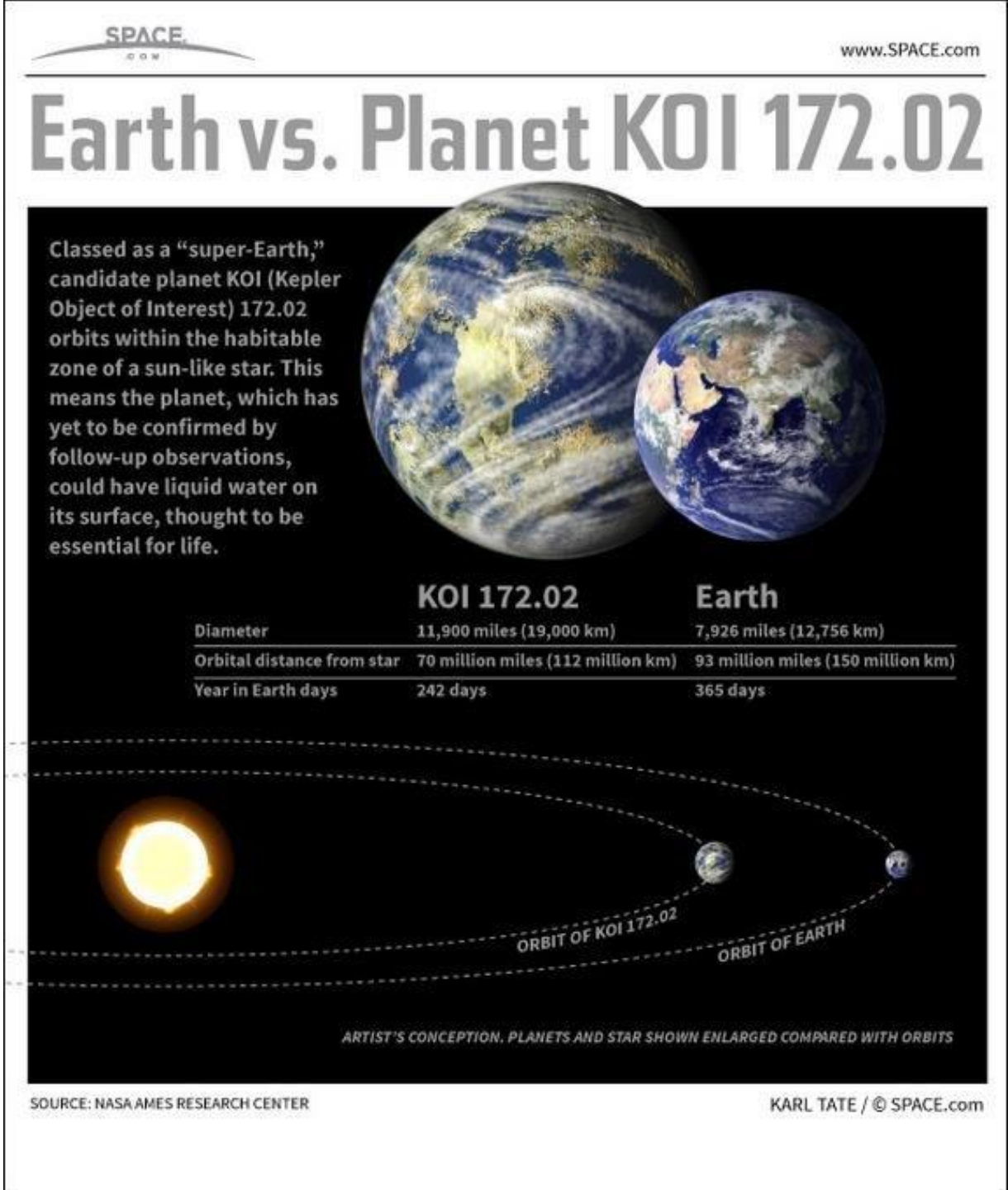
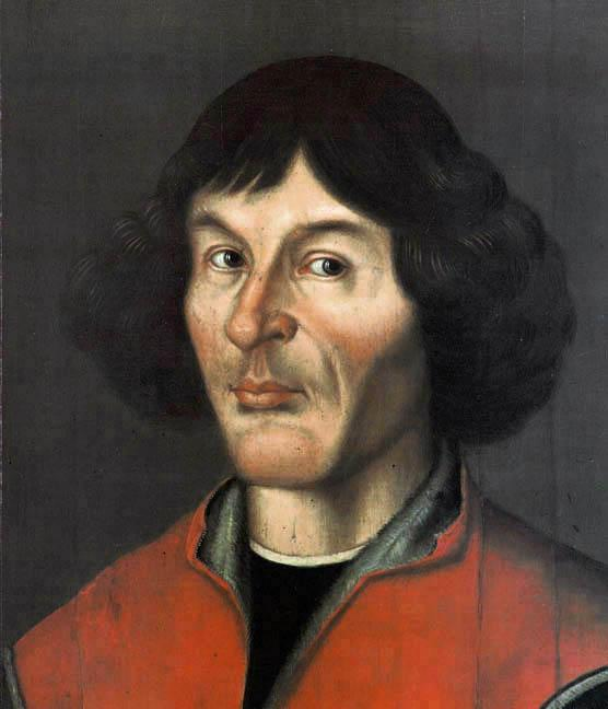
Extrasolar Planets and Kepler's Third Law

