BPhO Computational Challenge

Seminar 06: Exoplanets and Kepler's Third Law

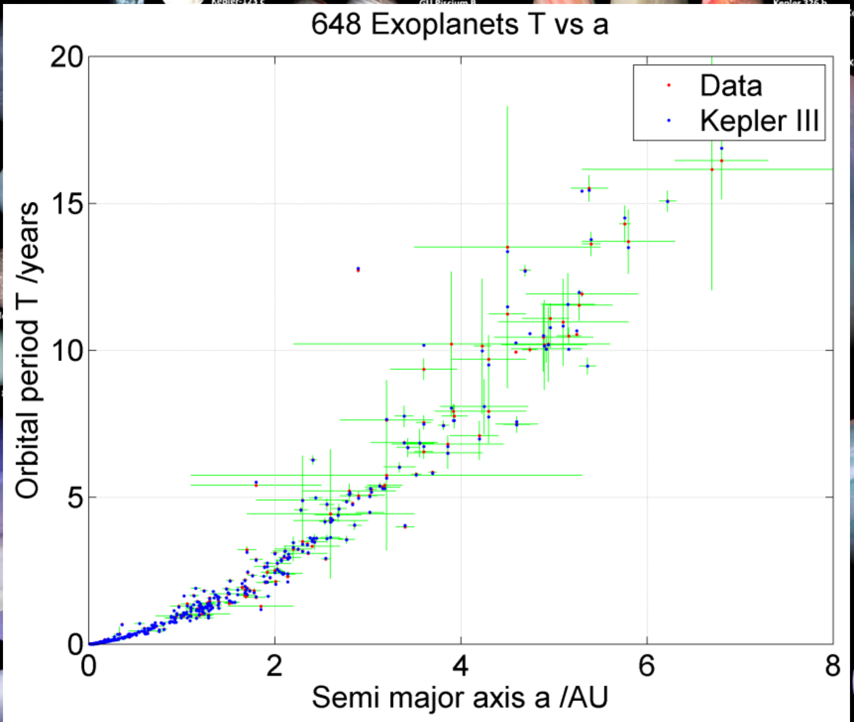
Dr Andrew French.
December 2021.

BPhO British Physics Olympiad

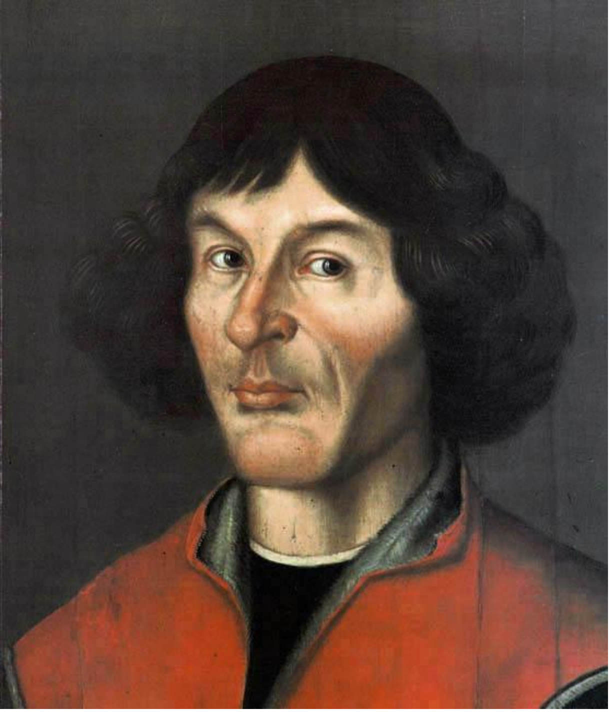
Image by Martin Vargic

Extrasolar Planets and Kepler's Third Law

$$T^2 = \frac{4\pi^2}{G(M+m)}a^3$$



Kepler’s laws of orbital motion

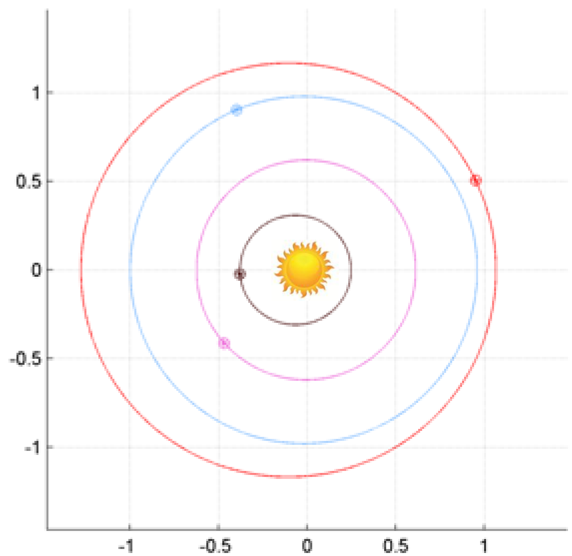


Nicolaus Copernicus
1473-1543

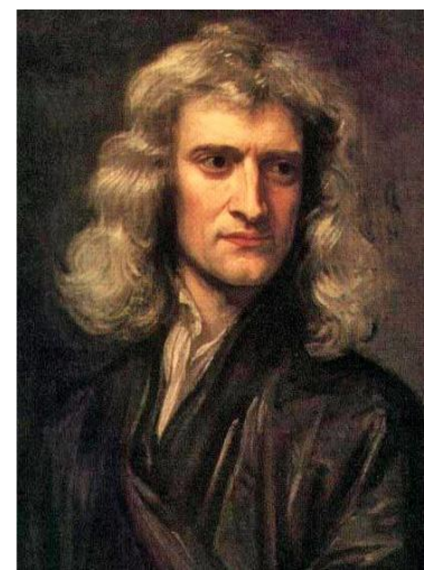
Tycho Brahe
1546-1601

Nose lost in 1566 following a sword duel with third cousin Manderup Parsberg over the legitimacy of a mathematical formula!

Johannes
Kepler
1571-1630



Isaac
Newton
1642-
1727



Kepler's three laws are:

1. *The orbit of every planet in the solar system is an ellipse with the Sun at one of the two foci.*
2. *A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.*
3. *The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.*

The wording of Kepler's laws implies a specific application to the solar system. However, the laws are more generally applicable to any system of two masses whose mutual attraction is an inverse-square law.

$$r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}$$

Polar equation of ellipse

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

Eccentricity of ellipse

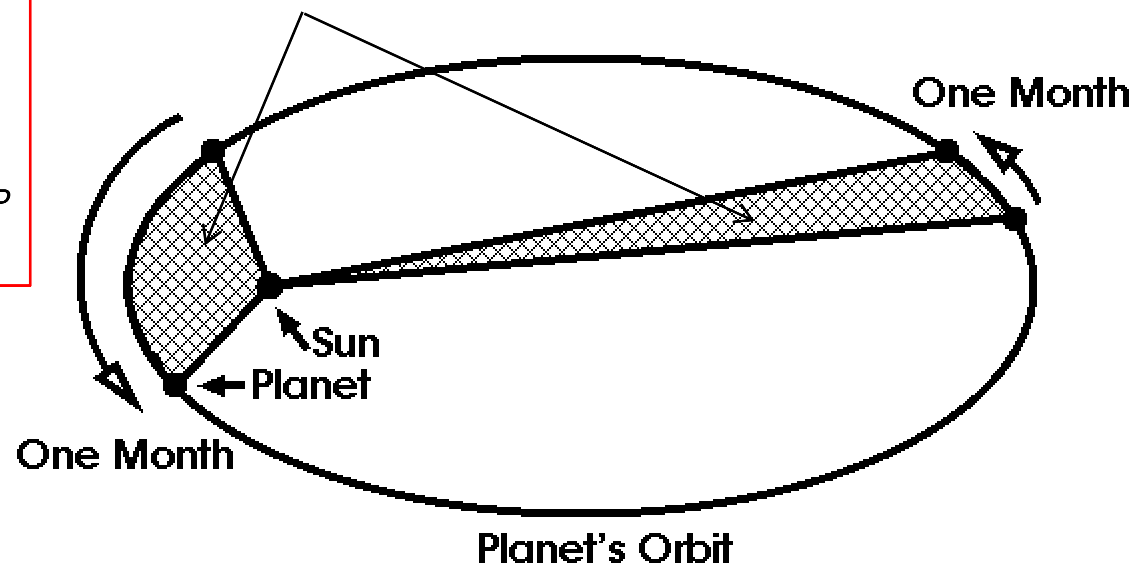
$$P^2 = \frac{4\pi^2}{G(m+M_\odot)}a^3$$

Orbital period *P*

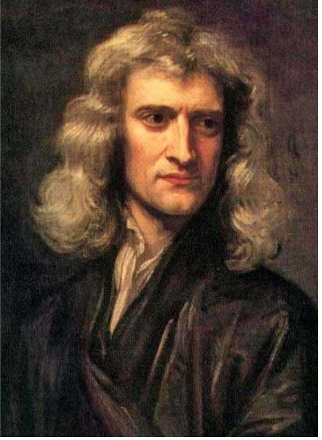
$$\frac{dA}{dt} = \tfrac{1}{2}\sqrt{G(m+M_\odot)\left(1-\varepsilon^2\right)a}$$

This is a constant

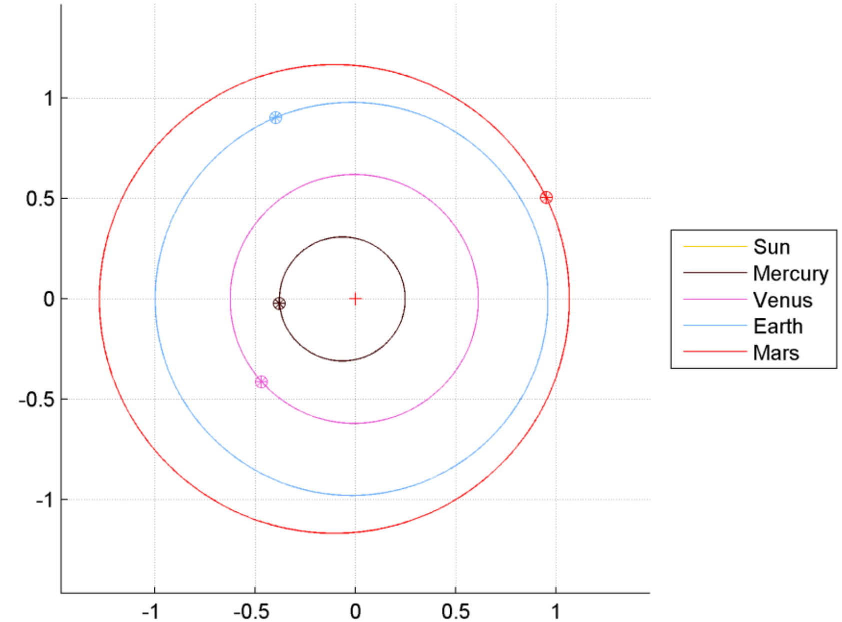
Equal areas swept out in equal times



Johannes Kepler
1571-1630



Isaac Newton (1642-1727) developed a mathematical model of Gravity which predicted the elliptical orbits proposed by Kepler



Planet and Solar masses

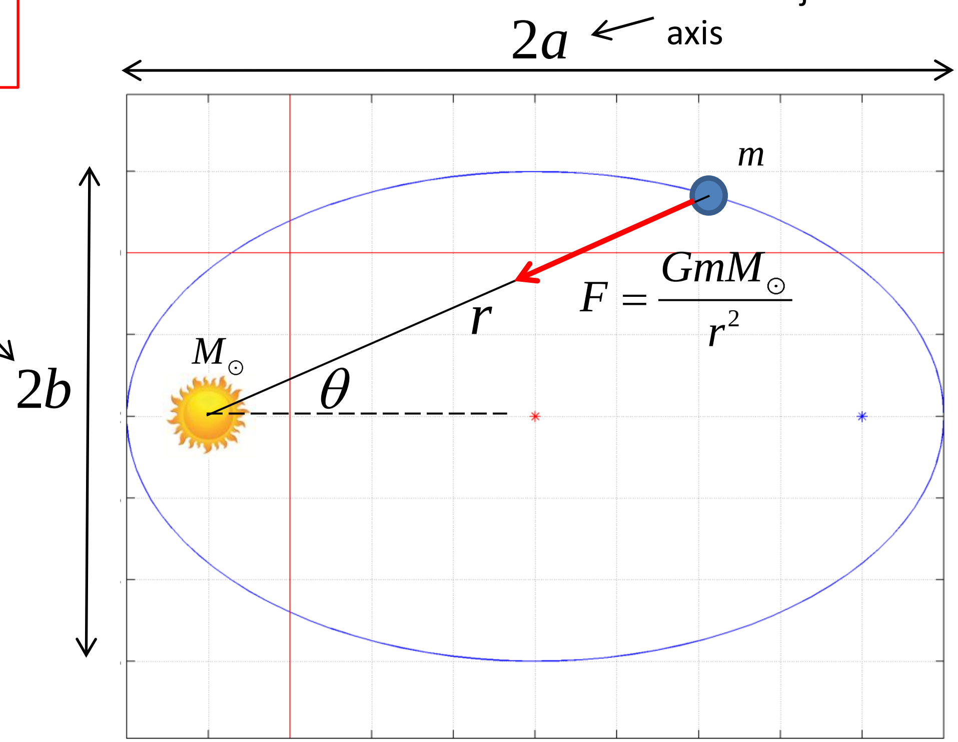
Force of gravity

$$F = \frac{GmM_{\odot}}{r^2}$$

$$G = 6.67 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$$

Semi-major axis

Semi-minor axis



$$r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}$$

Polar equation of ellipse

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

Eccentricity of ellipse

$$P^2 = \frac{4\pi^2}{G(m+M_{\odot})}a^3$$

Orbital period P

Object	Mass in Earth masses	Distance from Sun in AU	Radius in Earth radii	Rotational period /days	Orbital period /years	Gravitational field (in terms of g = 9.81 ms^-2)
Saturn	95.16	9.58	9.45	0.44	29.63	1.07
Uranus	14.50	19.29	4.01	0.72	84.75	0.90
Jupiter	317.85	5.20	11.21	0.41	11.86	2.53
Sun	332,837	-	109.12	-	-	27.95
Neptune	17.20	30.25	3.88	0.67	166.34	1.14
Pluto	0.00	39.51	0.19	6.39	248.35	0.09
Mars	0.107	1.523	0.53	1.03	1.88	0.38
Venus	0.815	0.723	0.95	243.02	0.62	0.90
Mercury	0.055	0.387	0.38	58.65	0.24	0.37
Earth	1.000	1.000	1.00	1.00	1.00	1.00

For our Solar System:

