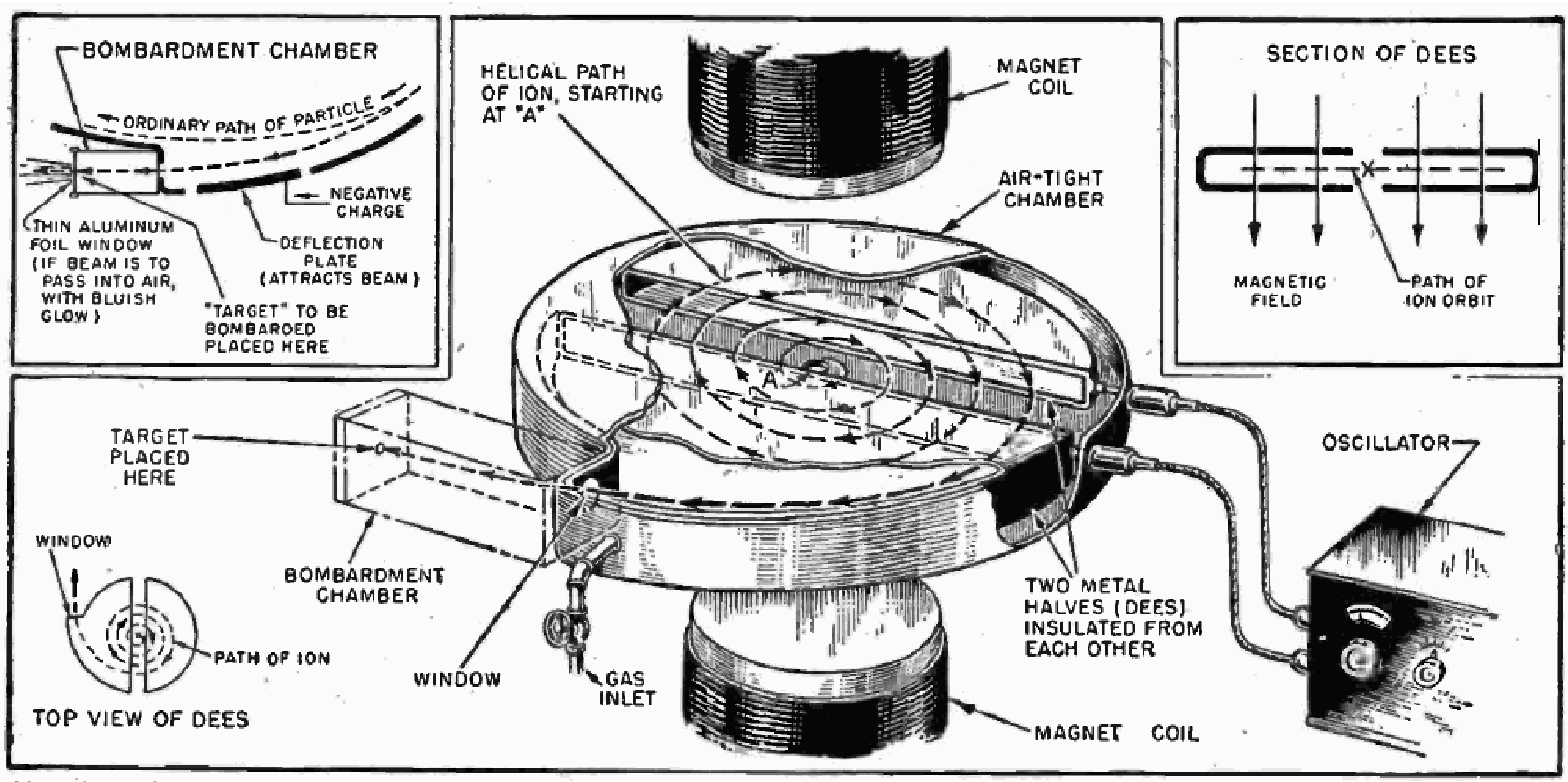
With ‘0.595 fudge factor’

i.e. assume the accelerating potential V is actually 59.5% of the voltage across the power supply.

This is proposed as the most obvious source of discrepancy between the measured and predicted lines.

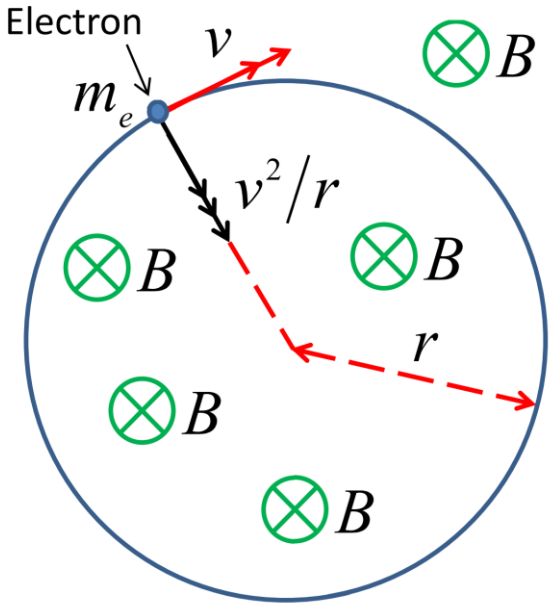
The ratio of the measured to actual e/m value is:
1.76/2.96 = 0.595

Cyclotron



$$f_c = \frac{1}{2\pi}\frac{qB}{m}$$

Cyclotron frequency



$$\frac{mv_n^2}{r_n} = qv_nB \rightarrow r_n = \frac{mv_n}{qB}$$

Newton II

Particle speed

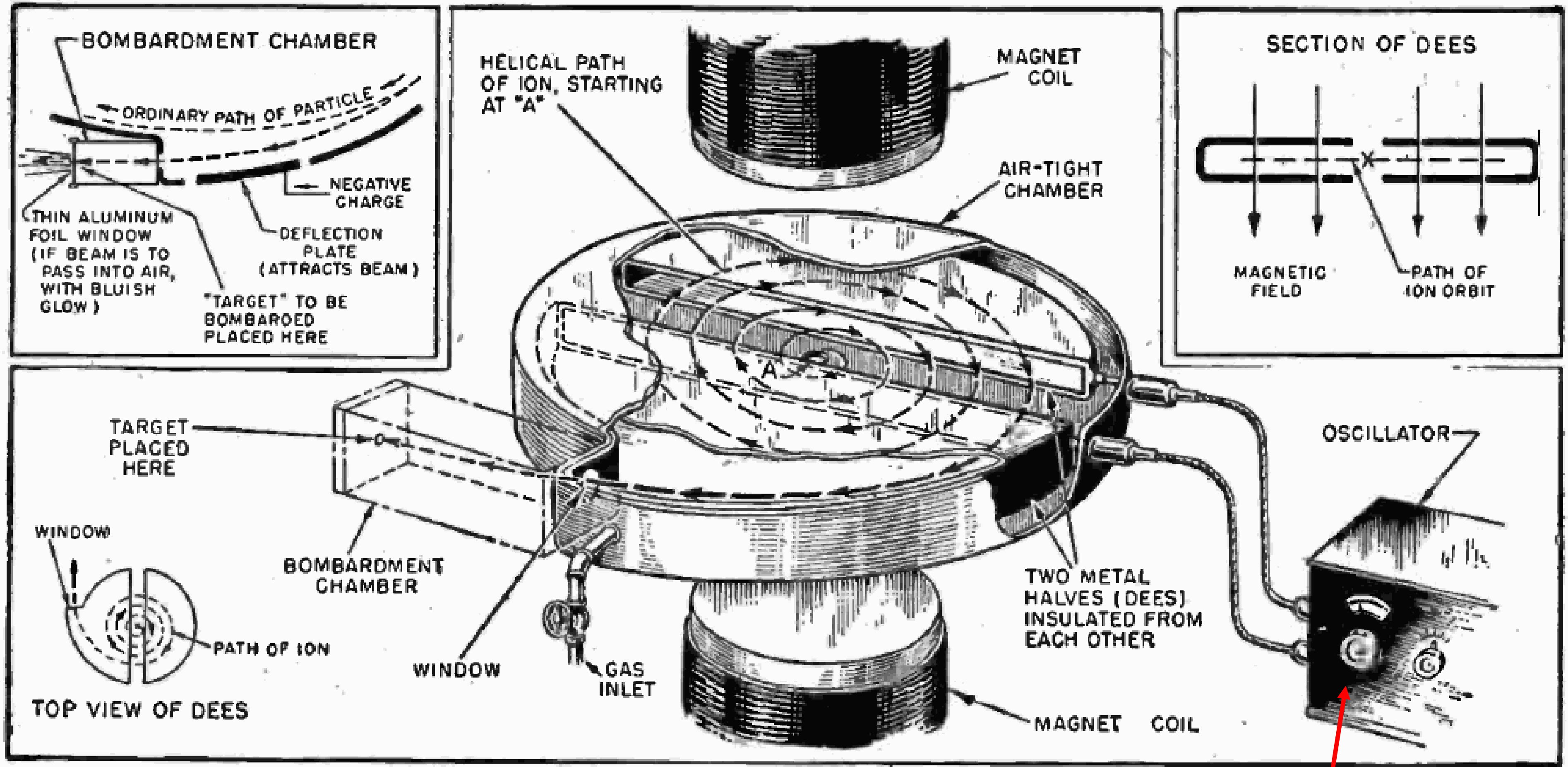
$$v_n = \frac{2\pi r_n}{T_n}$$

$$T_n v_n = \frac{2\pi m v_n}{qB}.$$

Cyclotron frequency

$$\therefore T_n = \frac{2\pi m}{qB} \rightarrow f_c = \frac{1}{2\pi}\frac{qB}{m}$$

$\frac{1}{2}T_n$ is the time to complete a half-circular orbit between boosts.