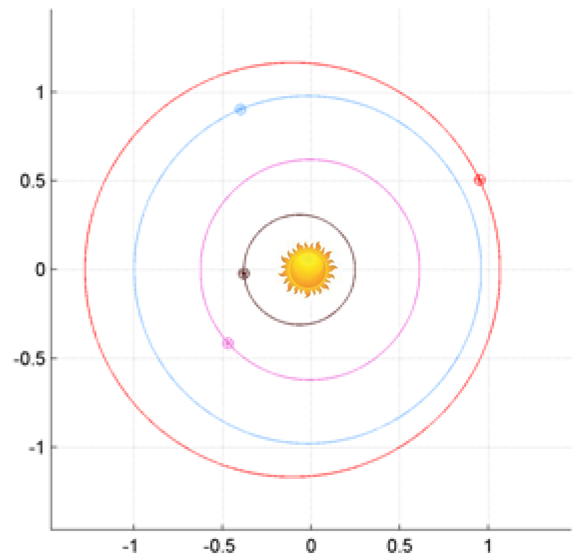
$m \ll M_{\odot}$

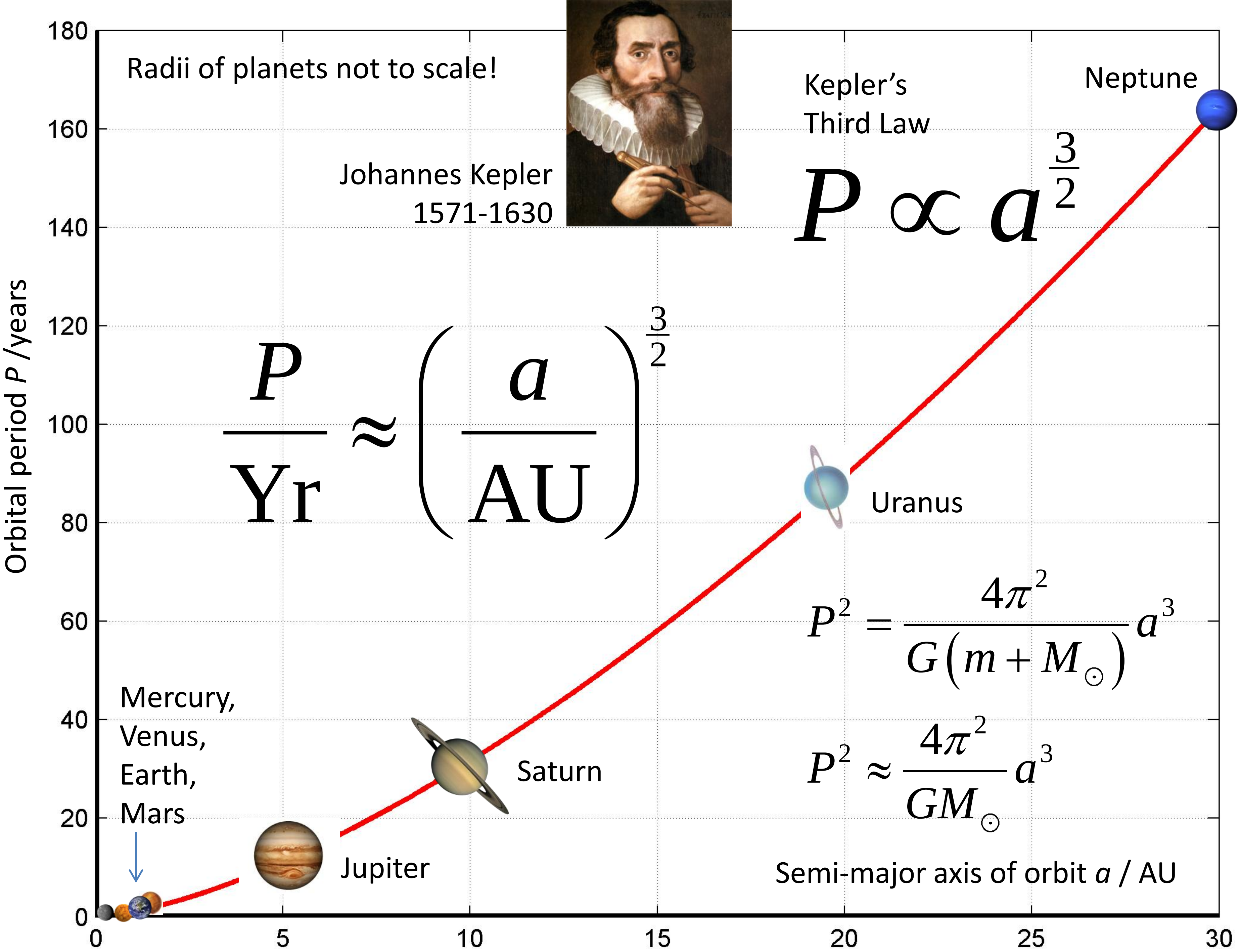
$$P^2 = \frac{4\pi^2}{G\left(m + M_{\odot}\right)} a^3$$

$$P^2 \approx \frac{4\pi^2}{GM_{\odot}} a^3$$

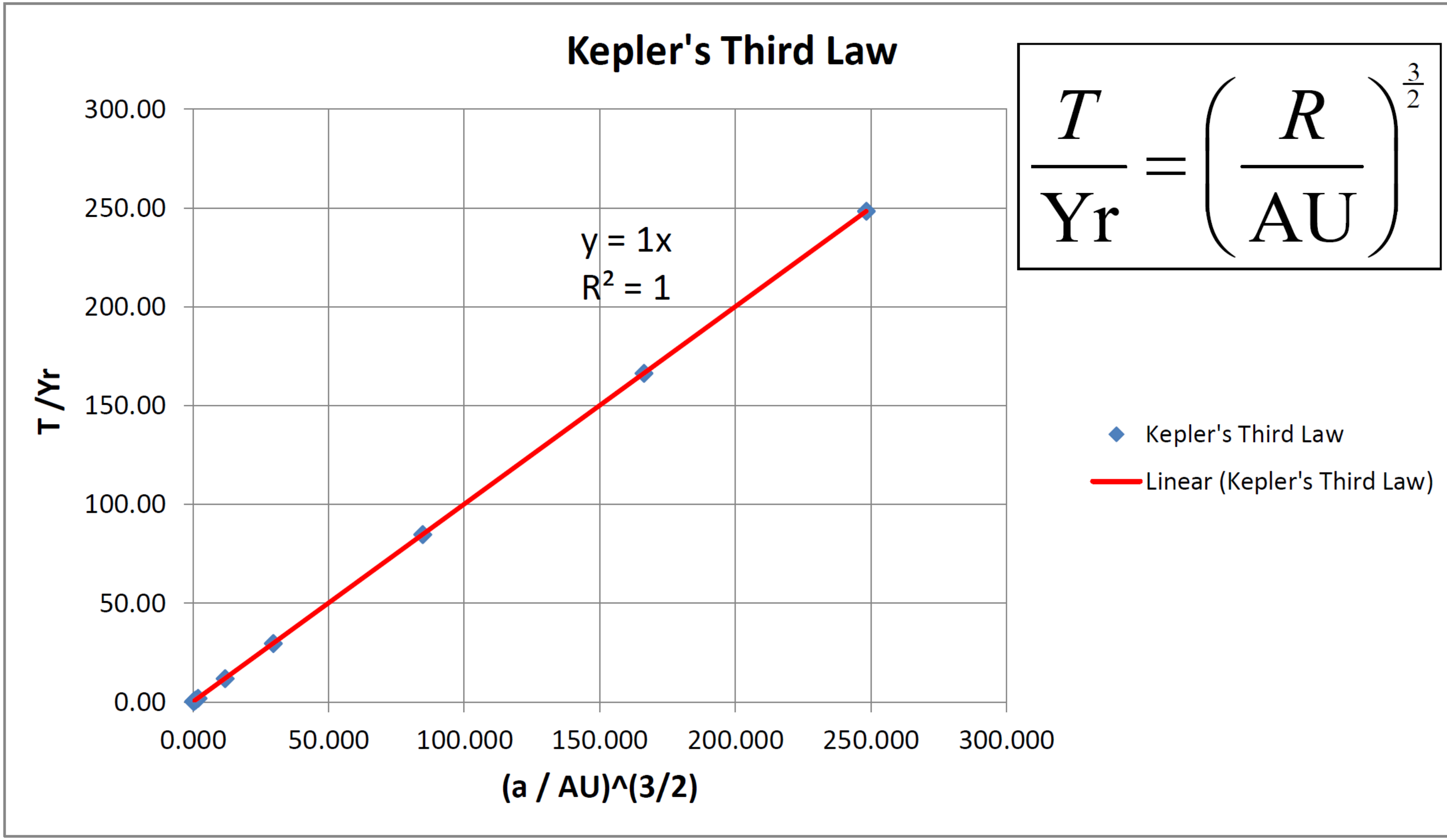
$$\mathrm{Yr}^2 = \frac{4\pi^2}{GM_{\odot}} \mathrm{AU}^3$$

$$\therefore \frac{P}{\mathrm{Yr}} \approx \left(\frac{a}{\mathrm{AU}}\right)^{\frac{3}{2}}$$





Radii of planets not to scale!
Johannes Kepler
1571-1630
Kepler's
Third Law
$P \propto a^{\frac{3}{2}}$
Neptune
$\frac{P}{\mathrm{Yr}} \approx \left(\frac{a}{\mathrm{AU}}\right)^{\frac{3}{2}}$
Uranus
$P^2 = \frac{4\pi^2}{G\left(m + M_\odot\right)} a^3$
$P^2 \approx \frac{4\pi^2}{GM_\odot} a^3$
Mercury,
Venus,
Earth,
Mars
Saturn
Jupiter
Semi-major axis of orbit a / AU
Orbital period P /years
0
20
40
60
80
100
120
140
160
180
0
5
10
15
20
25
30



$$\frac{T}{Yr} = \left(\frac{R}{AU}\right)^{\frac{3}{2}}$$

A *very strong* correlation of Kepler III to orbital data for planets in our solar system!

Excel Exoplanet Kepler III

Raw data (slightly processed!) from www.exoplanet.eu

660 Exoplanets

Exoplanet	Star	Star mass /solar mass	Exoplanet mass / mass of Jupiter	lower mass / mass of Jupiter	upper mass /mass of Jupiter	orbital_period /days	min orbital period /days	max orbital period /days	semi_major_axis a /AU	min a /AU	max a /AU
11 Com b	11 Com	2.7	19.4	17.9	20.9	326.03	325.71	326.35	1.29	1.24	1.34
11 UMi b	11 UMi	1.8	10.5	8.03	12.97	516.22	512.97	519.47	1.54	1.47	1.61
14 Her b	14 Her	0.9	4.64	4.45	4.83	1773.4	1770.9	1775.9	2.77	2.72	2.82
16 Cyg B b	16 Cyg B	1.01	1.68	1.61	1.75	799.5	798.9	800.1	1.68	1.65	1.71
1SWASP J1407 b	1SWASP J1407	0.9	20	14	26	3725	2825	4625	3.9	2.2	5.6
24 Sex b	24 Sex	1.54	1.99	1.61	2.25	452.8	448.3	454.9	1.333	1.324	1.337
24 Sex c	24 Sex	1.54	0.86	0.64	1.21	883	869	915	2.08	2.06	2.13
2M 0746+20 b	2M 0746+20	0.12	30	5	55	4640	4615	4665	2.897	2.892	2.902
2M 1936+4603 b	2M 1938+4603	0.6	1.9	1.8	2	416	414	418	0.92	0.9	0.94
2M 2140+16 b	2M 2140+16	0.08	20	0	100	7340	6756	7924	3.53	3.38	3.68
2M 2206-20 b	2M 2206-20	0.13	30	10	100	8686	8616.6	8755.4	4.48	4.08	4.88
30 Ari B b	30 Ari B	1.22	9.88	8.94	10.82	335.1	332.6	337.6	0.995	0.983	1.007
4 Uma b	4 Uma	1.234	7.1	5.5	8.7	269.3	267.34	271.26	0.87	0.83	0.91
42 Dra b	42 Dra	0.98	3.88	3.03	4.73	479.1	472.9	485.3	1.19	1.18	1.2
47 Uma b	47 Uma	1.03	2.53	2.47	2.6	1078	1076	1080	2.1	2.08	2.12
47 Uma c	47 Uma	1.03	0.54	0.467	0.606	2391	2304	2491	3.6	3.5	3.7
47 Uma d	47 Uma	1.03	1.64	1.16	1.93	14002	8907	18020	11.6	8.7	13.7
55 Cnc b	55 Cnc	0.905	0.8	0.788	0.812	14.651	14.6509	14.6511	0.1134	0.1128	0.114
55 Cnc c	55 Cnc	0.905	0.169	0.161	0.177	44.3446	44.3376	44.3516	0.2403	0.2386	0.242
55 Cnc d	55 Cnc	0.905	3.835	3.755	3.915	5218	4988	5448	5.76	5.7	5.82
55 Cnc e	55 Cnc	0.905	0.0261775	0.02495	0.027405	0.736542	0.736539	0.736545	0.0156	0.01549	0.01571
55 Cnc f	55 Cnc	0.905	0.144	0.104	0.184	260.7	259.6	261.8	0.781	0.775	0.787

Kepler's Third Law

In the exoplanet data, planet masses are given in Jupiter masses →

$$P^2 = \frac{4\pi^2}{G(M+m)} a^3$$

$$\mathrm{Yr}^2 = \frac{4\pi^2}{G(M_\odot + m_\oplus)} \mathrm{AU}^3 \approx \frac{4\pi^2}{GM_\odot} \mathrm{AU}^3$$

$$\therefore \left(\frac{P}{\mathrm{Yr}}\right)^2 = \left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right)^{-1} \left(\frac{a}{\mathrm{AU}}\right)^3$$

$$\therefore 2\log\left(\frac{P}{\mathrm{Yr}}\right) = -\log\left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right) + 3\log\left(\frac{a}{\mathrm{AU}}\right)$$

$$y = 2\log\left(\frac{P}{\mathrm{Yr}}\right) + \log\left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right)$$

$$x = \log\left(\frac{a}{\mathrm{AU}}\right)$$

$$M_\odot = 1.98847 \times 10^{30}\,\mathrm{kg}$$

$$m_J = 1.898 \times 10^{27}\,\mathrm{kg}$$

$$m_\oplus = 5.972 \times 10^{24}\,\mathrm{kg}$$

$$G = 6.67 \times 10^{-11}\,\mathrm{Nm^2kg^{-2}}$$

$$\mathrm{AU} = 1.49597871 \times 10^{11}\mathrm{m}$$

$$24 \times 3600\mathrm{s} = 1\mathrm{day}$$

$$1\mathrm{Yr} = 365.2422\,\mathrm{days}$$

Note in most cases:

$$\frac{M}{M_\odot} + \frac{m}{M_\odot} \approx \frac{M}{M_\odot}$$

So y vs x should be a **straight line** from the origin of **gradient 3**

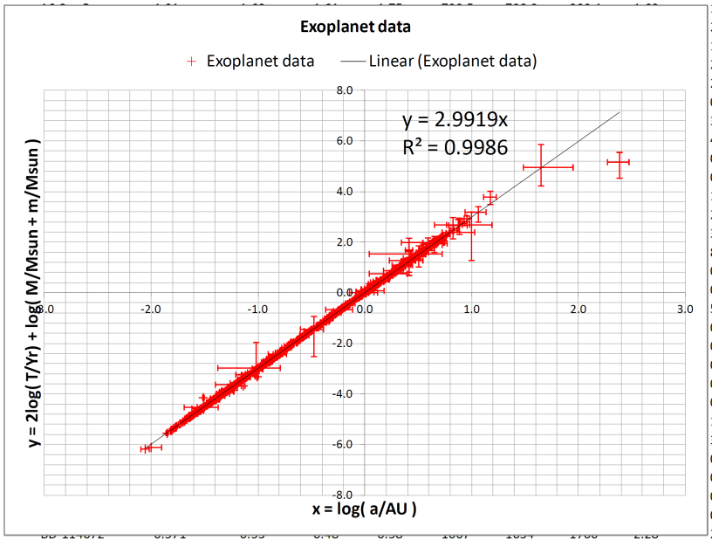
Analysis using Microsoft Excel

Use these to set the error bars

EXOPLANET ANALYSIS
A. FRENCH March 2019

Solar mass /kg	Jupiter mass /kg	Yr in days
1.98847E+30	1.90E+27	365.2422

Exoplanet	Star	Star mass /solar mass	Exoplanet mass / mass of Jupiter	lower mass / mass of Jupiter	upper mass /mass of Jupiter	orbital_period /days	min orbital period /days	max orbital period /days	semi_major_axis a /AU	min a /AU	max a /AU	x - xmin	x = log(a/AU)	xmax - x	2log(Tmin/Yr)	2log(T/Yr)	2log(Tmax/Yr)	lower	log((M+m)/Msun)	upper	y - ymin	y	ymax - y
11 Com b	11 Com	2.7	19.4	17.9	20.9	326.03	325.71	326.35	1.29	1.24	1.34	0.0172	0.1106	0.0165	-0.0995	-0.0986	-0.0978	0.4341	0.4343	0.4346	0.0011	0.3357	0.0011
11 UMi b	11 UMi	1.8	10.5	8.03	12.97	516.22	512.97	519.47	1.54	1.47	1.61	0.0202	0.1875	0.0193	0.2950	0.3005	0.3060	0.2571	0.2577	0.2582	0.0061	0.5582	0.0060
14 Her b	14 Her	0.9	4.64	4.45	4.83	1773.4	1770.9	1775.9	2.77	2.72	2.82	0.0079	0.4425	0.0078	1.3712	1.3725	1.3737	-0.0437	-0.0436	-0.0435	0.0013	1.3288	0.0013
16 Cyg B b										1.65	1.71	0.0078	0.2253	0.0077	0.6798	0.6805	0.6811	0.0050	0.0050	0.0050	0.0007	0.6855	0.0007
1SWASP J1407 b										2.2									-0.0366	-0.0339	0.2429	1.9804	0.1907
24 Sex b										1.324									0.1881	0.1881	0.0088	0.3747	0.0041
24 Sex c										2.06									0.1878	0.1878	0.0139	0.9545	0.0310
2M 0746+20 b										2.892									-0.8279	-0.7632	0.0807	1.3800	0.0693
2M 1936+4603 b										0.9									-0.2205	-0.2205	0.0043	-0.1075	0.0042
2M 2140+16 b										3.38									-1.0040	-0.7558	0.1650	1.6023	0.3146
2M 2206-20 b										4.08									-0.7996	-0.6469	0.0627	1.9529	0.1596
30 Ari B b										0.983									0.0897	0.0900	0.0068	0.0149	0.0068
4 Uma b										0.83									0.0937	0.0942	0.0069	-0.1710	0.0068
42 Dra b										1.18									-0.0071	-0.0068	0.0117	0.2286	0.0115
47 Uma b										2.08									0.0139	0.0139	0.0016	0.9539	0.0016
47 Uma c										3.5									0.0131	0.0131	0.0322	1.6451	0.0356
47 Uma d										8.7									0.0135	0.0136	0.3931	3.1807	0.2192
55 Cnc b										0.1128									-0.0430	-0.0430	0.0000	-2.8364	0.0000
55 Cnc c										0.2386									-0.0433	-0.0433	0.0001	-1.8748	0.0001
55 Cnc d										5.7									-0.0416	-0.0416	0.0392	2.2682	0.0375
55 Cnc e										0.01549									-0.0433	-0.0433	0.0000	-5.4341	0.0000
55 Cnc f										0.775									-0.0433	-0.0433	0.0037	-0.3362	0.0037
61 Vir b										0.050196									-0.0223	-0.0223	0.0001	-3.8978	0.0001
61 Vir c										0.2174									-0.0223	-0.0222	0.0008	-1.9874	0.0008
61 Vir d										0.475									-0.0222	-0.0222	0.0039	-0.9675	0.0039
7 CMa b										1.8									0.1826	0.1827	0.0197	0.8224	0.0193
BD+49 828										3.88									0.1823	0.1823	0.1070	1.8837	0.0584
BD-061339 b										0.0421									-0.1549	-0.1549	0.0001	-4.1040	0.0001
BD-061339 c										0.428									-0.1548	-0.1548	0.0031	-1.0796	0.0030
BD-082823 b										0.054									-0.1307	-0.1307	0.0031	-3.7595	0.0031
BD-082823 c										0.66	0.7	0.0130	-0.1675	0.0126	-0.3790	-0.3735	-0.3680	-0.1306	-0.1306	-0.1306	0.0055	-0.5041	0.0055
BD-114672 b										2.21	2.35	0.0135	0.3579	0.0131	1.3013	1.3187	1.3357	-0.2430	-0.2430	-0.2429	0.0174	1.0757	0.0171
BD-17 63 b	BD-17 63	0.74	5.1	4.98	5.22	655.6	655	656.2	1.34	1.32	1.36	0.0065	0.1271	0.0064	0.5073	0.5081	0.5089	-0.1280	-0.1279	-0.1279	0.0009	0.3802	0.0009
CFBDS 1458 b	CFBDS 1458	0.024	6.5	2	11	14600	5800	23400	2.6	2.4	2.8	0.0348	0.4150	0.0322	2.4017	3.2035	3.6133	-1.5865	-1.5199	-1.4622	0.8685	1.6836	0.4675
CoRoT-1 b	CoRoT-1	0.95	1.03	0.91	1.15	1.508956	1.508949	1.508962	0.0254	0.025	0.0258	0.0069	-1.5952	0.0068	-4.7678	-4.7678	-4.7678	-0.0219	-0.0218	-0.0218	0.0001	-4.7896	0.0001
CoRoT-10 b	CoRoT-10	0.89	2.75	2.61	2.89	13.2406	13.2404	13.2408	0.1055	0.1034	0.1076	0.0087	-0.9767	0.0086	-2.8814	-2.8813	-2.8813	-0.0494	-0.0493	-0.0493	0.0001	-2.9307	0.0001
CoRoT-11 b	CoRoT-11	1.56	2.33	2.06	2.6	2.994325	2.994304	2.994346	0.04351	0.04315	0.04387	0.0036	-1.3614	0.0036	-4.1726	-4.1726	-4.1726	0.1937	0.1937	0.1938	0.0001	-3.9788	0.0001



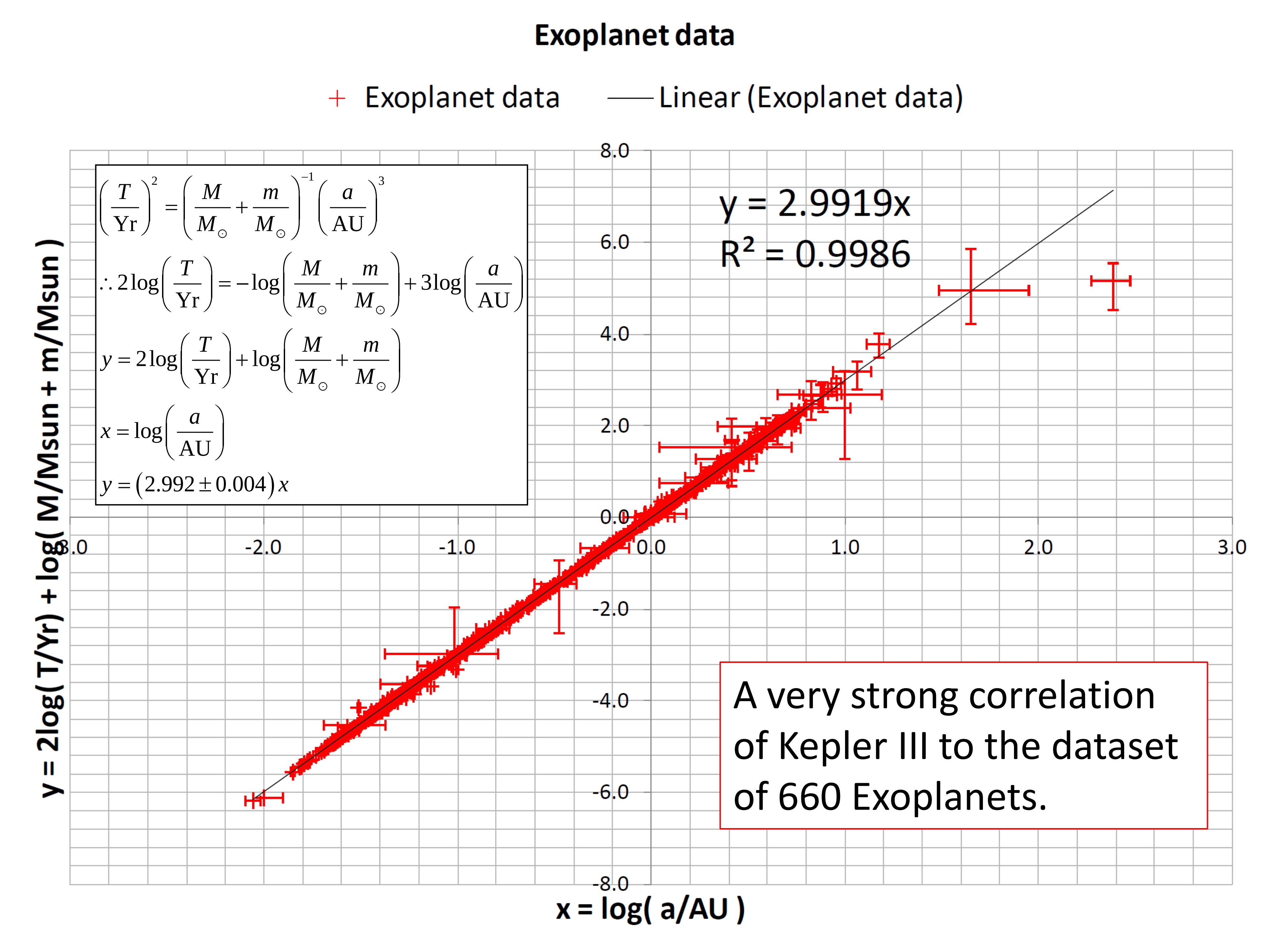
$$\left(\frac{T}{\mathrm{Yr}}\right)^2 = \left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right)^{-1} \left(\frac{a}{\mathrm{AU}}\right)^3$$

$$\therefore 2\log\left(\frac{T}{\mathrm{Yr}}\right) = -\log\left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right) + 3\log\left(\frac{a}{\mathrm{AU}}\right)$$

$$y = 2\log\left(\frac{T}{\mathrm{Yr}}\right) + \log\left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right)$$

$$x = \log\left(\frac{a}{\mathrm{AU}}\right)$$

$$y = (2.992 \pm 0.004)\,x$$



Exoplanet data
Exoplanet data
Linear (Exoplanet data)
$\left(\frac{T}{\mathrm{Yr}}\right)^2 = \left(\frac{M}{M_\odot}+\frac{m}{M_\odot}\right)^{-1}\left(\frac{a}{\mathrm{AU}}\right)^3$
$\therefore 2\log\left(\frac{T}{\mathrm{Yr}}\right) = -\log\left(\frac{M}{M_\odot}+\frac{m}{M_\odot}\right)+3\log\left(\frac{a}{\mathrm{AU}}\right)$
$y = 2\log\left(\frac{T}{\mathrm{Yr}}\right)+\log\left(\frac{M}{M_\odot}+\frac{m}{M_\odot}\right)$
$x = \log\left(\frac{a}{\mathrm{AU}}\right)$
$y = (2.992 \pm 0.004)x$
y = 2.9919x
$R^2 = 0.9986$
A very strong correlation of Kepler III to the dataset of 660 Exoplanets.
y = 2log(T/Yr) + log(M/Msun + m/Msun)
x = log(a/AU)
8.0
6.0
4.0
2.0
0.0
-2.0
-4.0
-6.0
-8.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0

LINE OF BEST FIT CALCULATOR y = mx

Dr Andy French. March 2019

paste as values x,y data here

x	y	x^2	y^2	xy	xfit	yfit	(y-fit)^2	ylower	yupper
0.111	0.336	0.012	0.113	0.037	0.111	0.331	0.000	0.330	0.331
0.188	0.558	0.035	0.312	0.105	0.188	0.561	0.000	0.560	0.562
0.442	1.329	0.196	1.766	0.588	0.442	1.324	0.000	1.322	1.326
0.225	0.685	0.051	0.470	0.154	0.225	0.674	0.000	0.673	0.675
0.591	1.980	0.349	3.922	1.171	0.591	1.768	0.045	1.766	1.771
0.125	0.375	0.016	0.140	0.047	0.125	0.373	0.000	0.373	0.374
0.318	0.955	0.101	0.911	0.304	0.318	0.952	0.000	0.950	0.953
0.462	1.380	0.213	1.904	0.637	0.462	1.382	0.000	1.380	1.384
-0.036	-0.108	0.001	0.012	0.004	-0.036	-0.108	0.000	-0.108	-0.109
0.548	1.602	0.300	2.567	0.878	0.548	1.639	0.001	1.637	1.641
0.651	1.953	0.424	3.814	1.272	0.651	1.949	0.000	1.946	1.951
-0.002	0.015	0.000	0.000	0.000	-0.002	-0.007	0.000	-0.007	-0.007
-0.060	-0.171	0.004	0.029	0.010	-0.060	-0.181	0.000	-0.181	-0.181
0.076	0.229	0.006	0.052	0.017	0.076	0.226	0.000	0.226	0.226
0.322	0.954	0.104	0.910	0.307	0.322	0.964	0.000	0.963	0.965
0.556	1.645	0.309	2.706	0.915	0.556	1.664	0.000	1.662	1.667
1.064	3.181	1.133	10.117	3.386	1.064	3.185	0.000	3.180	3.189
-0.945	-2.836	0.894	8.045	2.682	-0.945	-2.829	0.000	-2.824	-2.833
-0.619	-1.875	0.383	3.515	1.161	-0.619	-1.853	0.000	-1.850	-1.855
0.760	2.268	0.578	5.145	1.725	0.760	2.275	0.000	2.272	2.278
-1.807	-5.434	3.265	29.530	9.819	-1.807	-5.406	0.001	-5.398	-5.414
-0.107	-0.336	0.012	0.113	0.036	-0.107	-0.321	0.000	-0.321	-0.322
-1.299	-3.898	1.688	15.193	5.064	-1.299	-3.887	0.000	-3.882	-3.893
-0.663	-1.987	0.439	3.950	1.317	-0.663	-1.982	0.000	-1.979	-1.985
-0.322	-0.968	0.104	0.936	0.312	-0.322	-0.965	0.000	-0.963	-0.966
0.279	0.822	0.078	0.676	0.229	0.279	0.834	0.000	0.833	0.835
0.623	1.884	0.388	3.548	1.174	0.623	1.865	0.000	1.862	1.867
-1.369	-4.104	1.873	16.843	5.617	-1.369	-4.095	0.000	-4.089	-4.101
-0.362	-1.080	0.131	1.166	0.390	-0.362	-1.082	0.000	-1.080	-1.083
-1.252	-3.760	1.567	14.134	4.706	-1.252	-3.745	0.000	-3.740	-3.751
-0.167	-0.504	0.028	0.254	0.084	-0.167	-0.501	0.000	-0.500	-0.502
0.358	1.076	0.128	1.157	0.385	0.358	1.071	0.000	1.069	1.072
0.127	0.380	0.016	0.145	0.048	0.127	0.380	0.000	0.380	0.381
0.415	1.684	0.172	2.835	0.699	0.415	1.242	0.195	1.240	1.243
-1.595	-4.790	2.545	22.941	7.640	-1.595	-4.773	0.000	-4.766	-4.780
-0.977	-2.931	0.954	8.589	2.863	-0.977	-2.922	0.000	-2.918	-2.927
-1.361	-3.979	1.853	15.831	5.417	-1.361	-4.073	0.009	-4.067	-4.079
-1.396	-4.189	1.949	17.550	5.849	-1.396	-4.177	0.000	-4.171	-4.183
-1.292	-3.876	1.670	15.020	5.009	-1.292	-3.867	0.000	-3.861	-3.873
-1.569	-4.710	2.461	22.185	7.388	-1.569	-4.693	0.000	-4.686	-4.700
-1.209	-3.627	1.462	13.157	4.385	-1.209	-3.617	0.000	-3.612	-3.623
-1.336	-3.955	1.786	15.641	5.285	-1.336	-3.998	0.002	-3.992	-4.004
-1.530	-4.588	2.341	21.053	7.021	-1.530	-4.578	0.000	-4.572	-4.585
-1.286	-3.861	1.653	14.904	4.963	-1.286	-3.847	0.000	-3.841	-3.852
-1.551	-4.654	2.407	21.663	7.220	-1.551	-4.641	0.000	-4.635	-4.648

N	660

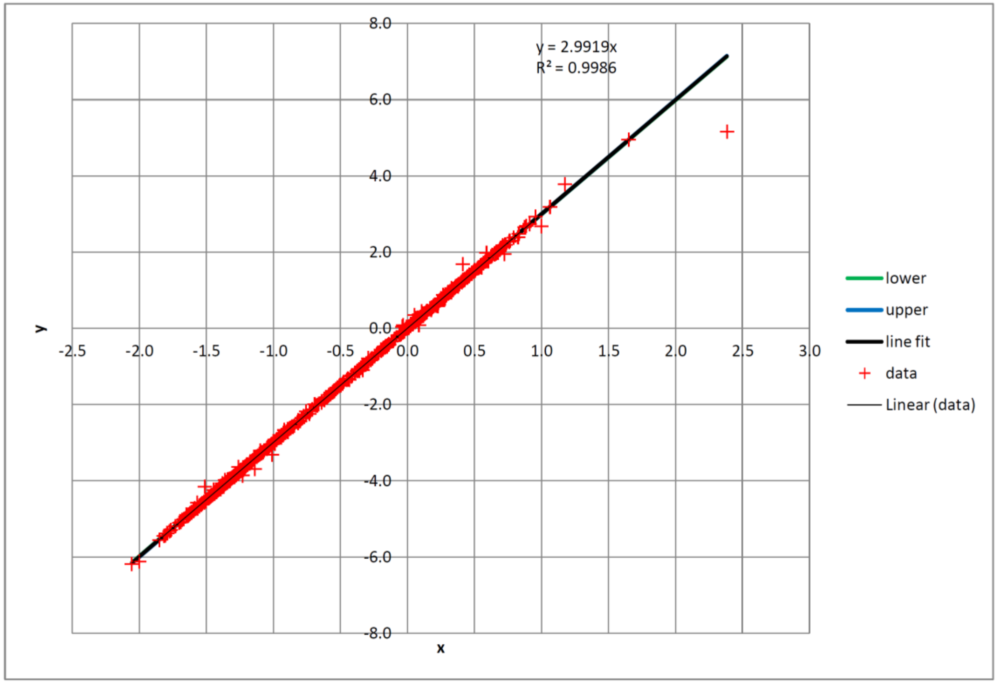
xbar	ybar	x^2 bar	y^2 bar	xy bar
-0.572	-1.716	0.950	8.508	2.841

Vx	Vy	Cov[x,y]	s
0.623	5.563	1.860	0.089

r	r^2
0.999	0.999

m
2.992

dm
0.004



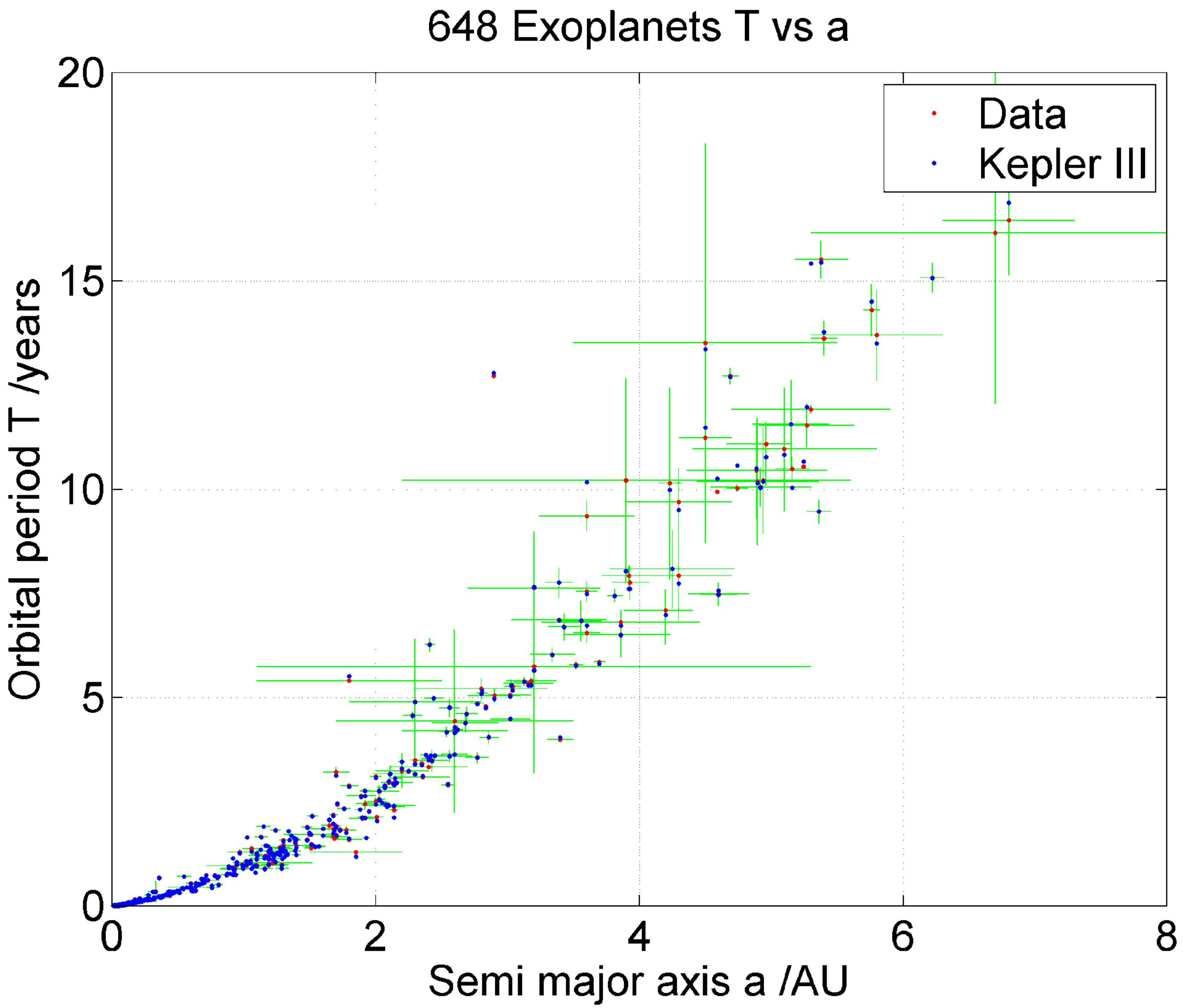
Note bug in old versions fo Excel (<2003), that will give an incorrect R^2 value for the built-in trend line function when 'set intercept at 0,0' is chosen

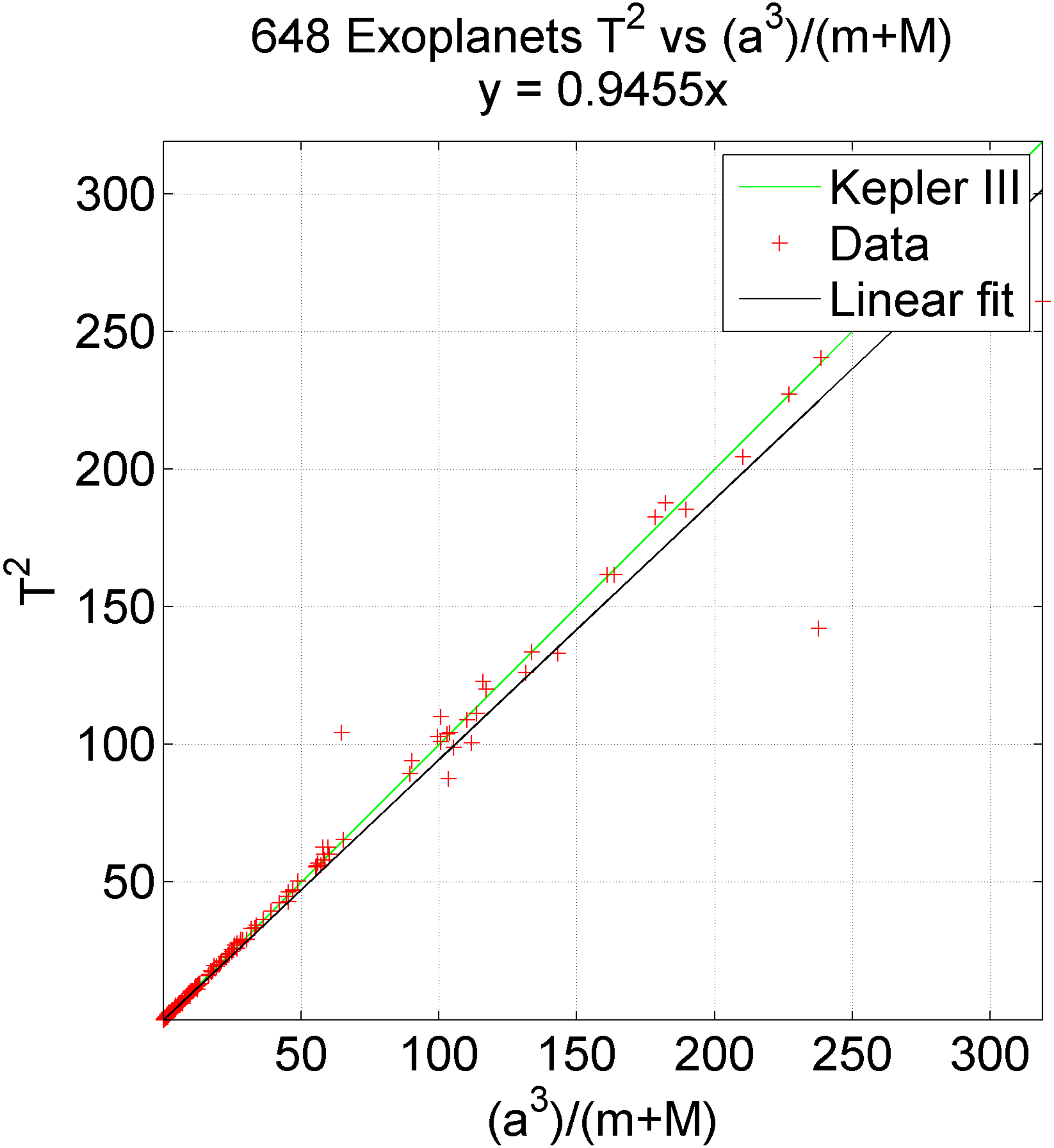
$$y = 2\log\left(\frac{T}{\mathrm{Yr}}\right) + \log\left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right), \quad x = \log\left(\frac{a}{\mathrm{AU}}\right)$$

$$y = \left(2.992 \pm 0.004\right) x$$

MATLAB Exoplanet Kepler III

Similar to the Excel analysis – but a bit easier to implement. Particularly the error bars!


648 Exoplanets T vs a
Data
Kepler III
Orbital period T /years
Semi major axis a /AU
0
2
4
6
8
0
5
10
15
20



648 Exoplanets T^2 vs (a^3)/(m+M)
y = 0.9455x
Kepler III
Data
Linear fit
300
250
200
150
100
50
T^2
50
100
150
200
250
300
(a^3)/(m+M)

Obtaining exoplanet parameters

As of January 2021, **4,395** confirmed exoplanets in **3,242** systems, with **720** systems having more than one planet.
https://en.wikipedia.org/wiki/Exoplanet#Methodology

Popular exoplanet detection methods:

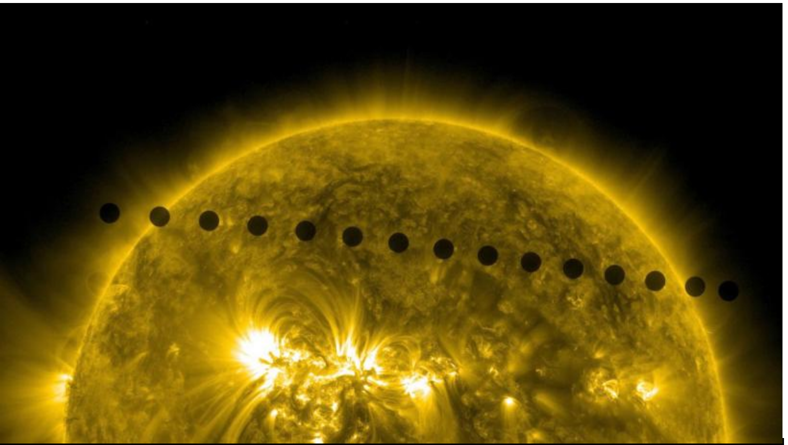
- *Detection of a dip in star **luminosity** during a **transit** of a planet*
- ***Doppler shift** in peak wavelength of star emission spectra due to **star motion** about centre of mass of star and planet*
- *Direct imaging*
- *Gravitational microlensing*

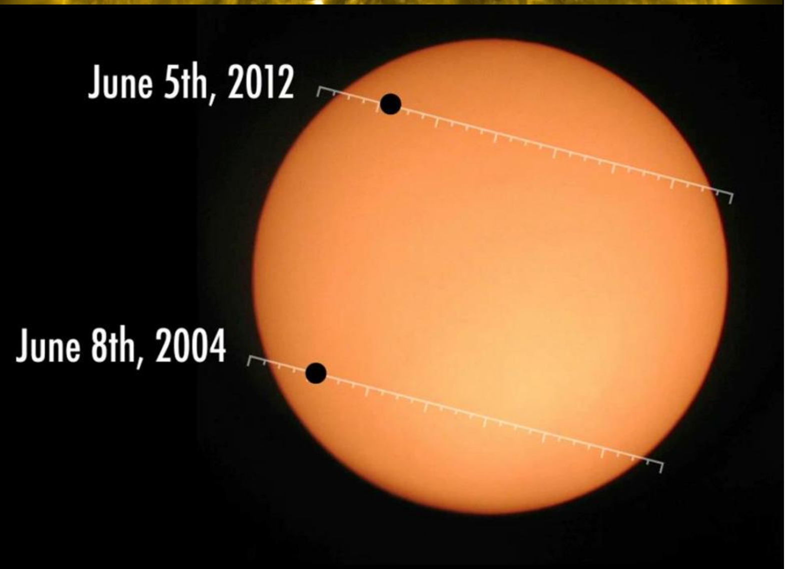
For multiple planet systems – *variations in orbital periods* due to multi-body gravitational interaction

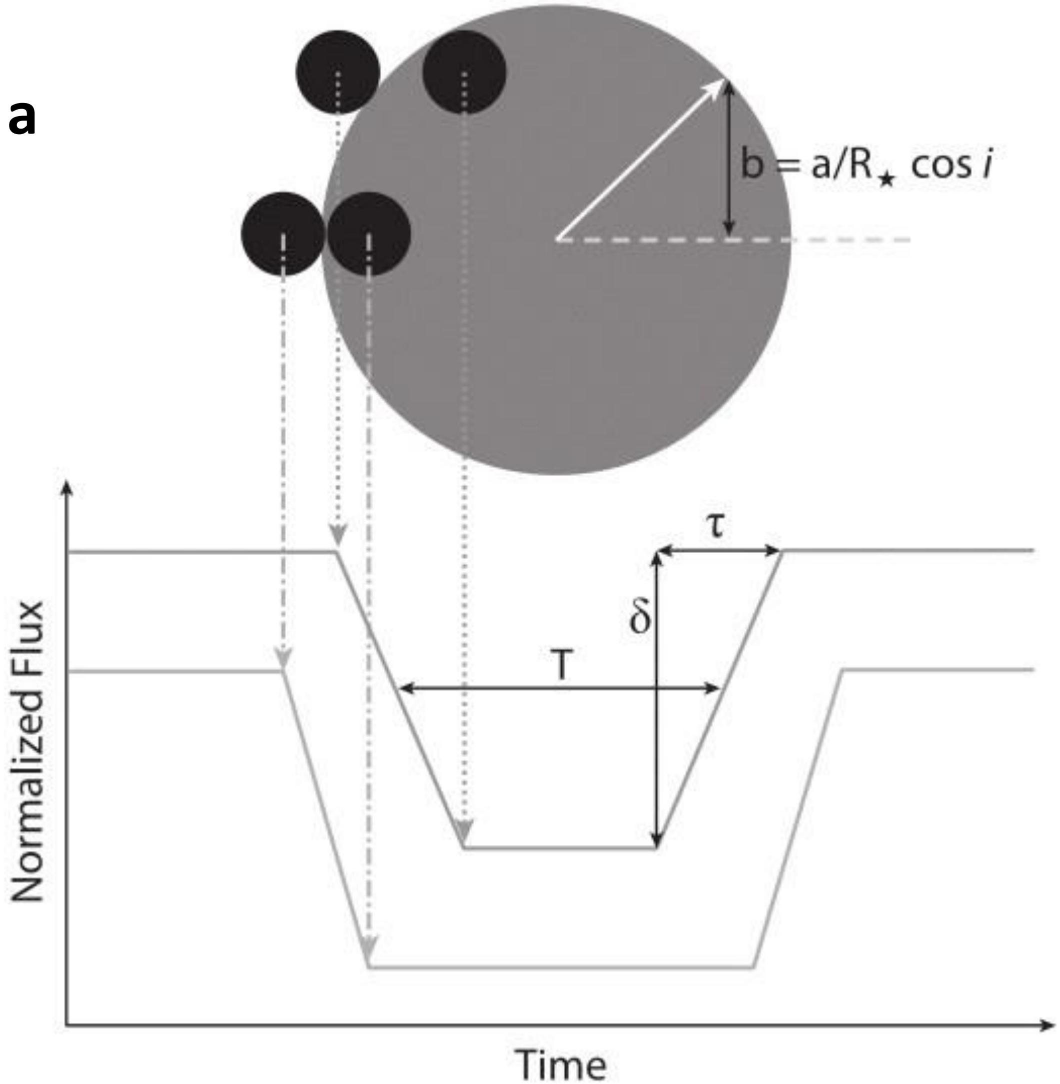
Any many more exotic techniques....
https://en.wikipedia.org/wiki/Methods_of_detecting_exoplanets