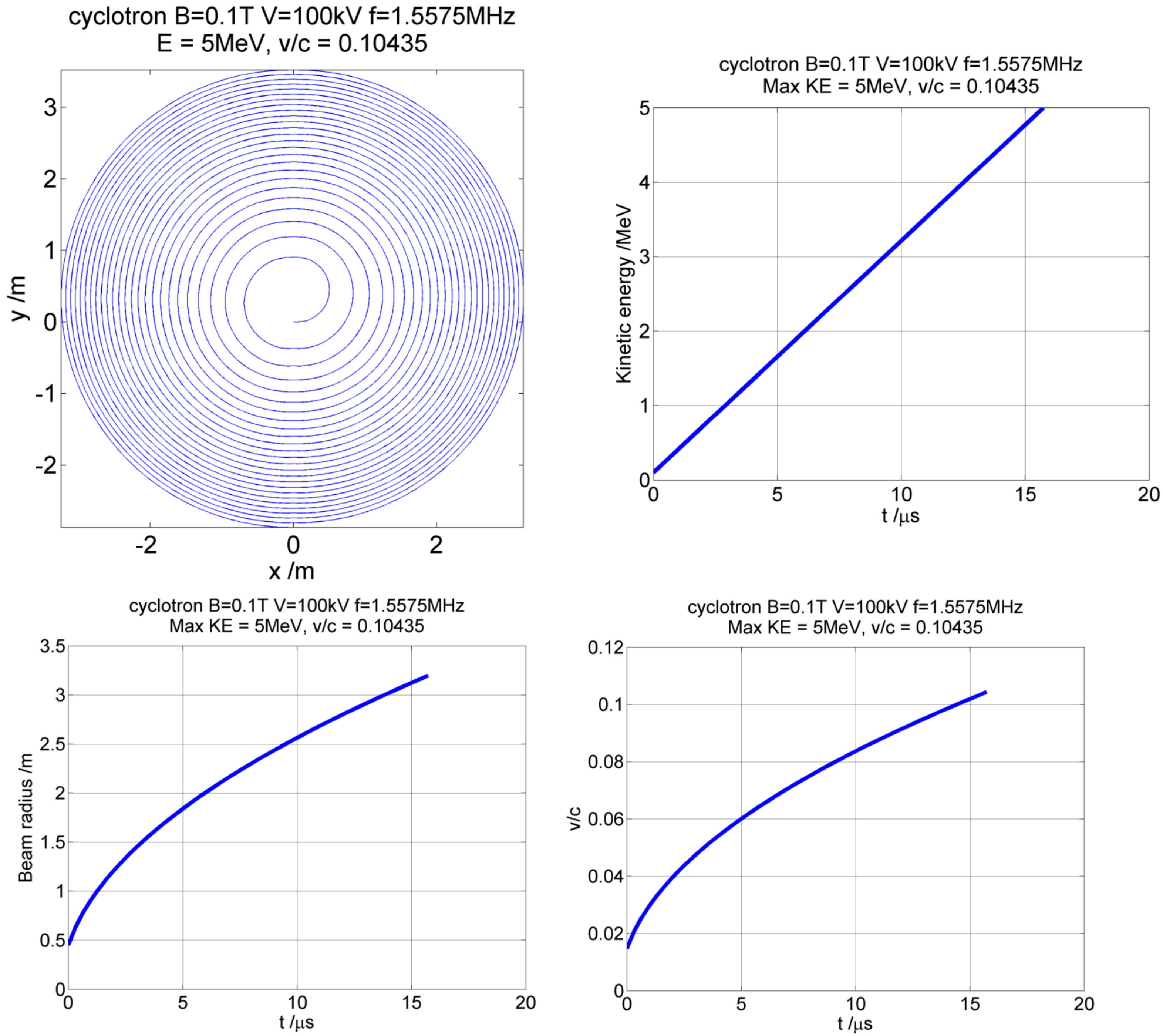
$$\frac{1}{2}mv_{n+1}^2 = \frac{1}{2}mv_n^2 + qV_0.$$

$$V(t) = V_0 \cos(2\pi f_c t) = V_0 \cos\left(\frac{qBt}{m}\right).$$

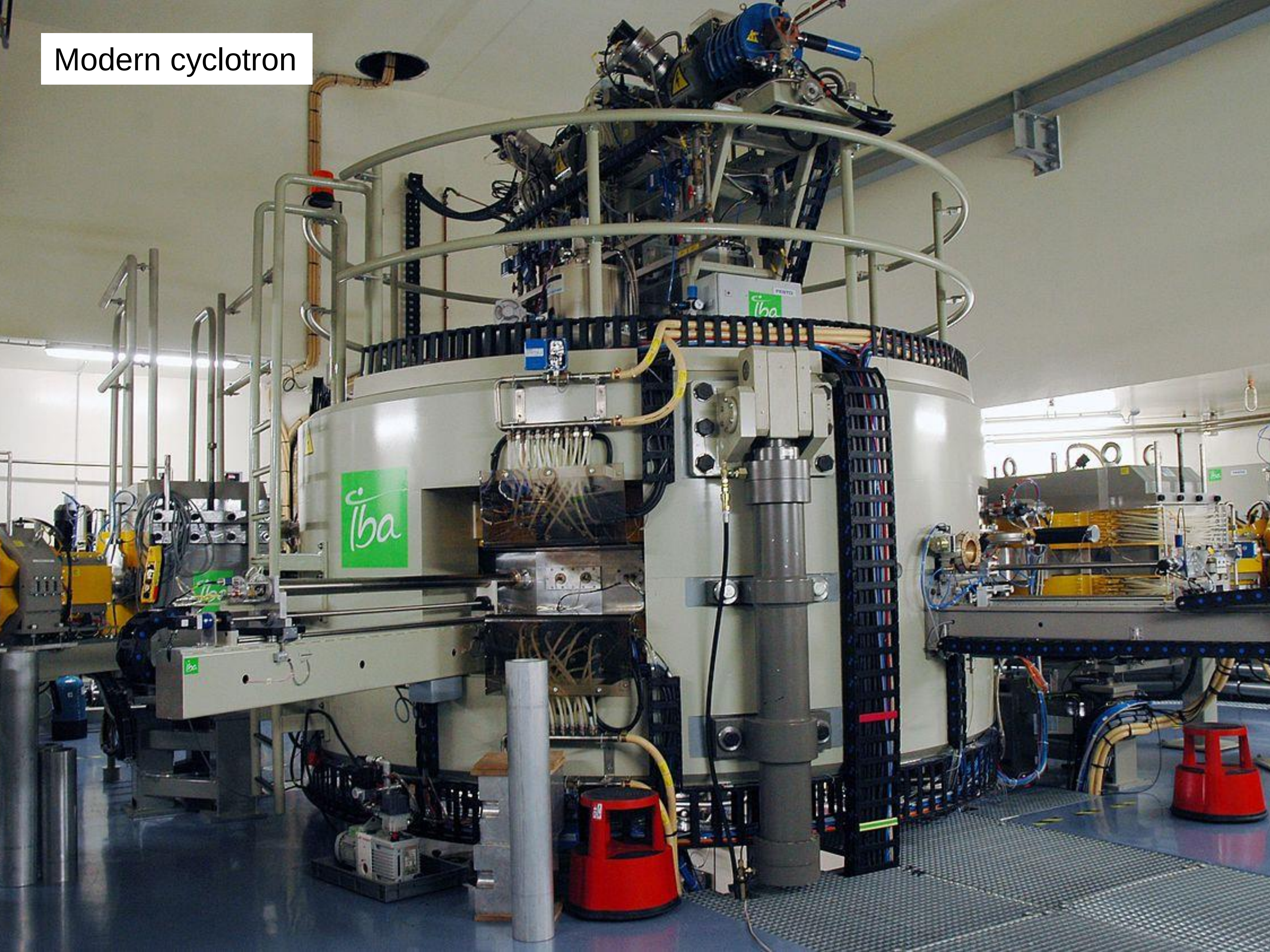
$$\therefore v_{n+1} = \sqrt{v_n^2 + \frac{2qV_0}{m}}.$$

Particle speeds get a boost every half turn

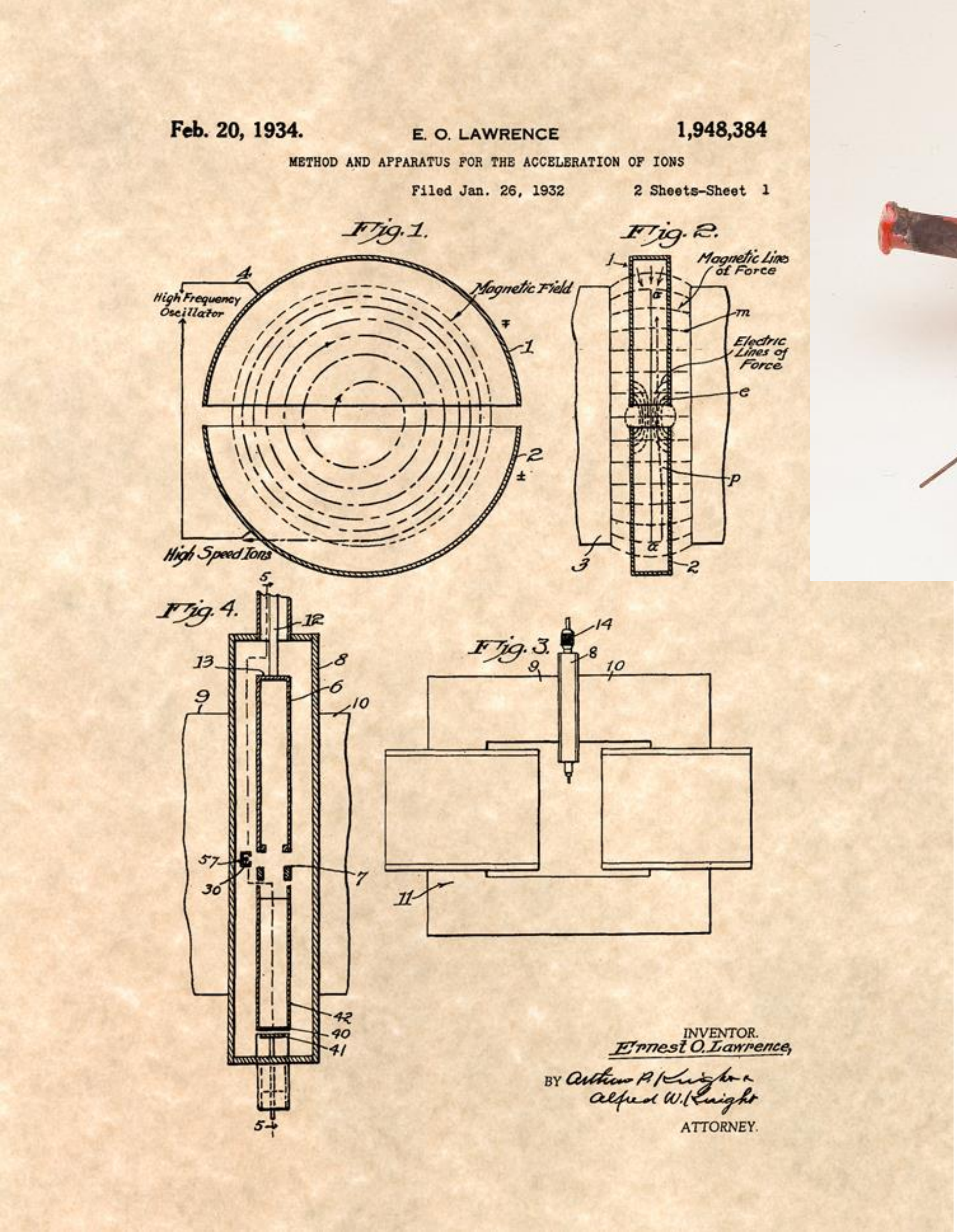
$$f_c = \frac{1}{2\pi}\frac{qB}{m}$$


cyclotron B=0.1T V=100kV f=1.5575MHz
E = 5MeV, v/c = 0.10435
y /m
x /m
cyclotron B=0.1T V=100kV f=1.5575MHz
Max KE = 5MeV, v/c = 0.10435
Kinetic energy /MeV
t /μs
cyclotron B=0.1T V=100kV f=1.5575MHz
Max KE = 5MeV, v/c = 0.10435
Beam radius /m
t /μs
cyclotron B=0.1T V=100kV f=1.5575MHz
Max KE = 5MeV, v/c = 0.10435
v/c
t /μs


ZURICH SUISSE



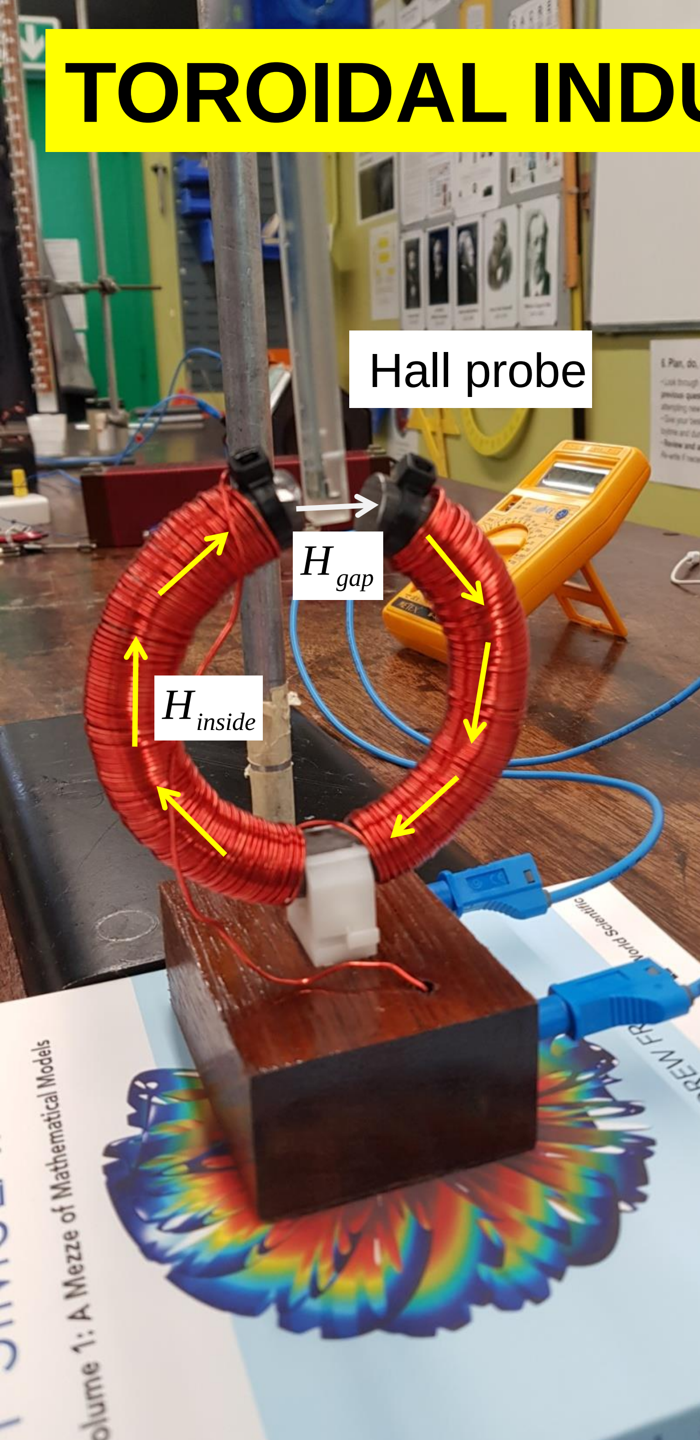
Modern cyclotron

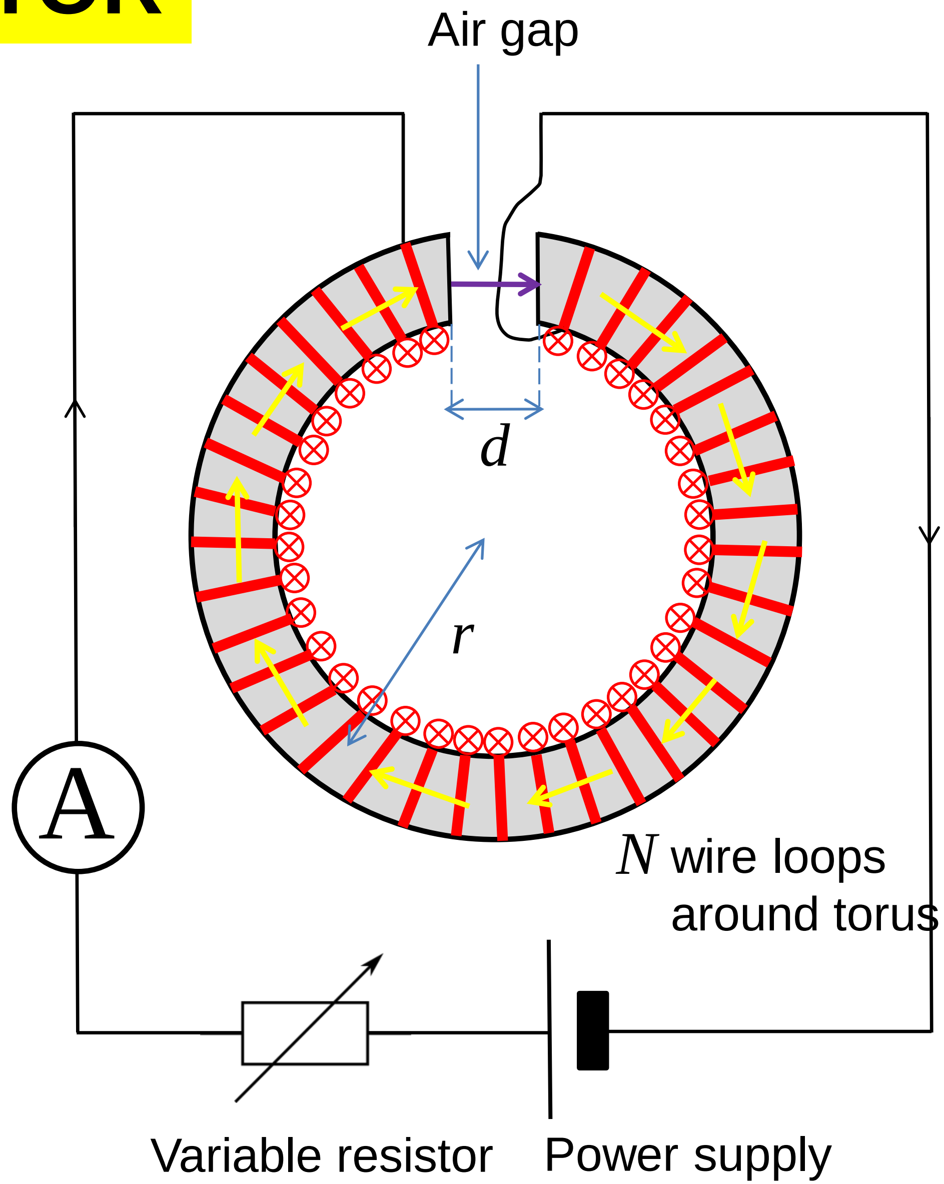


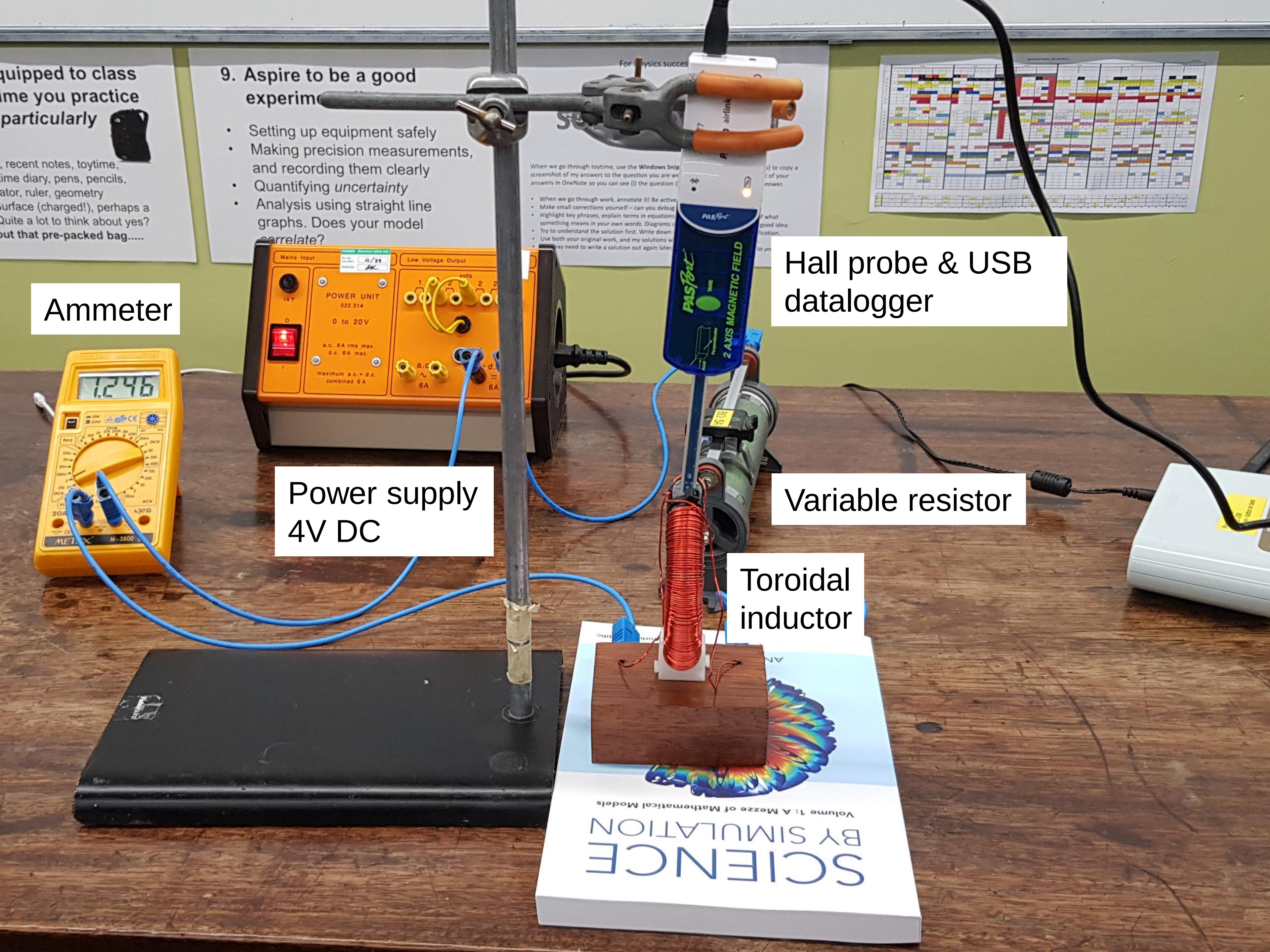
Ernest Lawrence
(1901-1958)