Detection of a dip in star luminosity during a transit of a planet

Transit of the Sun by Venus

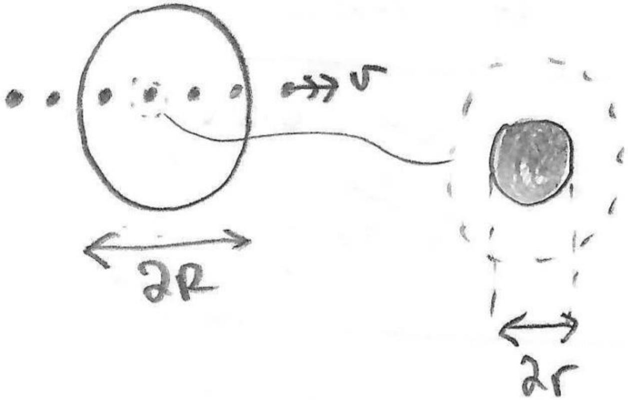


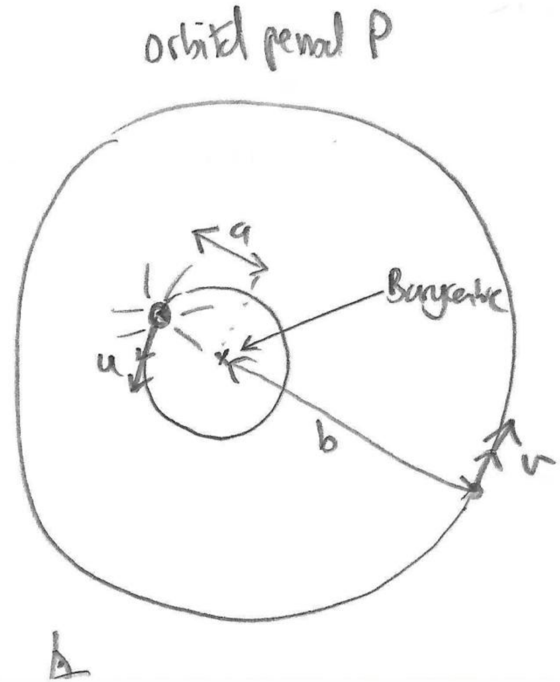




Fractional luminosity dip

$$\delta = \frac{\Delta L}{L_0} = \frac{\pi r^2}{\pi R^2} = \left(\frac{r}{R}\right)^2$$

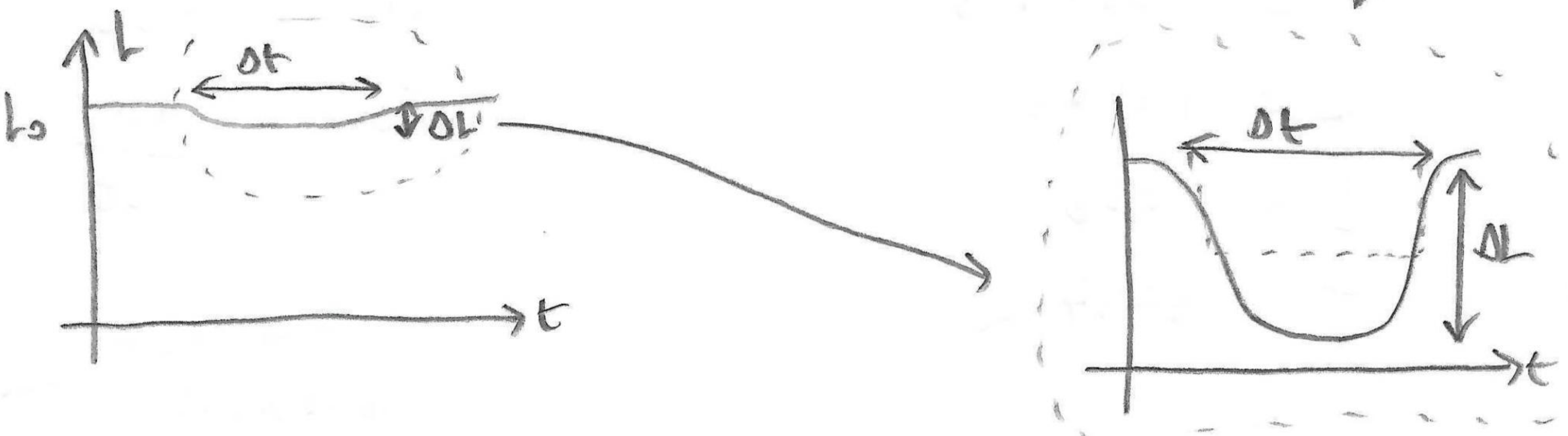




Fractional luminosity dip due to transit of planet across star

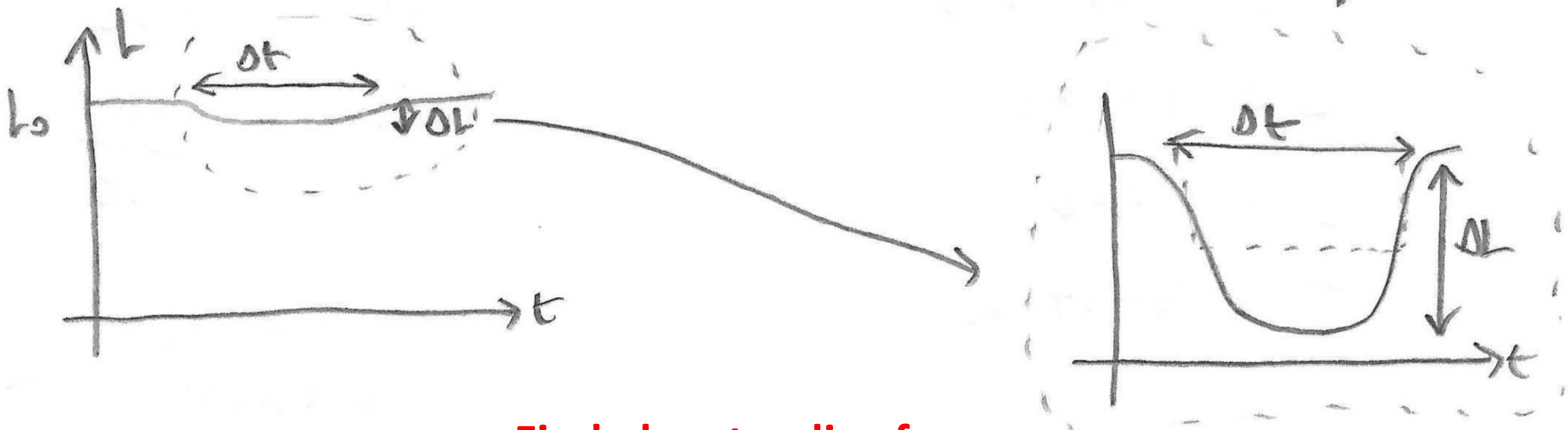
Also $\Delta t = \frac{2R}{v}$ and $v = \frac{2\pi b}{P}$

So $2R = v\Delta t$



Also $\Delta t = \frac{2R}{v}$ and $v = \frac{2\pi b}{P}$

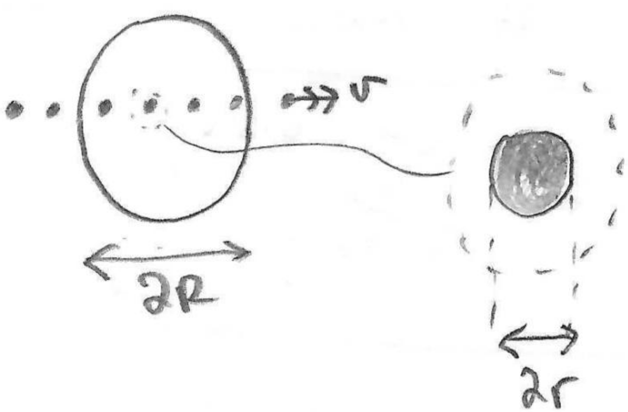
So $2R = v\Delta t$



$$\frac{\Delta L}{L_0} = \frac{\pi r^2}{\pi R^2} = \left(\frac{r}{R}\right)^2$$

Find planet radius from luminosity dip, if you know star radius and luminosity

$$\frac{r}{R_\odot} = \left(\frac{\Delta L}{L_0}\right)^{\frac{1}{2}} \left(\frac{R}{R_\odot}\right)$$



$$\frac{r}{R_\oplus} = \frac{r}{R_\odot} \times \frac{R_\odot}{R_\oplus}$$

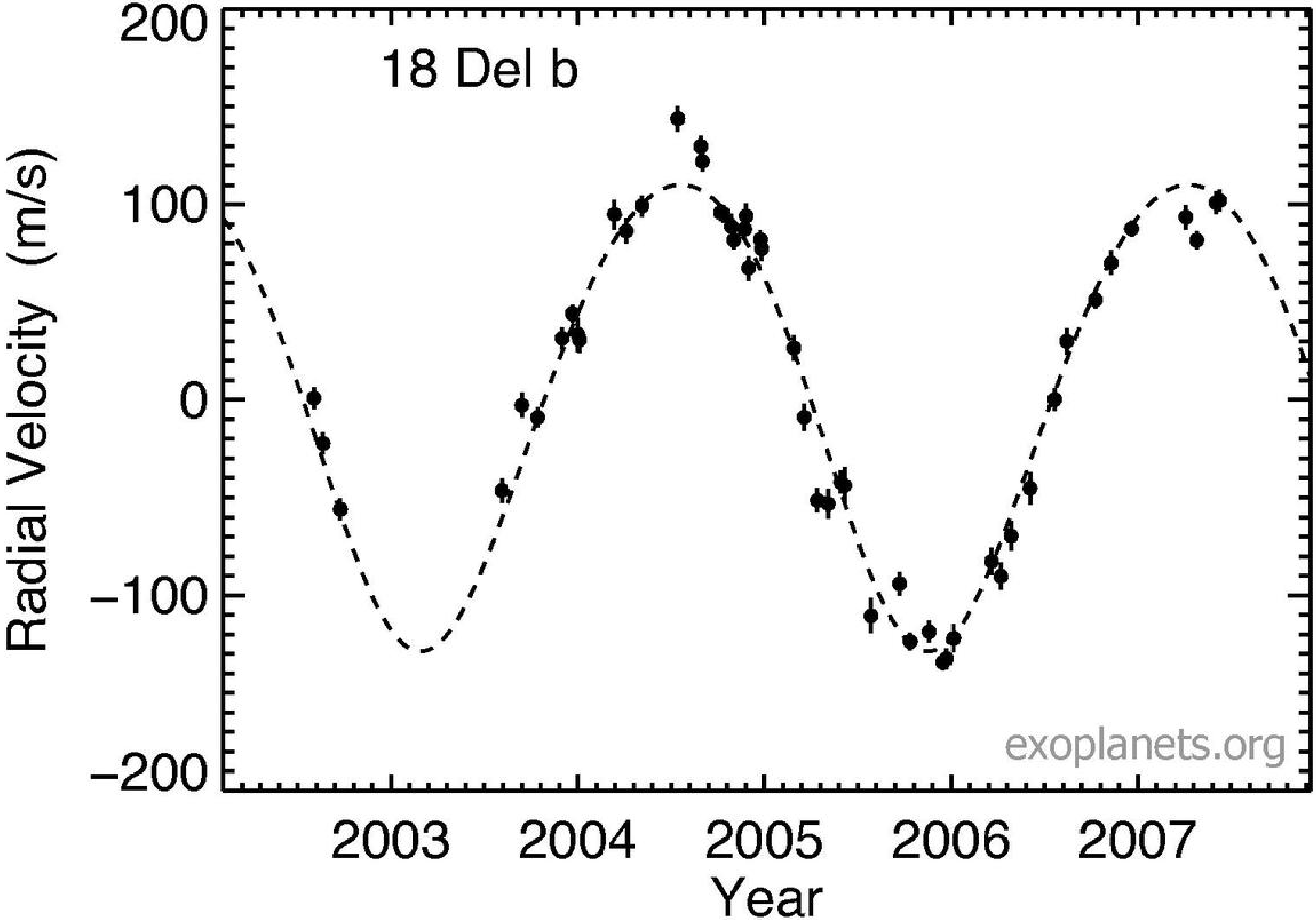
$$\boxed{\frac{r}{R_\oplus} = \left(\frac{\Delta L}{L_0}\right)^{\frac{1}{2}} \left(\frac{R}{R_\odot}\right) \frac{696340}{6371}}$$

This is amazingly a real image!

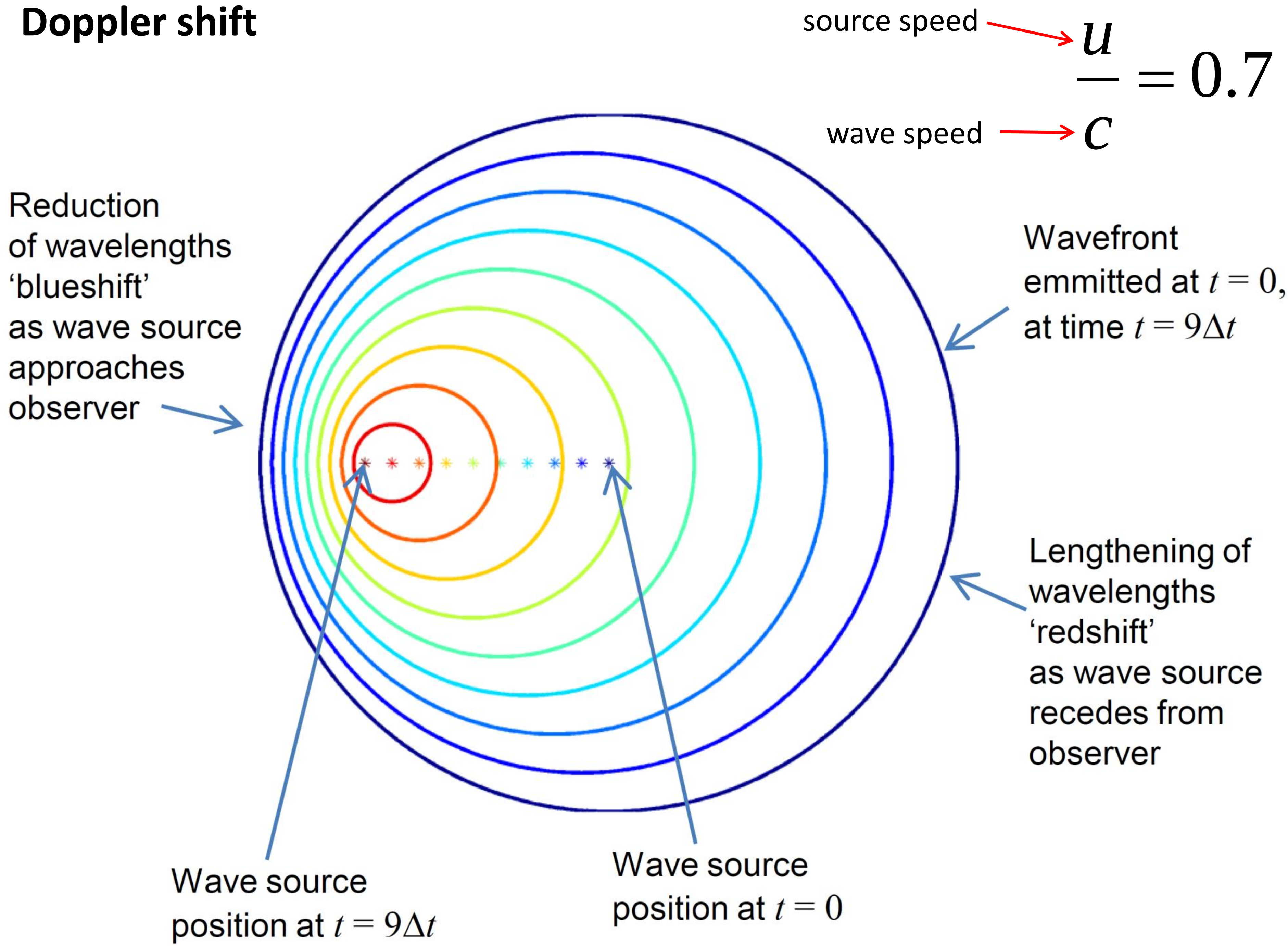
Moon transit captured by a NASA camera aboard the Deep Space Climate Observatory (DSCOVR) satellite

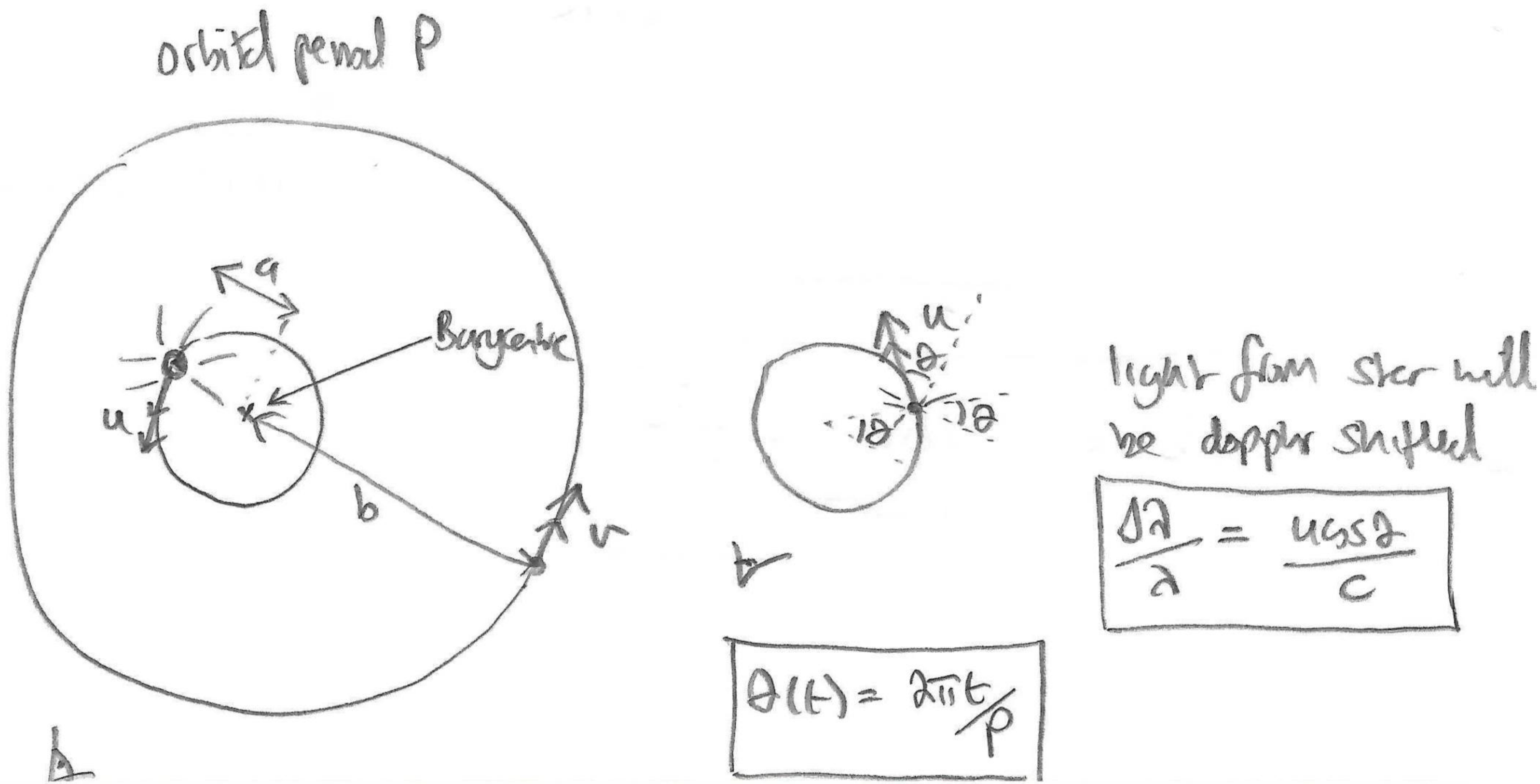
https://www.nasa.gov/feature/goddard/from-a-million-miles-away-nasa-camera-shows-moon-crossing-face-of-earth

Doppler shift in wavelength of star emission spectra due to star motion about centre of mass of star and planet

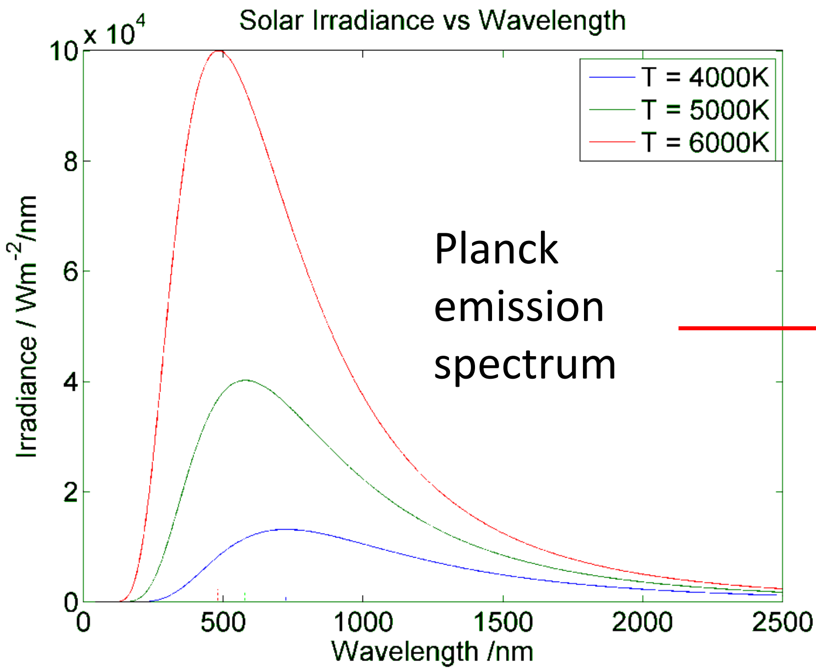


Doppler shift



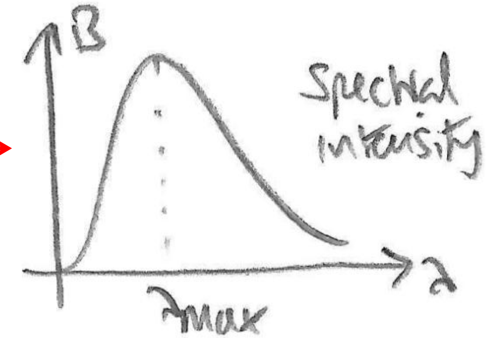


Note we are trying to measure the **orbital speed** u **of the** ***star***, *not the planet*, about the common centre of mass, the 'barycentre'.

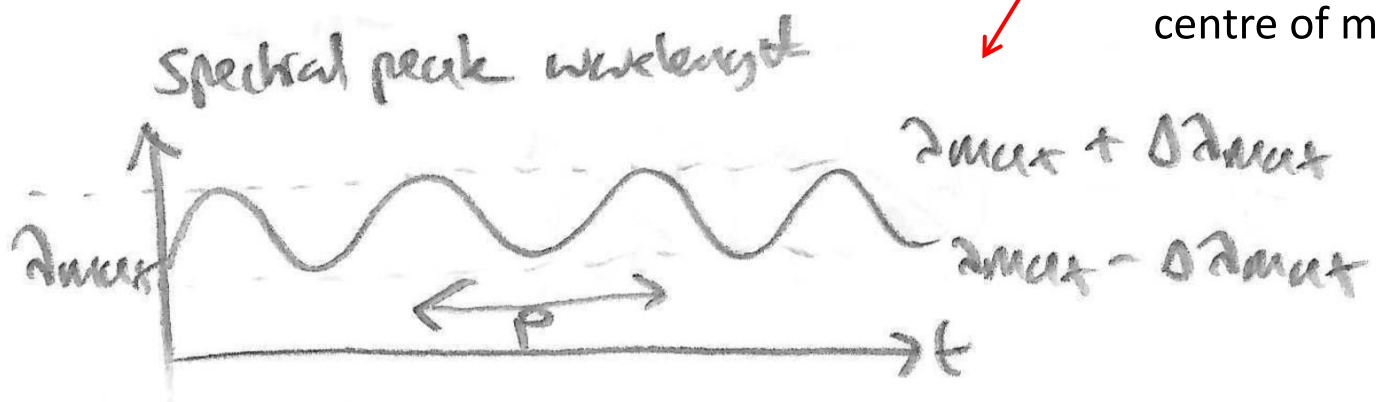


Wein's law: $(\lambda_{max}/\text{nm}) = \dfrac{2.899\times10^{6}}{(T/\text{K})}$

Planck emission spectrum

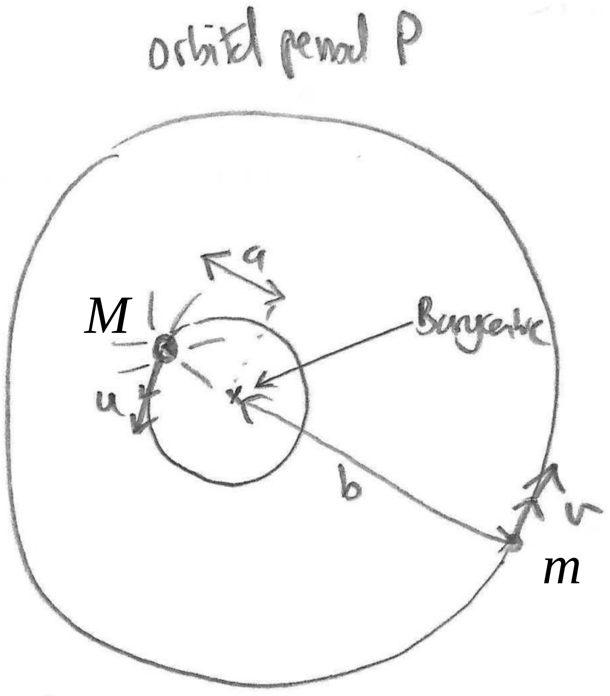


As star and planet orbit the common centre of mass



Find star orbital speed from peak of sinusoidal variation of spectral peak wavelength. Can also determine orbital period of planet-star system from this data. *Note the doppler shifts will be* **TINY.**

$$\frac{\Delta\lambda_{max}}{\lambda_{max}} = \frac{u}{c}$$



Now expect: $m \ll M,\ a \ll b$

and orbits to be *approximately circular*

Kepler III: $$P^2 = \frac{4\pi^2}{GM} b^3 \quad \therefore \left(\frac{b}{\text{AU}}\right) = \left(\frac{M}{M_\odot}\right)^{\frac{1}{3}} \left(\frac{P}{\text{Yr}}\right)^{\frac{2}{3}}$$

So if we know the star mass and the orbital period we can find the **orbital radius of the planet.**

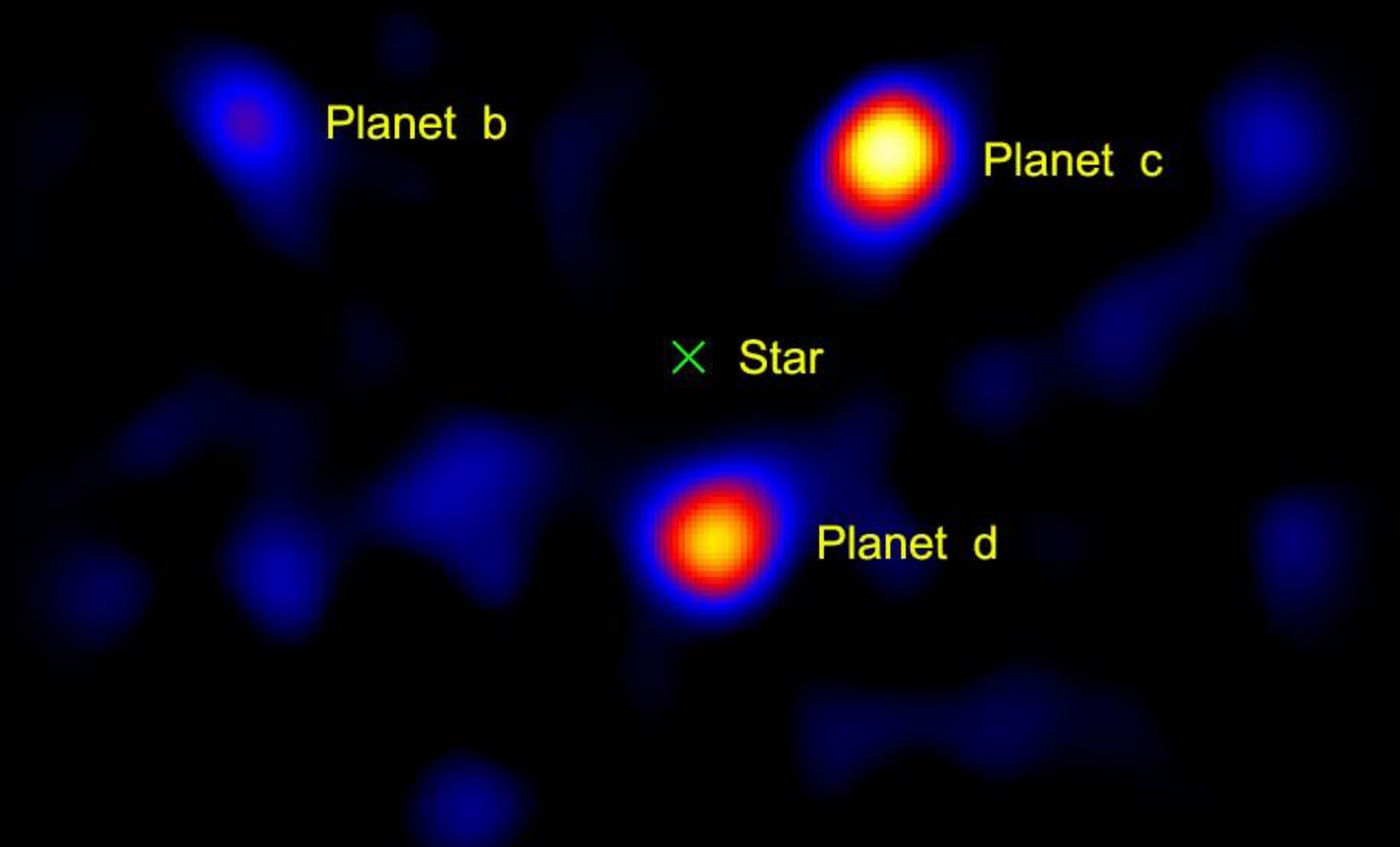
Newton II: $$\frac{Mu^2}{a} = \frac{GMm}{(a+b)^2} \Rightarrow \frac{u^2}{a} \approx \frac{Gm}{b^2} \quad \therefore m = \frac{u^2 b^2}{aG}$$

$$u = \frac{2\pi a}{P} \therefore a = \frac{uP}{2\pi}$$

$$\therefore m = \frac{u^2 b^2}{\frac{uP}{2\pi} G} \Rightarrow \boxed{m = \frac{2\pi b^2 u}{GP}}$$