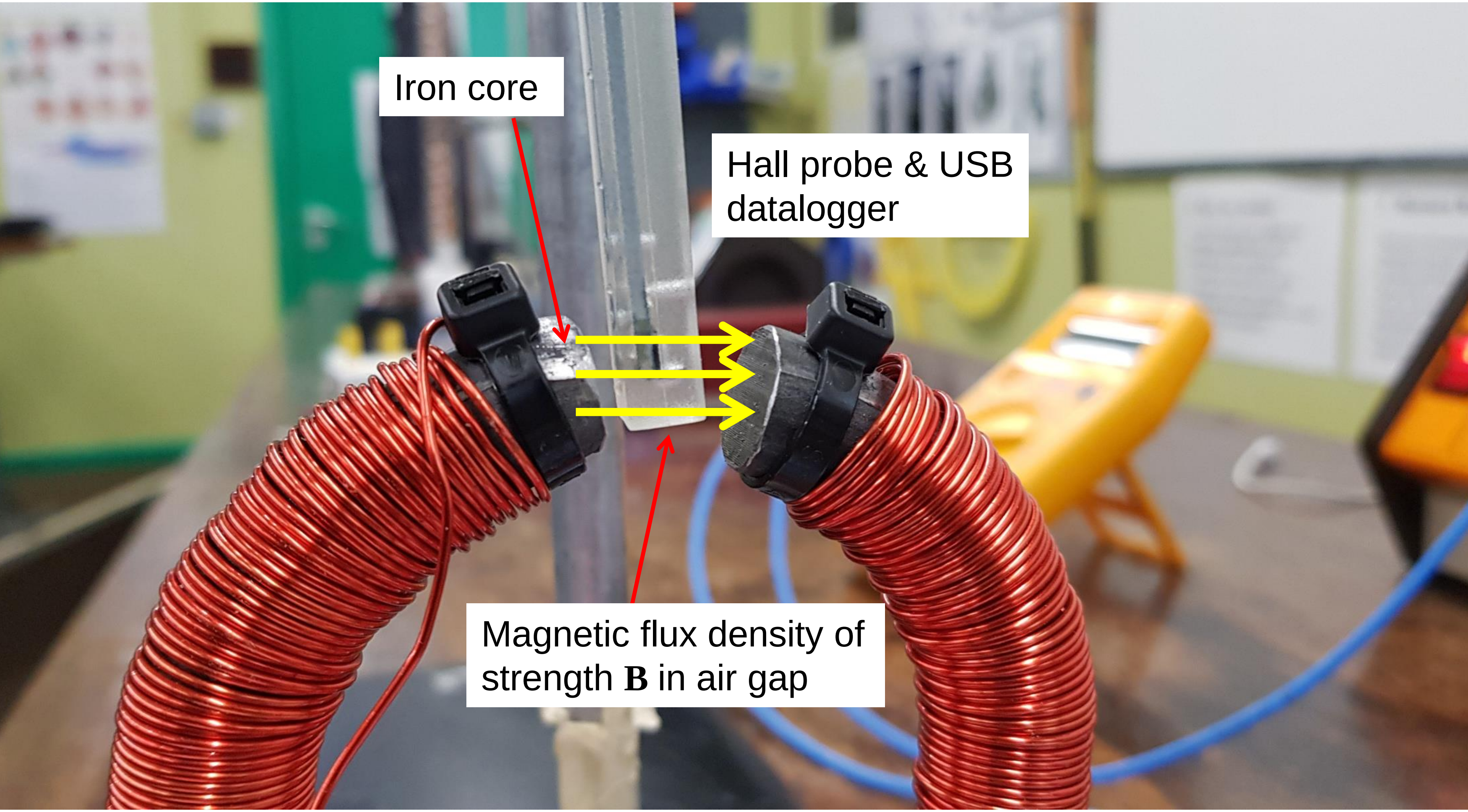
E. Lawrence’s 184-inch cyclotron

TOROIDAL INDUCTOR






Ammeter
Power supply 4V DC
Hall probe & USB datalogger
Variable resistor
Toroidal inductor



1. Vary current (range 0.4A to about 2.0A) in toroidal inductor by changing resistance of variable resistor.
2. Use Hall Probe and datalogger to measure magnetic flux density B in air gap.
3. Plot magnetic flux density (in T) vs current (in A). Use the graph to calculate the relative permeability μ of the iron core.

Application of the Lorentz force – the Hall Efffect

A semiconductor of width w and height h is placed in a magnetic field B. Current I passes through the semiconductor as shown.The Lorentz force on charges will cause a charge separation, which in turn will result in an electric field E perpendicular to both the magnetic field and the current direction.

Equilibrium is reached when the electric force and Lorentz magnetic forces balance.

Example calculation:

$n = 7\times10^{21}\text{m}^{-3}$

$q = e = 1.6\times10^{-19}\text{C}$

$h = 0.1\text{mm}$

$\therefore qnh = \boxed{0.112}$

$\therefore \frac{V_H}{I}$ can be a ratio of near-unity quantities, which are readily measureable i.e. B fields not too many orders of magnitude less than 1.0T can be easily measured.

$$qE = qvB$$

$$\therefore E = vB$$

$$E = \frac{V_H}{w}$$

The electric potential resulting from the separated charges is called the **Hall Voltage**

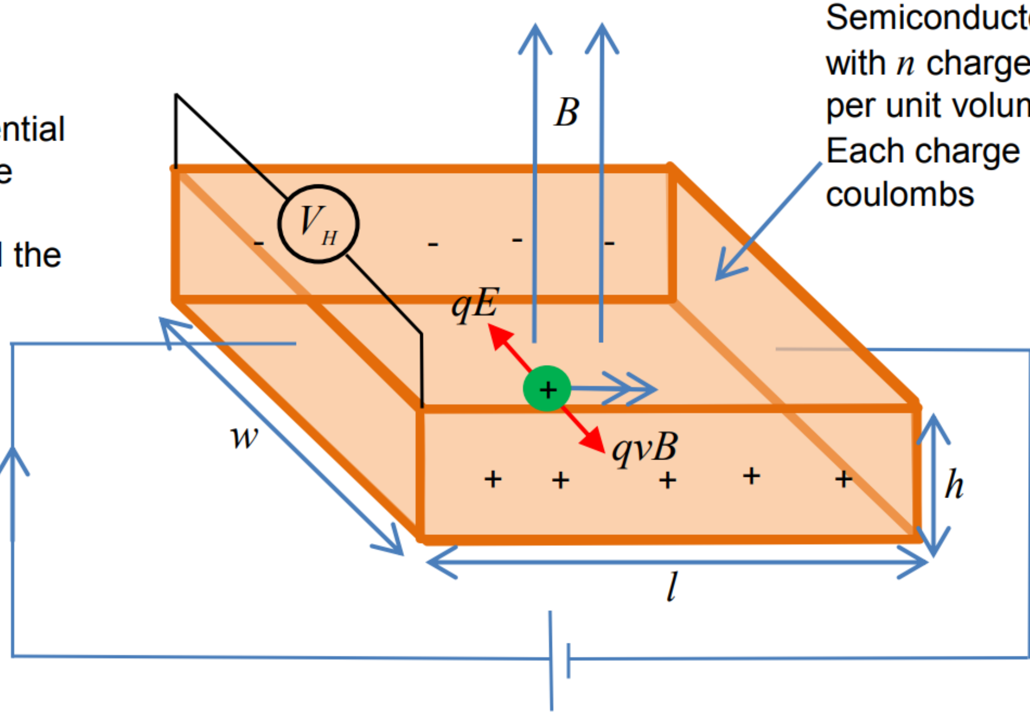
$$I = qnwhv$$

$$\therefore v = \frac{I}{qnwh}$$

$$\therefore \frac{V_H}{w} = \frac{IB}{qnwh}$$

$$V_H = \frac{IB}{qnh}$$

$$\boxed{B = \frac{qnhV_H}{I}}$$



It is possible to measure the Hall effect in a small semiconductor, so the effect can be used to determine the how a non uniform magnetic field varies in time and space.

Edwin Hall
1855-1938

Ampère's Theorem:

$$\oint_{loop} \mathbf{H} \cdot d\mathbf{l} = NI$$

Magnetic field strength inside torus is tangential to circular loop

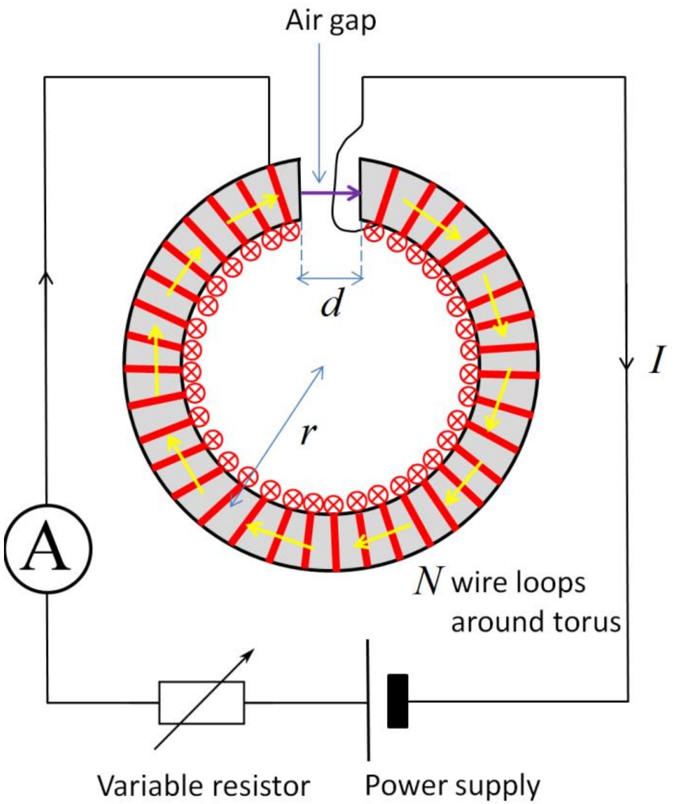
$$\therefore H_{inside}\left(2\pi r - d\right) + H_{gap} d = NI$$

if gap is small $d \ll 2\pi r$

$$B_{gap} = \mu_0 H_{gap}$$

$$B_{inside} = \mu \mu_0 H_{inside}$$

Permeability of free space $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{Hm}^{-1}$



Magnetic flux density **B** is continuous perpendicular to the iron, air boundary. (*Maxwell Equation* result).