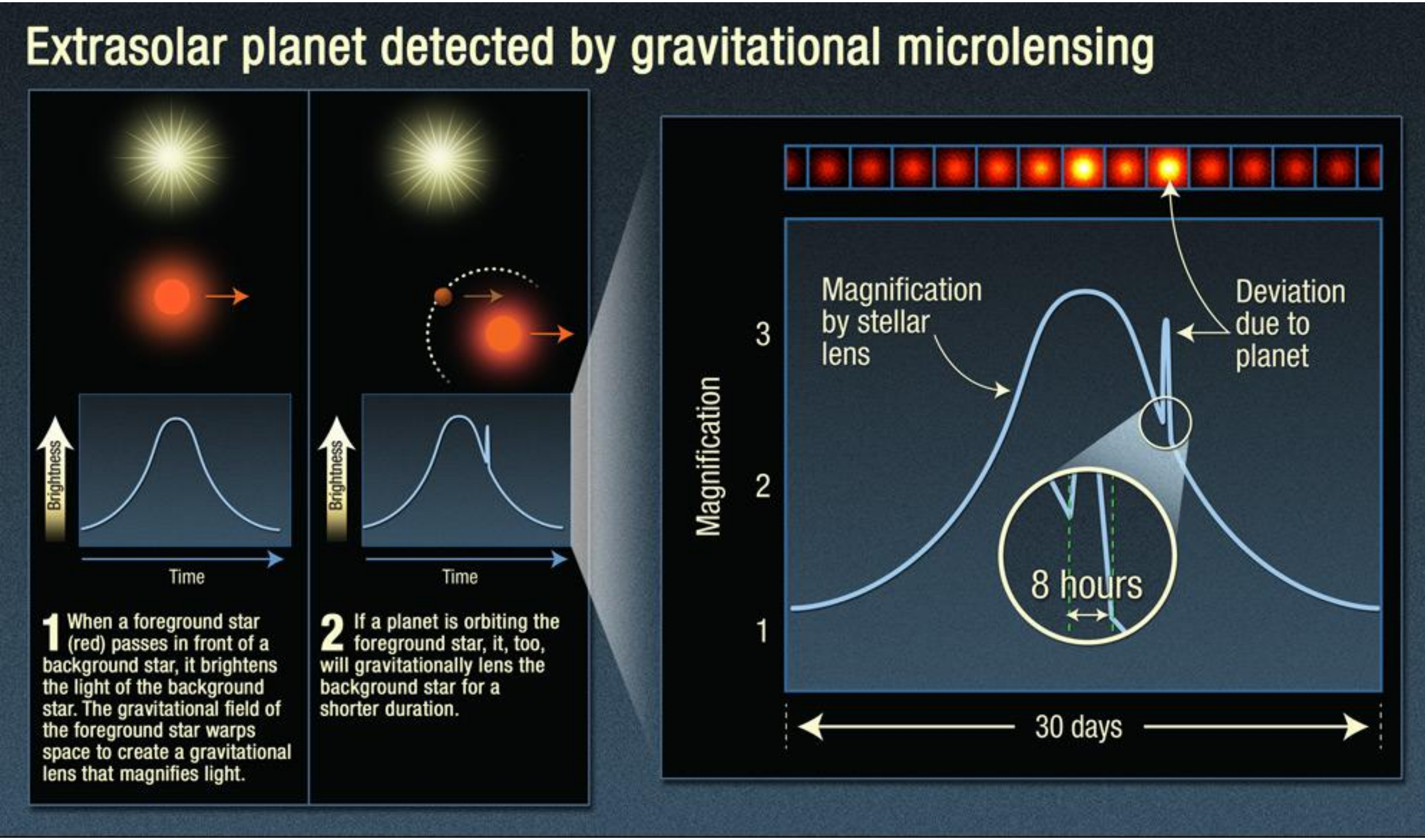
So if we know the star orbital speed, the planet orbital radius and the orbital period, we can find the **mass of the planet.**

Direct image of exoplanets around the star HR8799 using a *vector vortex coronagraph* on a 1.5m portion of the Hale telescope
http://www.nasa.gov/topics/universe/features/exoplanet20100414-a.html

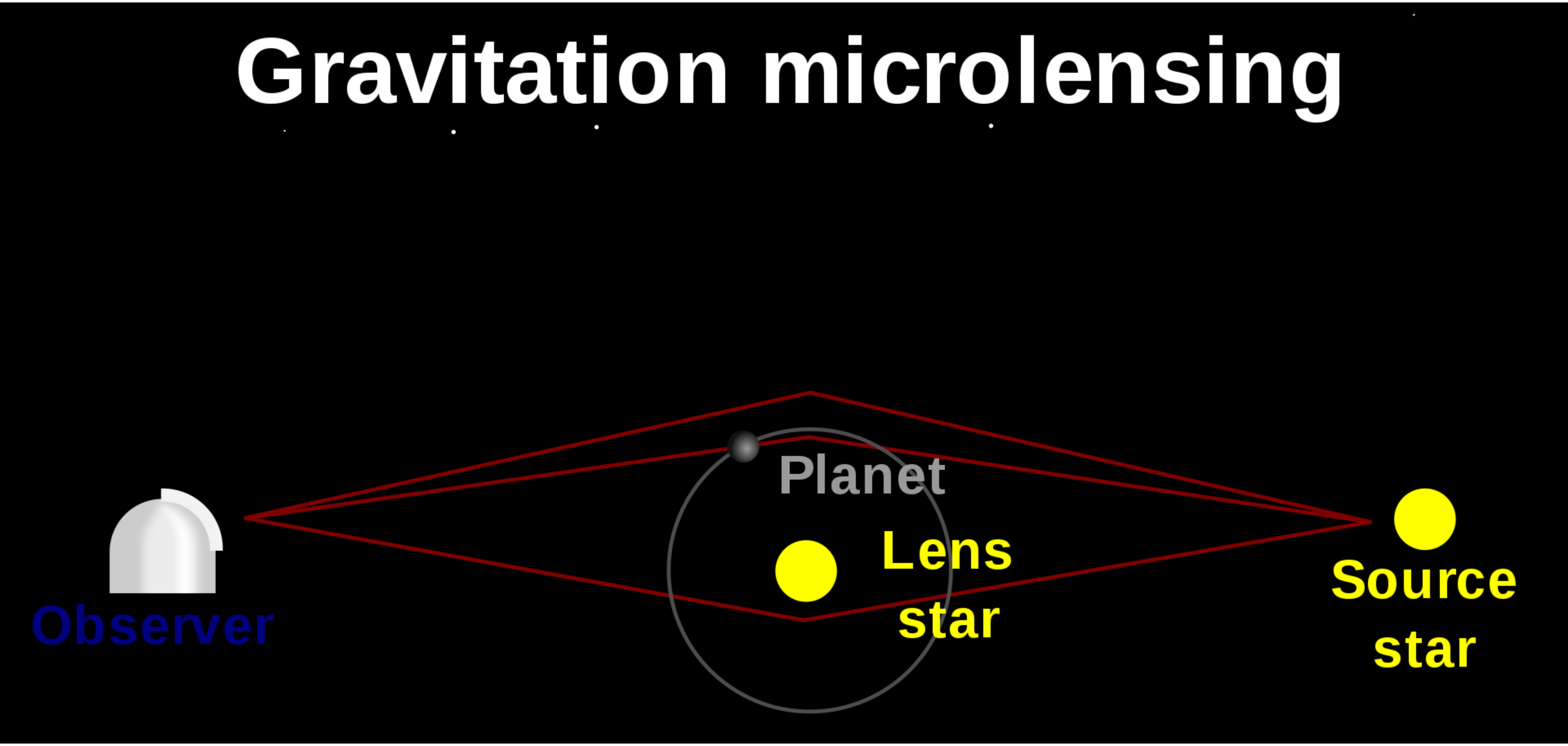


DIRECT IMAGING

Gravitational microlensing



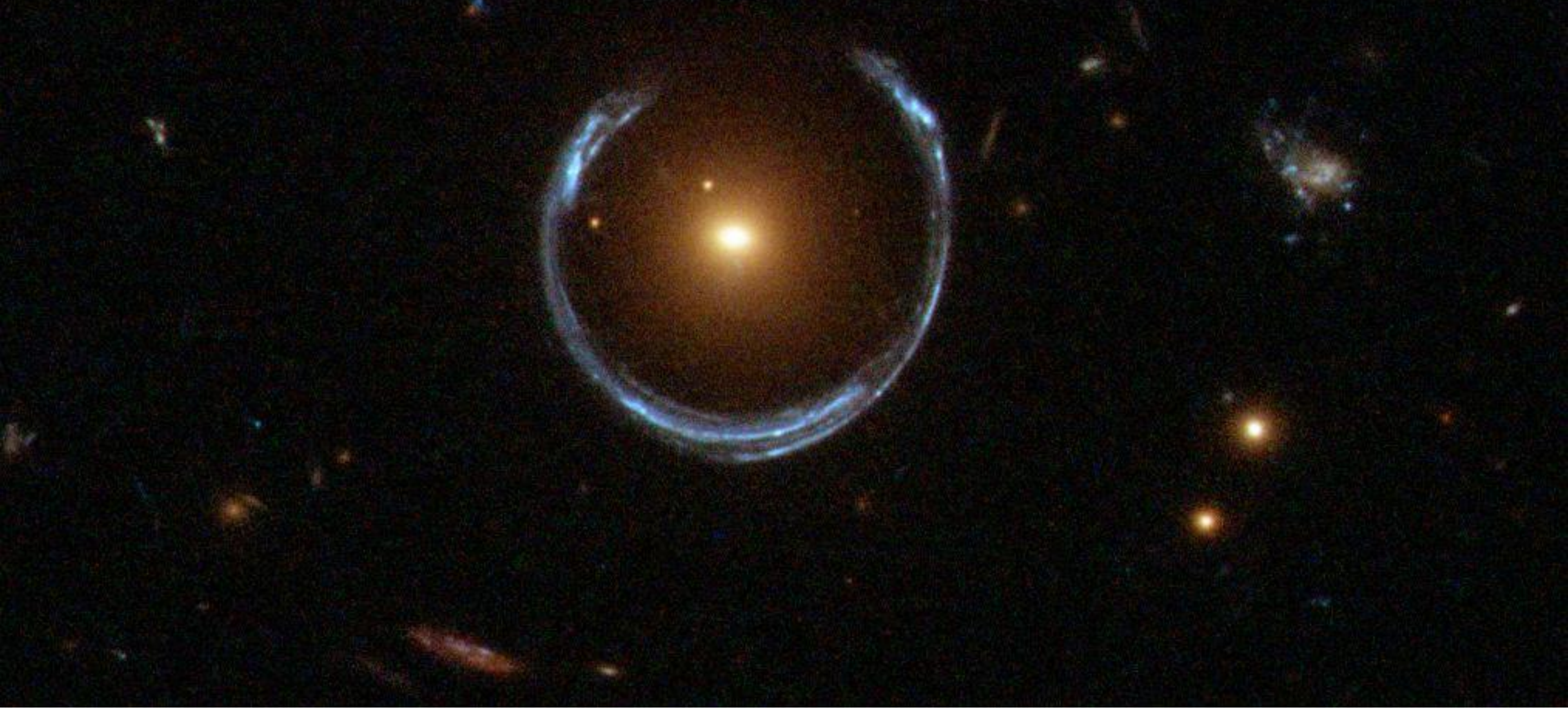
https://exoplanets.nasa.gov/resources/53/extrasolar-planet-detected-by-gravitational-microlensing/



https://en.wikipedia.org/wiki/Gravitational_microlensing

Gravitational lensing

In **this case the** gravity of a luminous red galaxy has gravitationally distorted the light from a much more distant blue galaxy.



https://en.wikipedia.org/wiki/Gravitational_microlensing#/media/File:A_Horseshoe_Einstein_Ring_from_Hubble.JPG

So how do you determine the **mass, radius** and **luminosity** of the *star*?

Good question! Probably via a *variety of methods* – but you can use *correlations* from the **Hertzsprung Russell diagram** (for *Main Sequence* stars) to have a *sensible initial guess*. All you need to start off is the *spectral peak wavelength*. From **Wein's law** this yields the star surface temperature.

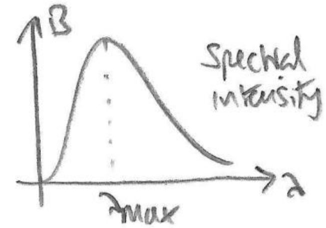
$$T/T_\odot = \left(\frac{\lambda_{max}}{502\,\text{nm}}\right)^{-1}$$