Hence:

$$B_{gap} = B_{inside} = B$$

$$\therefore H_{gap} = \mu H_{inside}$$

$$\therefore H_{inside} = \frac{B}{\mu \mu_0}$$

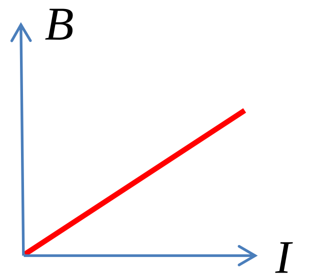
$$\therefore H_{inside}\left(2\pi r - d\right) + \mu H_{inside} d = NI$$

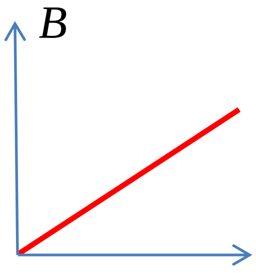
$$\therefore H_{inside}\left(2\pi r - d + \mu d\right) = NI$$

$$\therefore \frac{B}{\mu \mu_0}\left(2\pi r - d + \mu d\right) = NI$$

$$\therefore B = \frac{N \mu \mu_0}{2\pi r - d + \mu d} I$$

André-Marie Ampère (1775-1836)





$$B = kI$$

$$k = \frac{N\mu\mu_0}{2\pi r - d + \mu d}$$

$$k\left(2\pi r - d + \mu d\right) = N\mu\mu_0$$

$$k\left(2\pi r - d\right) = \mu\left(N\mu_0 - kd\right)$$

$$\therefore \mu = \frac{k\left(2\pi r - d\right)}{N\mu_0 - kd}$$

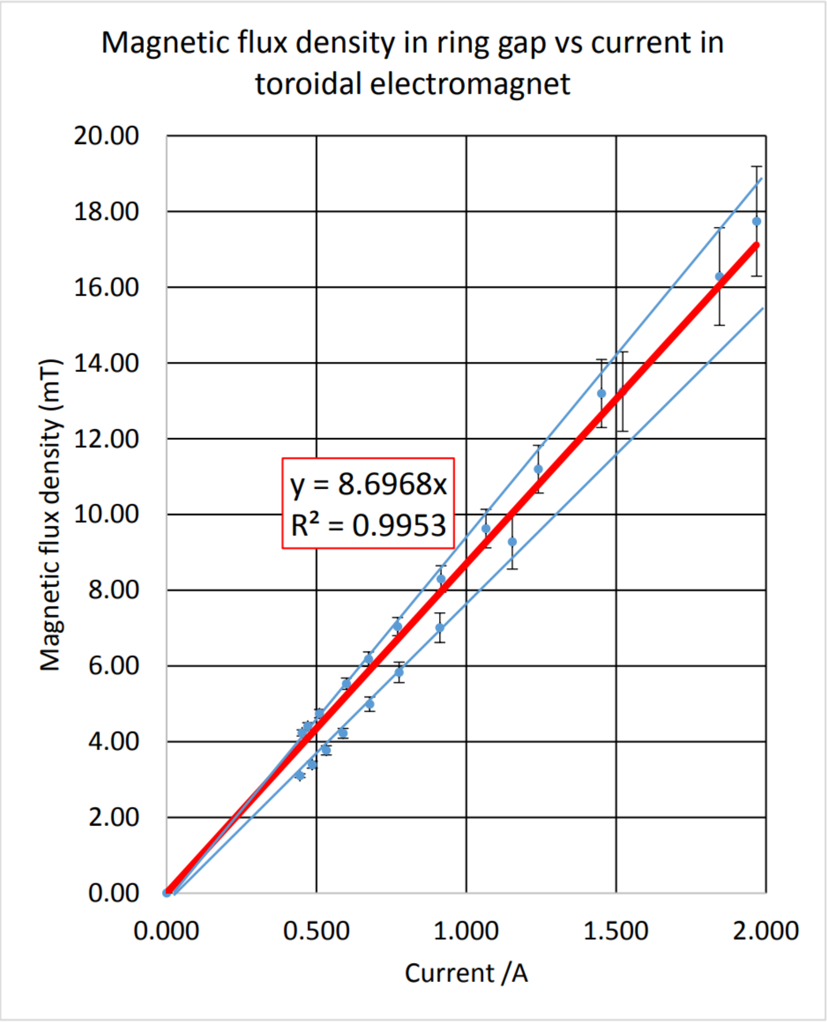
TOROIDAL ELECTROMAGNET EXPERIMENT

21/11/2022

Radius of ring /m	**0.08**
Gap in m	**0.018**
Number of coils N	**220**
µ0	**1.25664E-06**
k = dB/dI	**0.00870**
k max	**0.00950**
k min	**0.00775**
µ mean	**35.15**
µ max	**43.66**
µ min	**27.42**

Magnetic flux density B in air gap (mT)

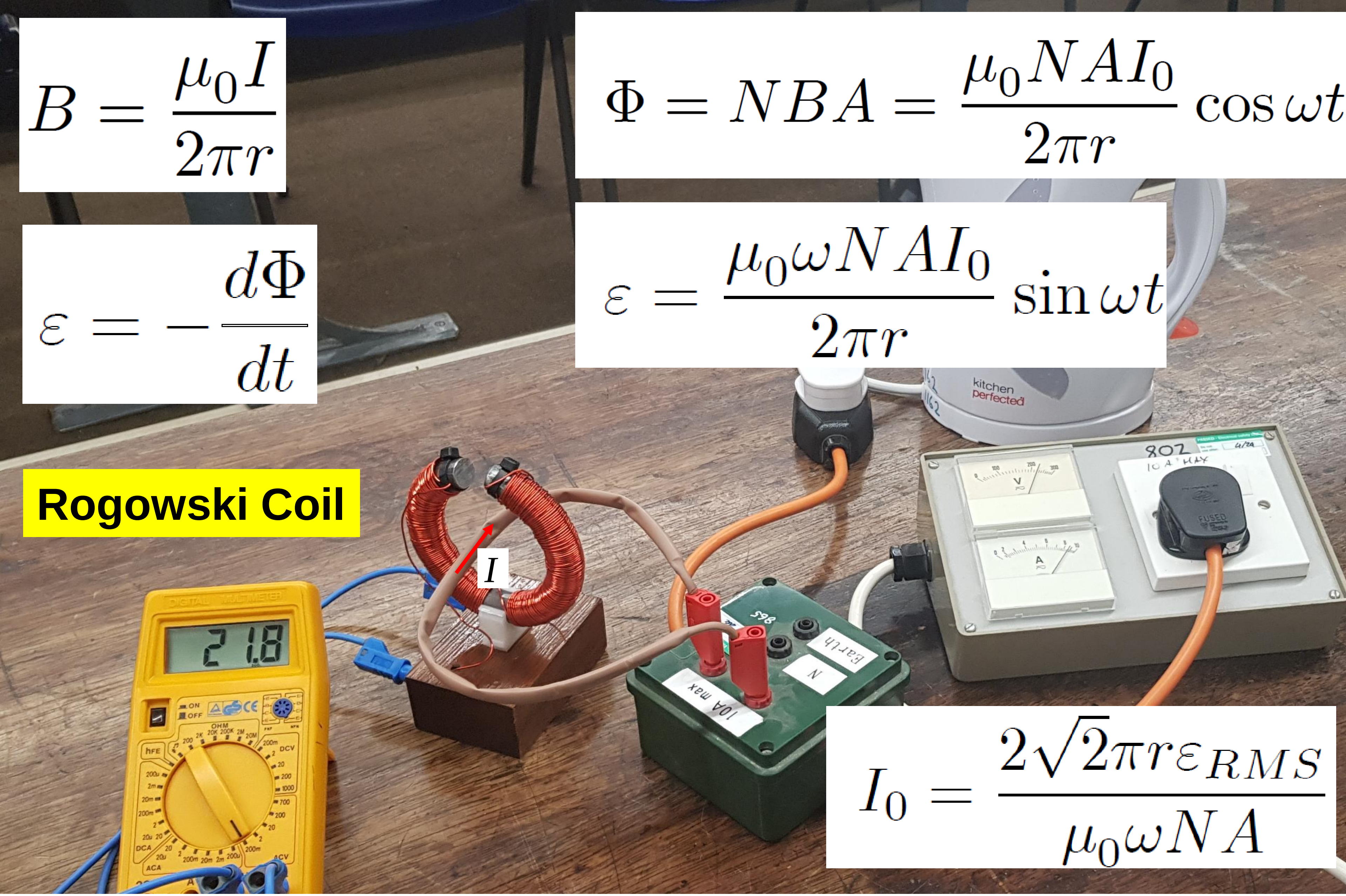
B (mT)	B error /mT	I (A)
0.00	0.01	0.000
3.10	0.05	0.445
3.39	0.09	0.486
3.77	0.12	0.533
4.22	0.13	0.589
4.99	0.19	0.678
5.83	0.27	0.776
7.01	0.39	0.912
9.28	0.72	1.154
13.25	1.05	1.523
16.29	1.29	1.846
17.75	1.45	1.970
13.20	0.90	1.452
11.20	0.63	1.241
9.63	0.51	1.066
8.30	0.35	0.916
7.04	0.24	0.771
6.18	0.19	0.674
5.53	0.15	0.600
4.74	0.11	0.510
4.41	0.09	0.471
4.23	0.08	0.452



In many literature sources μ is quoted as being about 1,000.

So the ring metal is probably not pure iron!

Current raised then lowered to investigate *hysteresis* - only very marginal in this experiment.



For our split-iron toroidal inductor we might assume $B = \frac{\mu\mu_0 I}{2\pi r}$, so we can multiply the induced EMF by core relative permeability μ which means $I_0 = \frac{2\sqrt{2}\pi r \varepsilon_{RMS}}{\mu\mu_0 \omega N A}$.

Walter Rogowski
(1881-1947)

Clamp ammeter – essentially a **Rogowski Coil**

$$I_0 = \frac{2\sqrt{2}\pi r \varepsilon_{RMS}}{\mu_0 \omega N A}$$

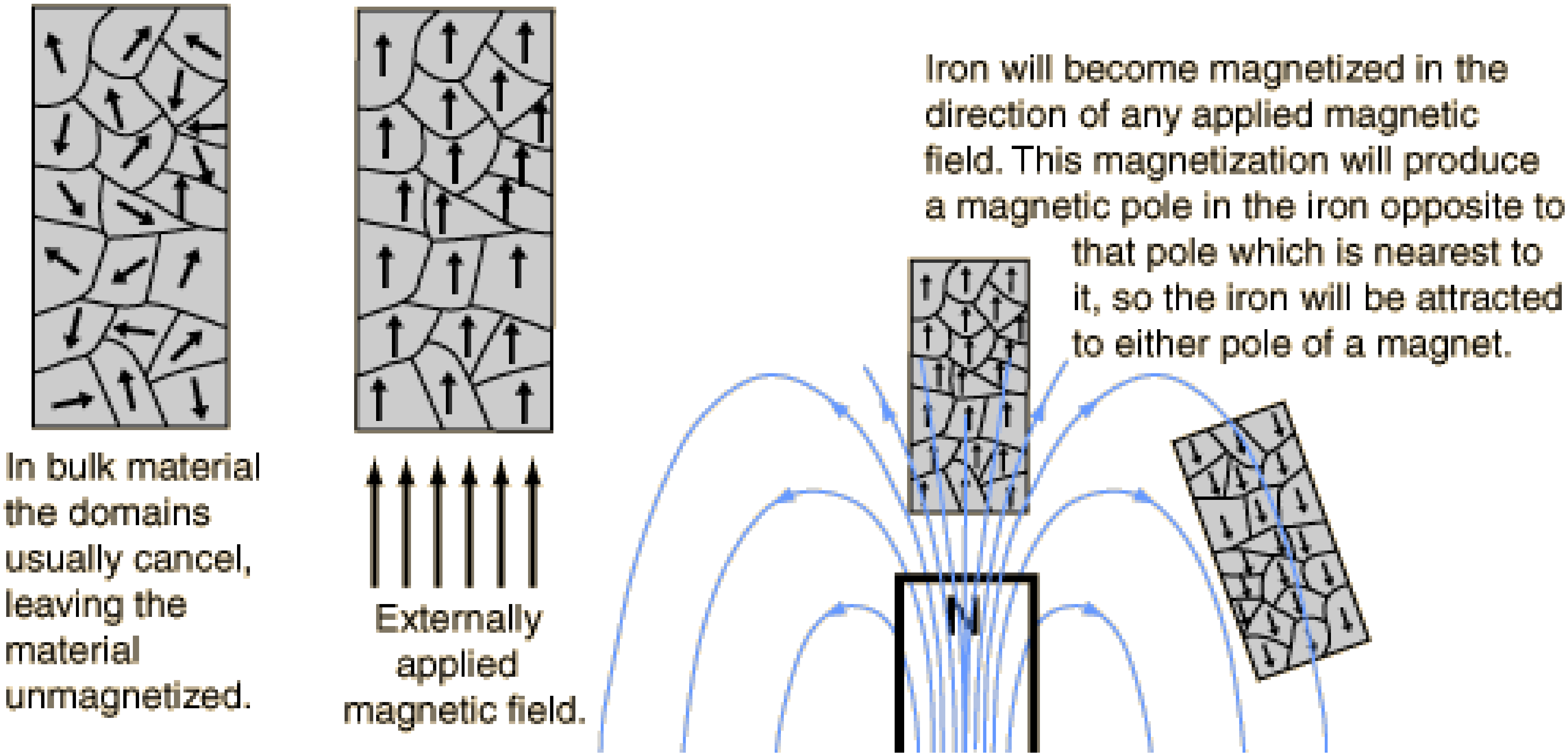
The Ising Model of Ferromagnetism

All atoms will respond in some fashion to **magnetic fields.** The angular momentum (and spin) properties of electrons imply a circulating charge, which means they will be subject to a Lorentz force in a magnetic field. **However the effects of *diamagnetism, paramagnetism* and *anti-ferromagnetism* are typically very small. Ferromagnetic materials** (iron, cobalt, nickel, some rare earth metal compounds) respond strongly to magnetic fields and can intensify them by orders of magnitude. i.e. the *relative permeability* can be tens or hundreds, or possibly thousands.

The Ising model is a simplified model of a **ferromagnet** which exhibits a **phase transition** above the **Curie temperature**. Below this, magnetic dipole alignment will tend to cluster into **domains**, and its is these micro-scale groupings which give rise to ferromagnetic behaviour.

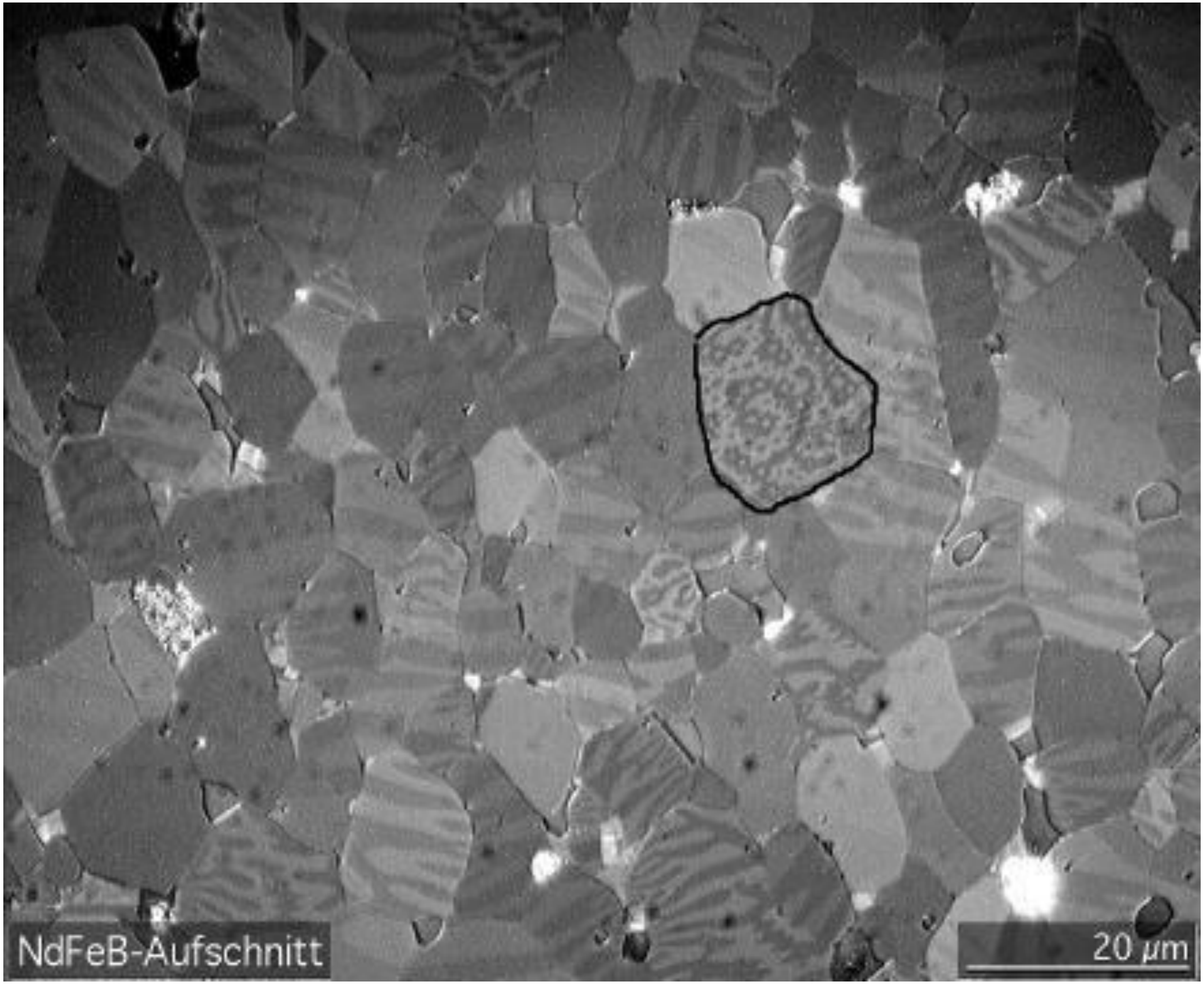
Ernst Ising (1900-1998)

“Soft” magnetism - Ferromagnets



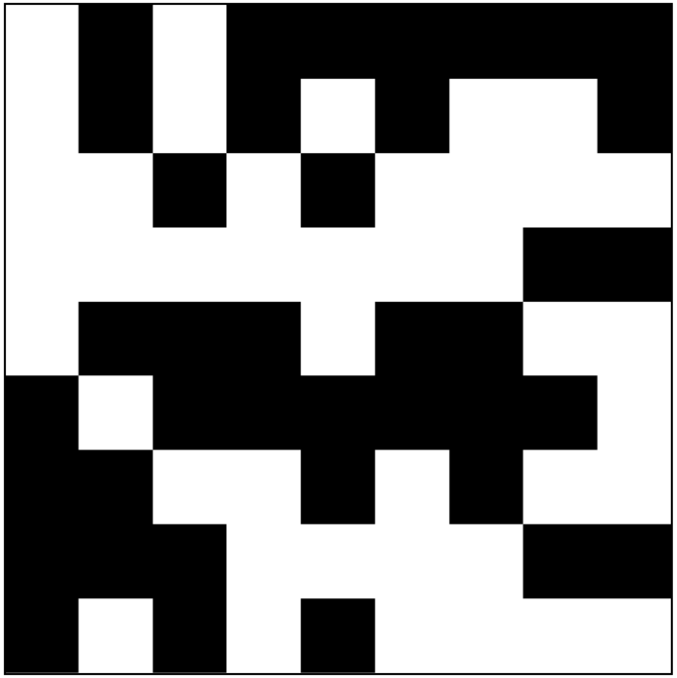
Unlike permanent “hard” magnets, once the applied field is removed, the domain alignment will randomize again, effectively zeroing the net magnetism.

Magnetic domains



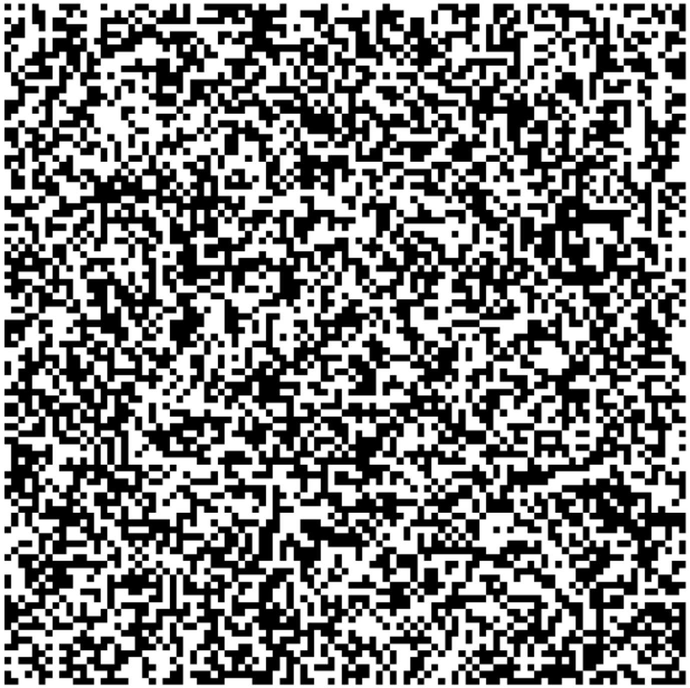
The **Ising model** can be used to demonstrate spontaneous mass alignment of magnetic dipoles, and possibly a mechanism for domain formation.

Perhaps the simplest model which yields characteristic behaviour is an $N \times N$ square grid, where each square is initially randomly assigned a +1 or -1 value, with equal probability. The +/-1 values correspond to a single direction of magnetic dipole moment in a rectangular lattice of ferromagnetic atoms, or in the case of individual electrons, *spin*.