$$M/M_\odot = \left(T/T_\odot\right)^{1.95}$$

$$R/R_\odot = \left(T/T_\odot\right)^{\frac{6.81}{2}-2}$$

$$L/L_\odot = \left(T/T_\odot\right)^{6.81}$$

$$(\lambda_{max}/\text{nm}) = \frac{2.899\times10^6}{(T/\text{K})}$$

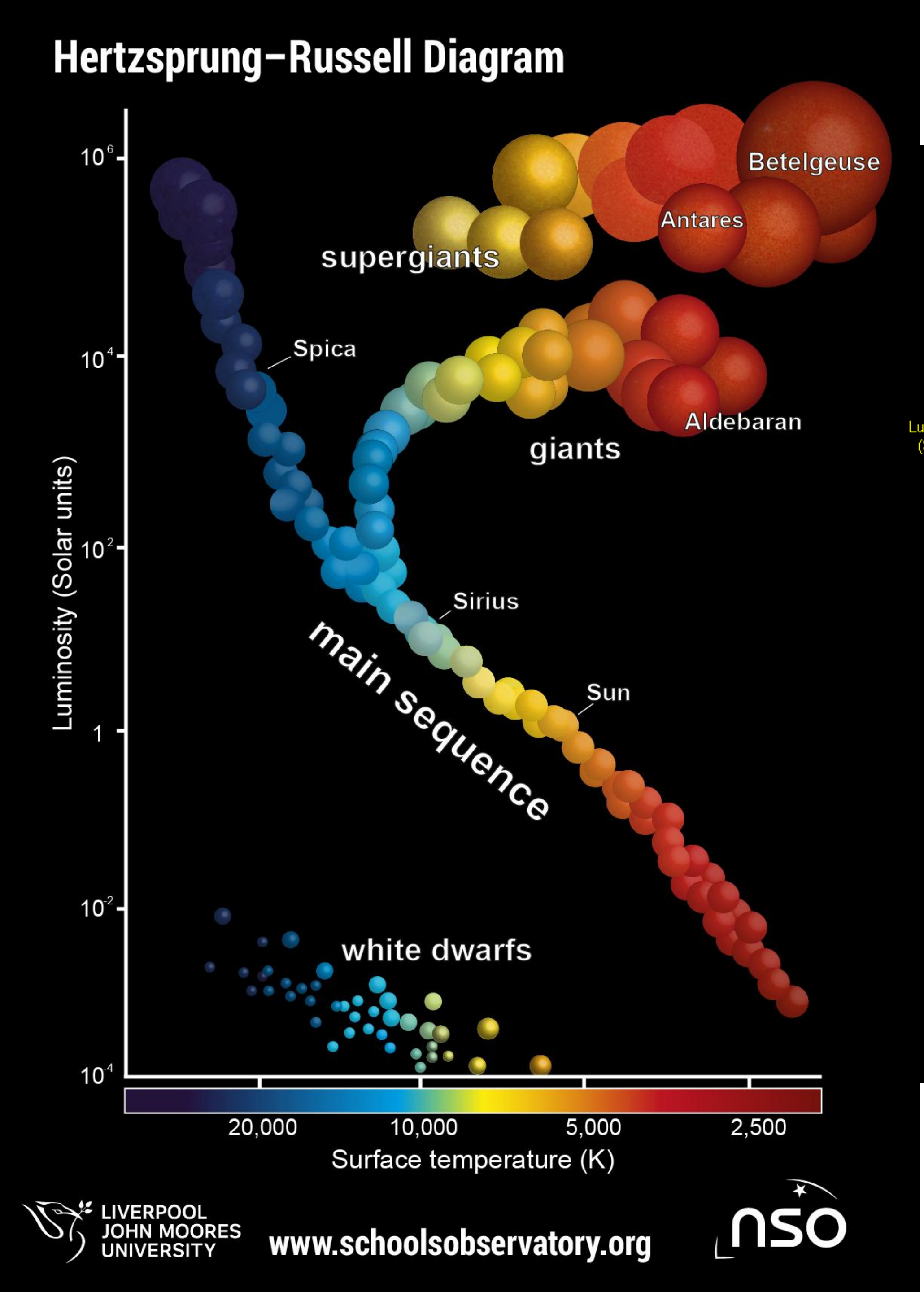


Note Stefan's Law: $L = 4\pi R^2 \sigma T^4$

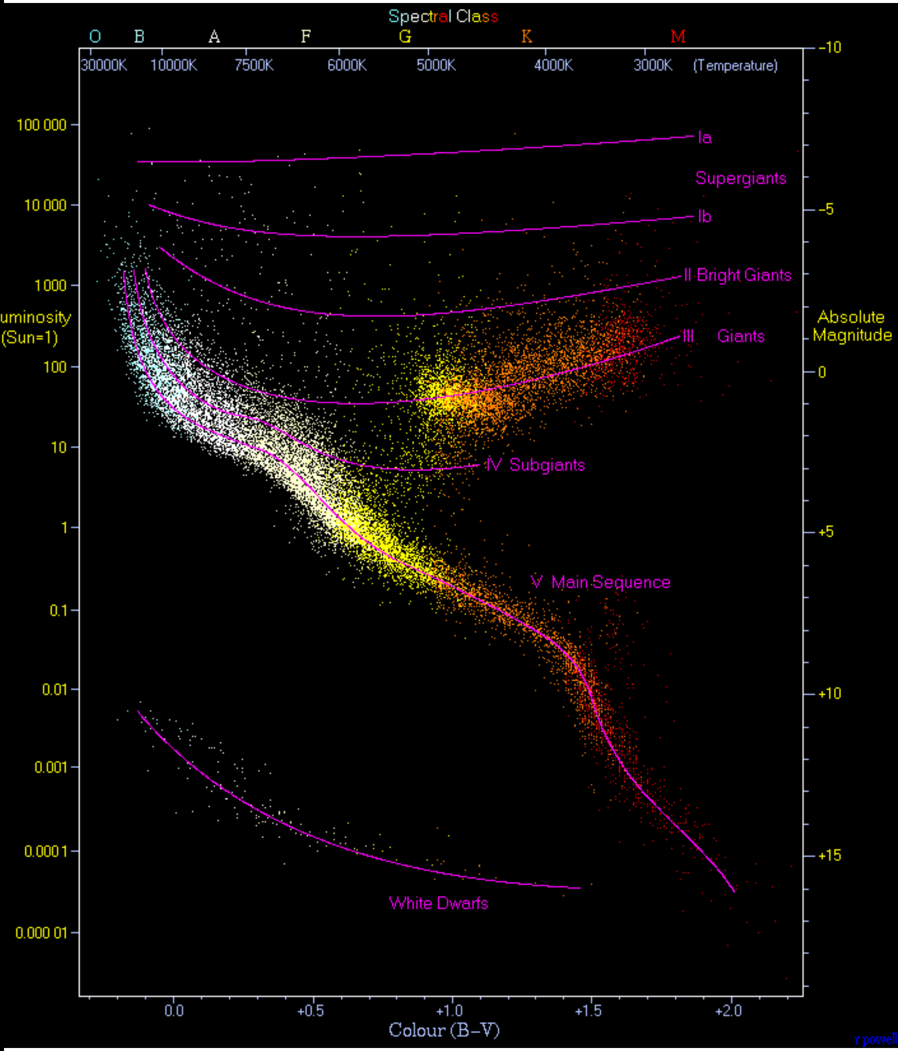
$$\sigma = \frac{2\pi^5 k_B^4}{15c^2h^3} = 5.67\times10^{-8}\,\text{Wm}^{-2}\text{K}^{-4}.$$

$$\therefore \frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2\left(\frac{T}{T_\odot}\right)^4 \qquad \therefore \frac{R}{R_\odot} = \left(\frac{L}{L_\odot}\right)^{\frac{1}{2}}\left(\frac{T}{T_\odot}\right)^{-2}$$

$$R_\odot = 696{,}340\text{km}, \quad M_\odot = 1.99\times10^{30}\,\text{kg}, \quad L_\odot = 3.846\times10^{26}\,\text{Wm}^{-2}.$$



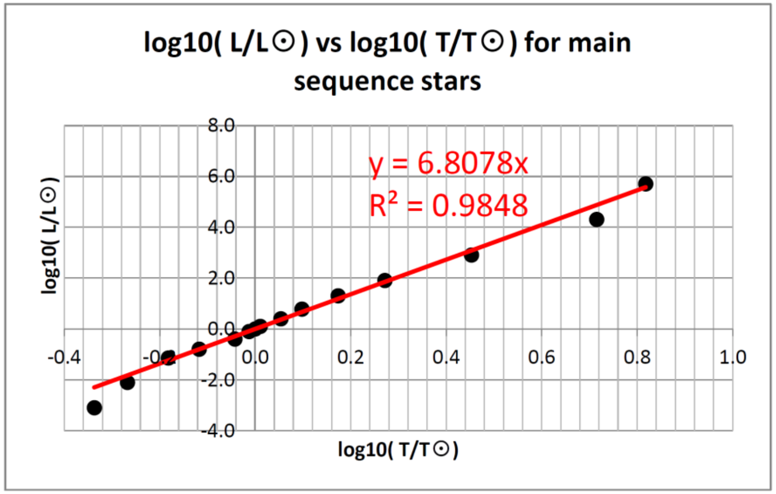
Hertzsprung–Russell Diagram
Luminosity (Solar units)
10^6
10^4
10^2
1
10^{-2}
10^{-4}
Betelgeuse
Antares
supergiants
Spica
giants
Aldebaran
Sirius
main sequence
Sun
white dwarfs
20,000
10,000
5,000
2,500
Surface temperature (K)
LIVERPOOL JOHN MOORES UNIVERSITY
www.schoolsobservatory.org
nso

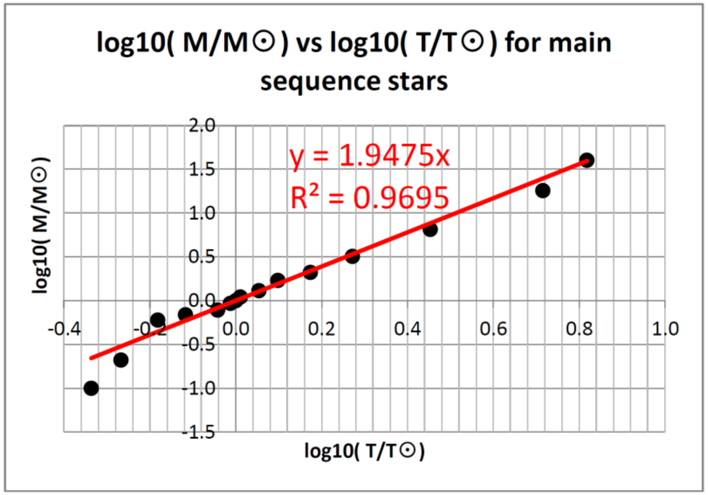


Spectral Class
O B A F G K M
30000K 10000K 7500K 6000K 5000K 4000K 3000K (Temperature)
Luminosity (Sun=1)
100 000
10 000
1000
100
10
1
0.1
0.01
0.001
0.0001
0.000 01
Ia
Supergiants
Ib
II Bright Giants
III Giants
IV Subgiants
V Main Sequence
White Dwarfs
Absolute Magnitude
−10
−5
0
+5
+10
+15
0.0
+0.5
+1.0
+1.5
+2.0
Colour (B–V)
r.powell

MAIN SEQUENCE EXAMPLE STARS https://en.wikipedia.org/wiki/Main_sequence

Stellar classification	Radius R/R☉	Mass M/M☉	Luminosity L/L☉	Surface temperature /K	Star	log10(L/L☉)	log10 (T/T☉)	log10(M/M☉)	log10 ((R/R☉)^2 * (T/5780K)^4)
M8	0.13	0.1	8.00E-04	2,660	Van Biesbroeck's star	-3.097	-0.337	-1.000	-3.120
M5	0.32	0.21	7.90E-03	3,120	EZ Aquarii A	-2.102	-0.268	-0.678	-2.061
M0	0.51	0.6	7.20E-02	3,800	Lacaille 8760	-1.143	-0.182	-0.222	-1.313
K5	0.74	0.69	1.60E-01	4,410	61 Cygni A	-0.796	-0.117	-0.161	-0.731
K0	0.85	0.78	4.00E-01	5,240	70 Ophiuchi A	-0.398	-0.043	-0.108	-0.312
G5	0.93	0.93	7.90E-01	5,610	Alpha Mensae	-0.102	-0.013	-0.032	-0.115
G2	**1**	**1**	**1.00E+00**	5,780	Sun	0.000	0.000	0.000	0.000
G0	1.05	1.1	1.26E+00	5,920	Beta Comae Berenices	0.100	0.010	0.041	0.084
F5	1.2	1.3	2.50E+00	6,540	Eta Arietis	0.398	0.054	0.114	0.373
F0	1.3	1.7	6.00E+00	7,240	Gamma Virginis	0.778	0.098	0.230	0.619
A5	1.7	2.1	2.00E+01	8,620	Beta Pictoris	1.301	0.174	0.322	1.155
A0	2.5	3.2	8.00E+01	10,800	Alpha Coronae Borealis A	1.903	0.271	0.505	1.882
B5	3.8	6.5	8.00E+02	16,400	Pi Andromedae A	2.903	0.453	0.813	2.971
B0	7.4	18	2.00E+04	30,000	Phi1 Orionis	4.301	0.715	1.255	4.599
O6	18	40	5.00E+05	38,000	Theta1 Orionis C	5.699	0.818	1.602	5.782





Putting these calculations all together in a spreadsheet.....

EXOPLANET ANALYSIS MODEL

A. French Jan 2021

Star	**Kepler 452**	
Planet	**Kepler 452b**	"Earth 2.0"

Planet mass /earth masses	5
Planet radius /earth radii	1.5
Orbital period /days	384.84
Orbital radius /AU	1.046

Star mass /solar masses	1.037
Star radius /solar radii	1.11
Luminosity /solar luminosity	1.2
Star surface temperature /K	5757
Distance from Earth /ly	1402
Parallax /milli arc seconds	1.7838

1AU /m	1.50E+11
Solar mass /kg	1.99E+30
Earth mass /kg	5.97E+24
Solar radius /km	696340
Earth radius /km	6371
Solar luminosity /Wm^-2	3.85E+26
Speed of light /ms^-1	3.00E+08
1 year /s	31536000
1 light year (ly) /m	9.45E+15
Gravitational constant G / Nm^2kg^-2	6.67E-11
Stefan-Boltzmann constant /Wm^-2K^-4	5.67E-08
Sun surface temperature /K	5780
Sun peak spectral intensity wavelength /nm	502

Peak star spectral intensity wavelength /nm	504

Star calculation based upon main sequence correlations, and quoted star surface temperature

		Published value
Calculated star mass / solar masses	0.992	1.037
Calculated star radius /solar radii	0.994	1.11
Calculated star luminosity /solar luminosity	0.973	1.2

Star calculation based upon main sequence correlations, and quoted star mass

Calculated star surface temperature / Sun surface temperature	1.019	0.996
Calculated star surface temperature /K	5889	5757
Calculated star radius /solar radii	1.027	1.11
Calculated star luminosity /solar luminosity	1.135	1.2

Calculated orbital radius /AU from star mass and period	1.049	1.046
Orbital speed of star /ms^-1 about barycentre	0.430	
Calculated planet mass /earth mass	5.02	5.00
Maximum Doppler wavelength shift of star due to orbit about barycentre /nm	7.23E-07	
Orbital speed of planet /kms^-1 about barycentre (effectively the star centre of mass)	29.64	29.57

Time for centre of mass of planet to transit star /hours	14.49
Luminosity dip (%) during transit of planet in front of star	0.0153

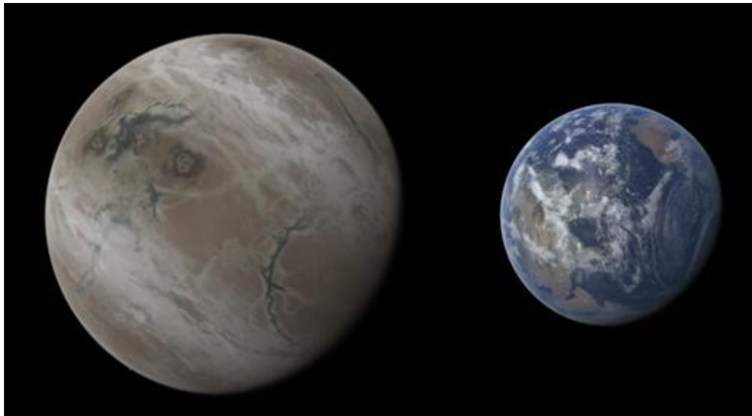
https://en.wikipedia.org/wiki/Exoplanet

https://en.wikipedia.org/wiki/Doppler_spectroscopy

Kepler 452-b exoplanet

Kepler-452 star

Notice how tiny the star doppler shift is! Apparently modern astronomical telescope systems can resolve star orbital speeds to a **precision of about 0.1m/s.** An *extraordinary* feat.



And the numbers for for another ‘earth-like’ exoplanet which will receive enough radiation to support liquid water, and *perhaps* alien life. Alas, it is 101.61 light years away so we may never know.

EXOPLANET ANALYSIS MODEL

A. French Jan 2021

Star	**TOI 700**	Red dwarf
Planet	**TOI 700d**	Liquid water?

Planet mass /earth masses	1.72
Planet radius /earth radii	1.19
Orbital period /days	37.43
Orbital radius /AU	0.163

Star mass /solar masses	0.416
Star radius /solar radii	0.42
Luminosity /solar luminosity	0.0233
Star surface temperature /K	3480
Distance from Earth /ly	101.61
Parallax /milli arc seconds	32.098

1AU /m	1.50E+11
Solar mass /kg	1.99E+30
Earth mass /kg	5.97E+24
Solar radius /km	696340
Earth radius /km	6371
Solar luminosity /Wm^-2	3.85E+26
Speed of light /ms^-1	3.00E+08
1 year /s	31536000
1 light year (ly) /m	9.45E+15
Gravitational constant G / Nm^2kg^-2	6.67E-11
Stefan-Boltzmann constant /Wm^-2K^-4	5.67E-08
Sun surface temperature /K	5780
Sun peak spectral intensity wavelength /nm	502

Peak star spectral intensity wavelength /nm	834

Star calculation based upon main sequence correlations, and quoted star surface temperature

		Published value
Calculated star mass / solar masses	0.372	0.416
Calculated star radius /solar radii	0.490	0.42
Calculated star luminosity /solar luminosity	0.032	0.0233

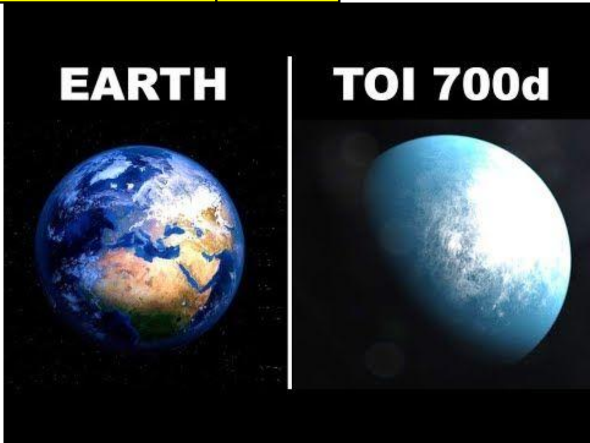
Star calculation based upon main sequence correlations, and quoted star mass

		Published value
Calculated star surface temperature / Sun surface temperature	0.638	0.602
Calculated star surface temperature /K	3686	3480
Calculated star radius /solar radii	0.532	0.42
Calculated star luminosity /solar luminosity	0.047	0.0233

		Published value
Calculated orbital radius /AU from star mass and period	0.164	0.163
Orbital speed of star /ms^-1 about barycentre	0.593	
Calculated planet mass /earth mass	1.73	1.72
Maximum Doppler wavelength shift of star due to orbit about barycentre /nm	1.65E-06	
Orbital speed of planet /kms^-1 about barycentre (effectively the star centre of mass)	47.54	47.38

Time for centre of mass of planet to transit star /hours	3.42
Luminosity dip (%) during transit of planet in front of star	0.0672

https://en.wikipedia.org/wiki/Exoplanet
https://en.wikipedia.org/wiki/Doppler_spectroscopy
https://en.wikipedia.org/wiki/TOI_700_d
https://en.wikipedia.org/wiki/TOI_700



BPhO Computational Challenge

- Suggested homework
- Q&A

BPhO British Physics Olympiad