10 x 10 grid

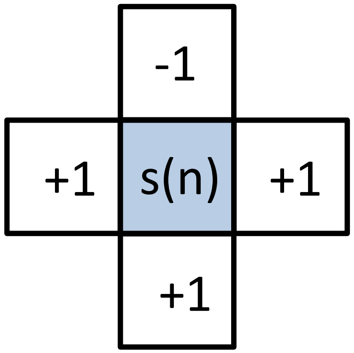
White squares represent +1
Black squares represent -1



100 x 100 grid

Metropolis algorithm

1. Choose one square at random from the N x N grid. Let its spin be s(n) = +1 or -1.
2. Find the spins of the nearest neighbours. Use *circular boundary conditions* e.g. if s(n) is at the edge of the grid, use the nearest neighbour to be that of the other end.



3. Compute a sum of **spin-coupling energies** for s(n) and its neighbours, and work out the energy change if s(n) were to **change sign**

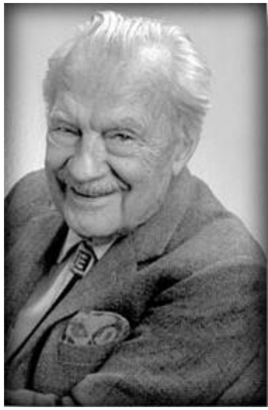
$$\Delta E = 2 \times \left(F + J \sum_{k=1}^{4} s_n(k) \right) s(n)$$

J is the spin coupling energy in eV and F is the energy in eV associated with the alignment of spin s(n) with an applied external magnetic field. Let us ignore any energy contributions from non-nearest neighbours.

$$r \sim \mathrm{U}(0,1)$$

Now change the sign of spin s(n) according to the following rule:

$$s(n) \to -s(n) \quad \text{if} \quad e^{-\frac{\Delta E}{k_B T}} \geq r \quad \text{or} \quad \Delta E < 0$$



Nicholas Metropolis 1915-1999

Apply the Metropolis method for I x N x N iterations, and then compute from the N x N grid the following parameters

$$\langle s \rangle = \frac{1}{N^2} \sum_{n=1}^{N^2} s(n)$$

Mean spin

$$\langle E \rangle = -\tfrac{1}{2} \frac{1}{N^2} \sum_{n=1}^{N^2} \left(J \sum_{k=1}^{4} s_n(k) + F \right) s(n)$$

Mean energy per spin

$$k_B T^2 \langle C \rangle = \tfrac{1}{4} \frac{1}{N^2} \sum_{n=1}^{N^2} \left(J s(n) \sum_{k=1}^{4} s_n(k) + F s(n) \right)^2 - \langle E \rangle^2$$

Heat capacity in eV per K

This is a well known result in Statistical Thermodynamics

$$k_B T^2 \langle C \rangle = \mathrm{Var}\left[E \right]$$

For a 2D Ising model, Lars Onsager determined in 1944 the relationship between the **phase transition Curie temperature** and **coupling energy** J

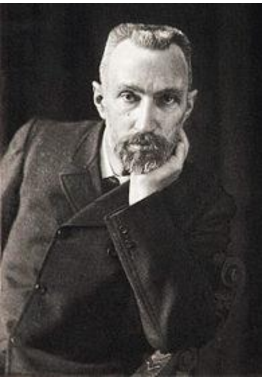
$$J = \frac{1}{2} k_B T_C \ln\left(1 + \sqrt{2}\right)$$

$T_C = 1{,}043\text{K}$ Iron

(Note this expression assumes Coupling energy J is in joules)

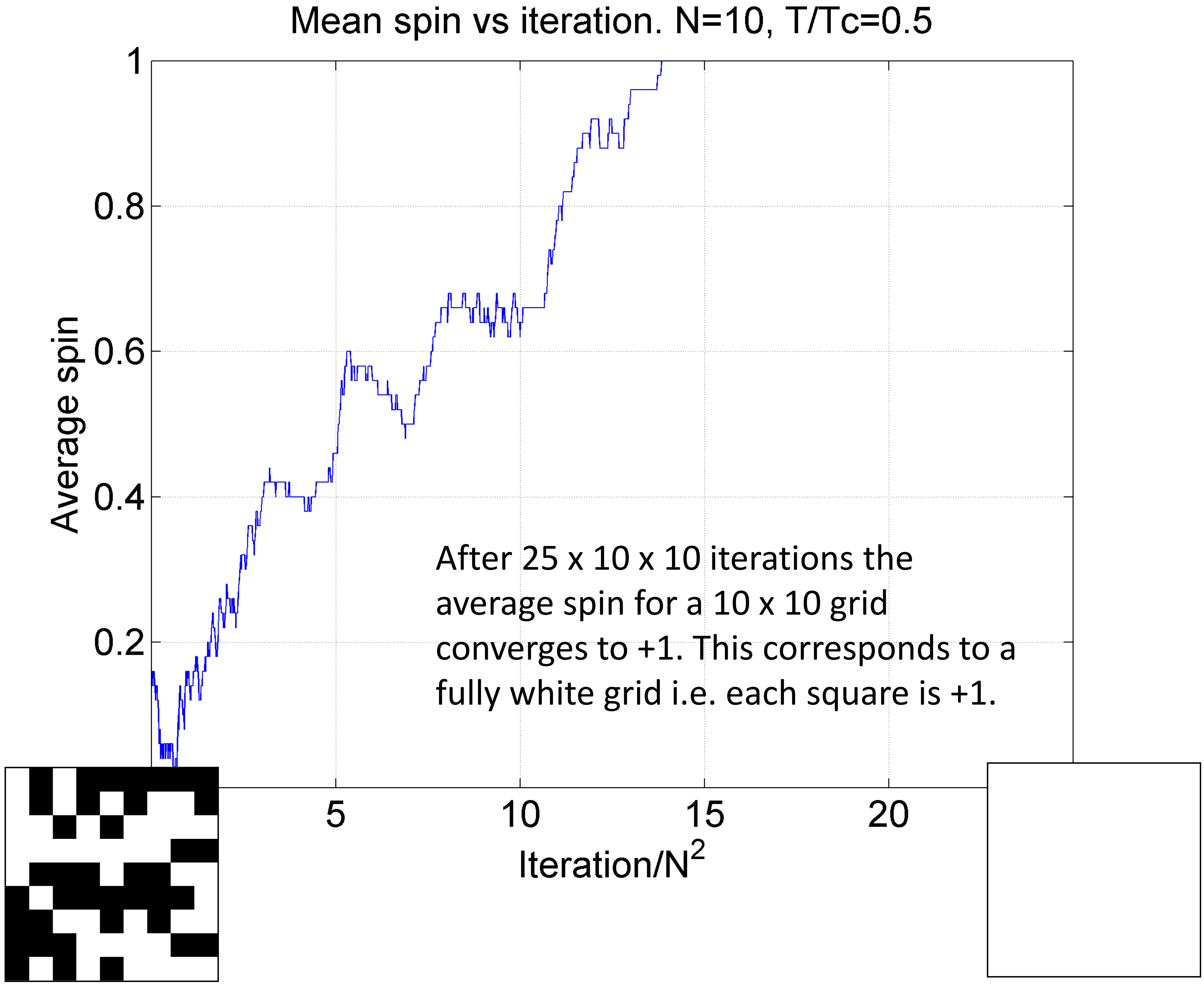
Boltzmann's constant

$k_B = 1.38 \times 10^{-23}\ \text{JK}^{-1}$

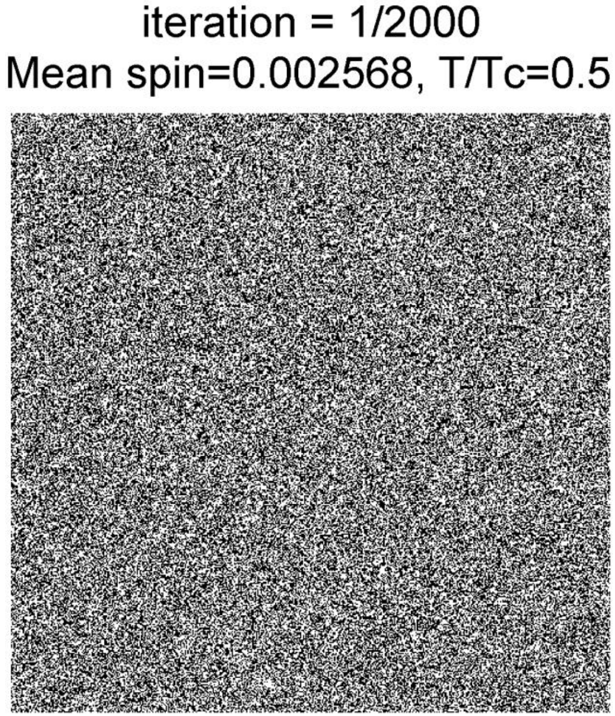


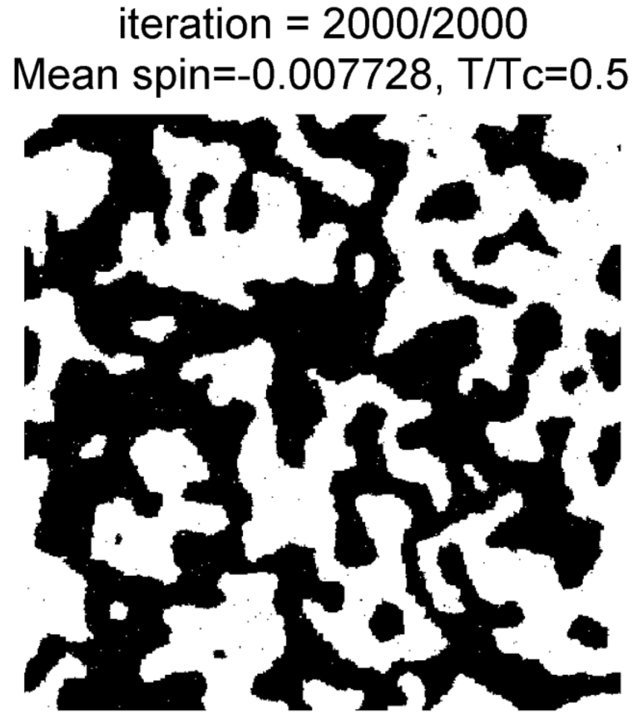
Peter Curie
(1859-1906)

Lars Onsager
(1903-1976)



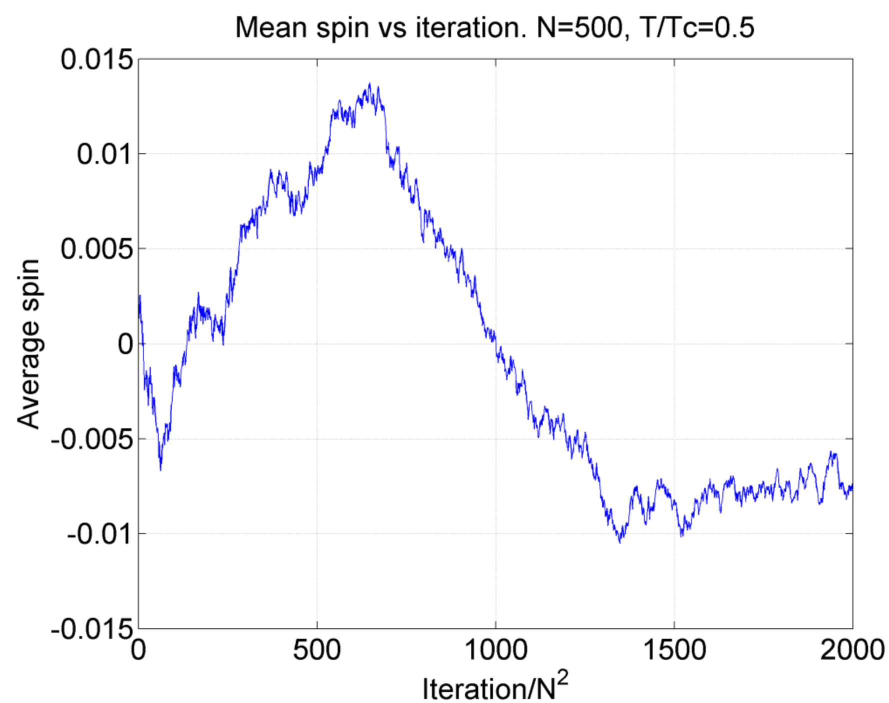
Mean spin vs iteration. N=10, T/Tc=0.5
Average spin
1
0.8
0.6
0.4
0.2
5
10
15
20
Iteration/N^2
After 25 x 10 x 10 iterations the average spin for a 10 x 10 grid converges to +1. This corresponds to a fully white grid i.e. each square is +1.





For a 500 x 500 grid, a similar equilibrium is not yet reached, even after I = 2000 x 500 x 500 iterations.

However, domain-like structures are clearly visible in this intermediate state.

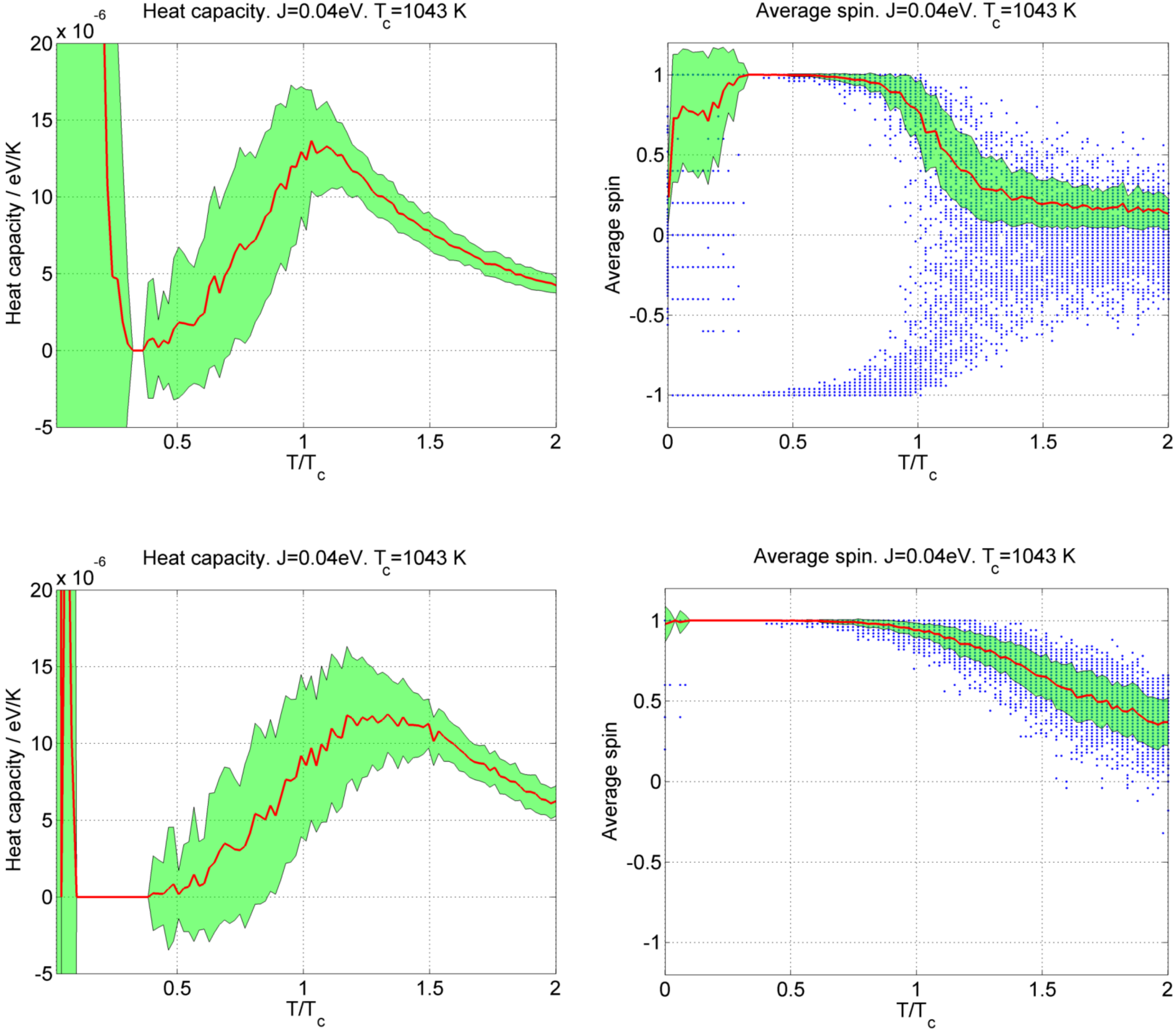


Results of a MATLAB simulation:

10 x 10 grid
I = 2000 (x 10 x 10) iterations of Metropolis algorithm
R = 100 repeats for each temperature
100 different temperatures from T/Tc = 0.0 ... 2.0
21 different F/J values from -2 to 2

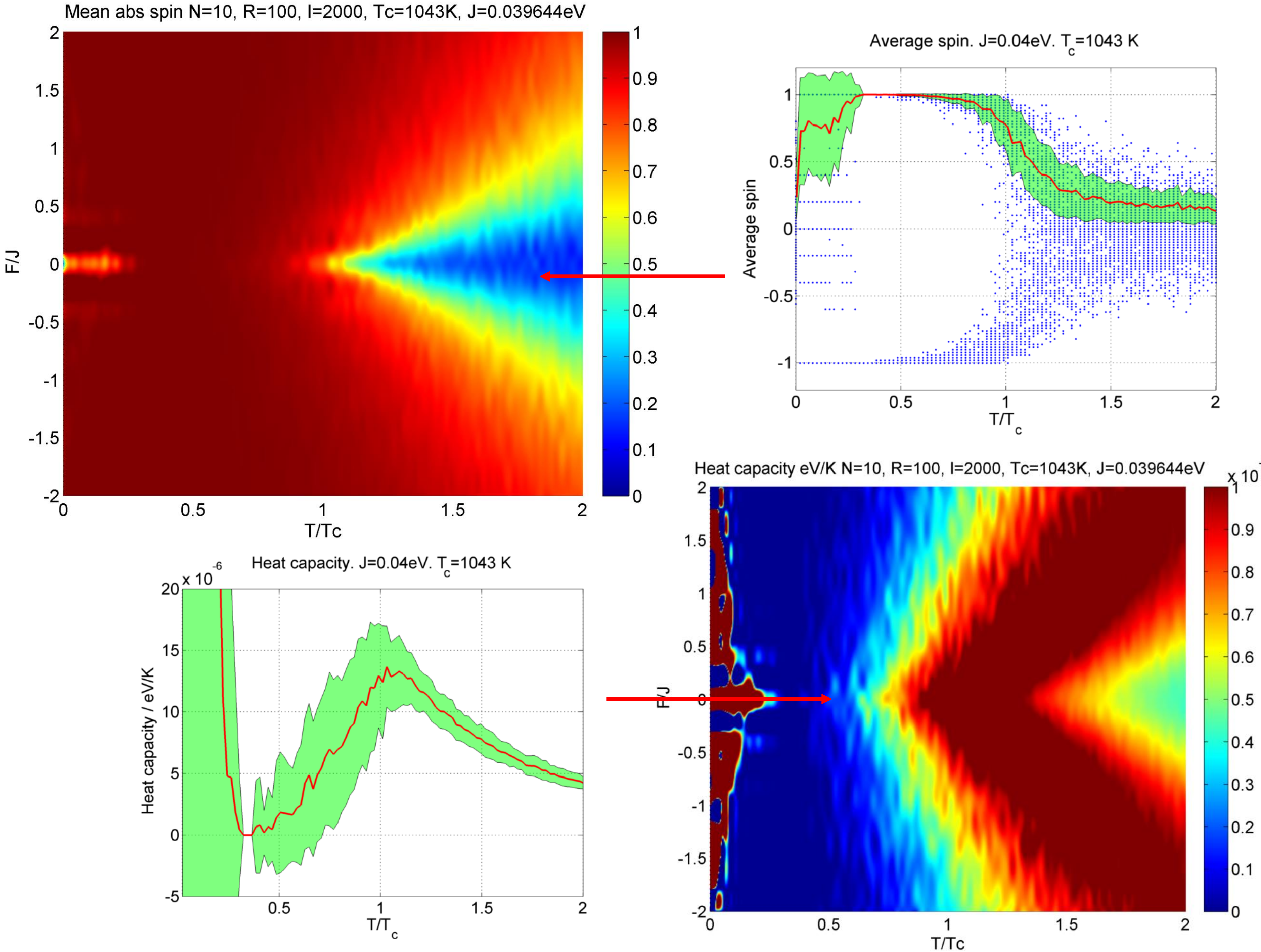
i.e. 2000 x 10 x 10 x 100 x 100 x 21 =
42 billion iterations of the Metropolis algorithm

Running time on an i5 PC was about five days! Opportunity for *parallel processing.*


Heat capacity. J=0.04eV. T_c=1043 K
Average spin. J=0.04eV. T_c=1043 K
Heat capacity / eV/K
x 10^-6
Average spin
T/T_c
Heat capacity. J=0.04eV. T_c=1043 K
Average spin. J=0.04eV. T_c=1043 K
Heat capacity / eV/K
x 10^-6
Average spin
T/T_c

$F/J = 0$

$F/J = 0.53$



Mean abs spin N=10, R=100, I=2000, Tc=1043K, J=0.039644eV
F/J
T/Tc
Average spin. J=0.04eV. T_c=1043 K
Average spin
T/T_c
Heat capacity. J=0.04eV. T_c=1043 K
x 10^{-6}
Heat capacity / eV/K
T/T_c
Heat capacity eV/K N=10, R=100, I=2000, Tc=1043K, J=0.039644eV
x 10^{-5}
F/J
T/Tc