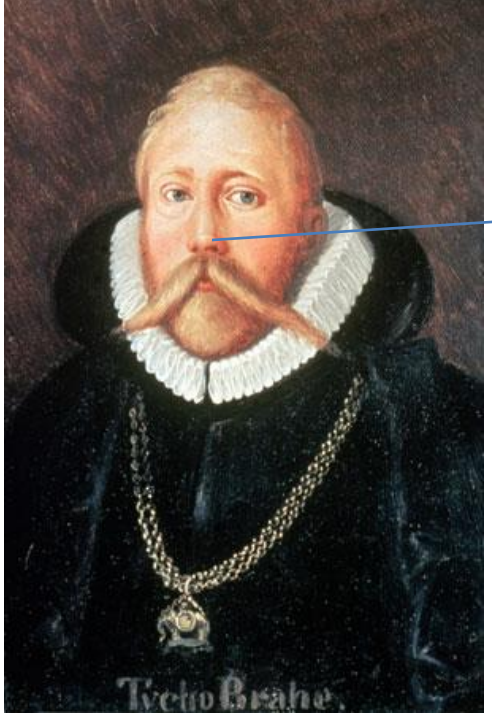
$$T^2 = \frac{4\pi^2}{G(M+m)} a^3$$



AF. March 2019



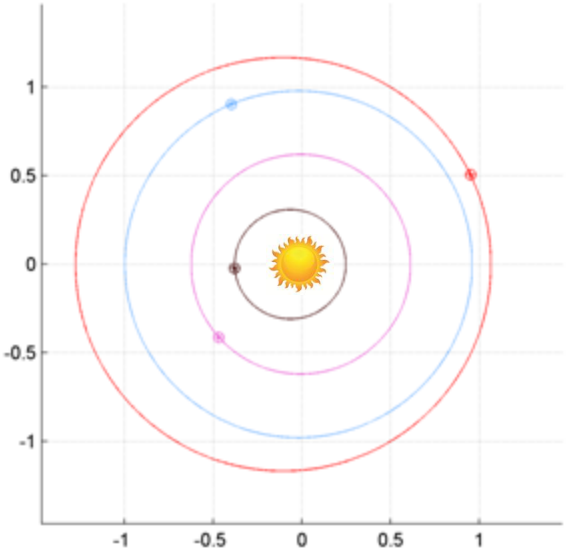
Nicolaus Copernicus
1473-1543



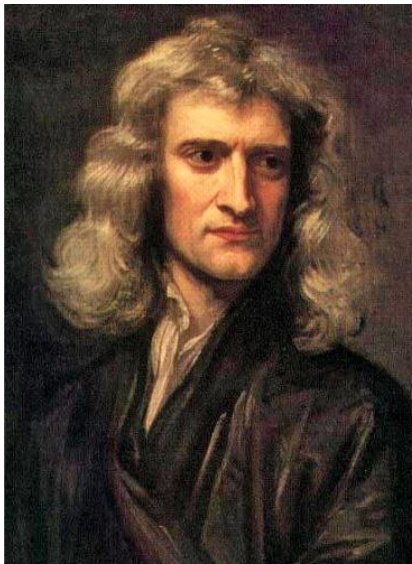
Tycho Brahe
1546-1601

Nose lost in 1566 following a sword duel with third cousin Manderup Parsberg over the legitimacy of a mathematical formula!

Johannes
Kepler
1571-1630



Isaac
Newton
1642-
1727



Kepler's three laws are:

1. *The orbit of every planet in the solar system is an ellipse with the Sun at one of the two foci.*
2. *A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.*
3. *The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.*

The wording of Kepler's laws implies a specific application to the solar system. However, the laws are more generally applicable to any system of two masses whose mutual attraction is an inverse-square law.

$$r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}$$ *Polar* equation of ellipse

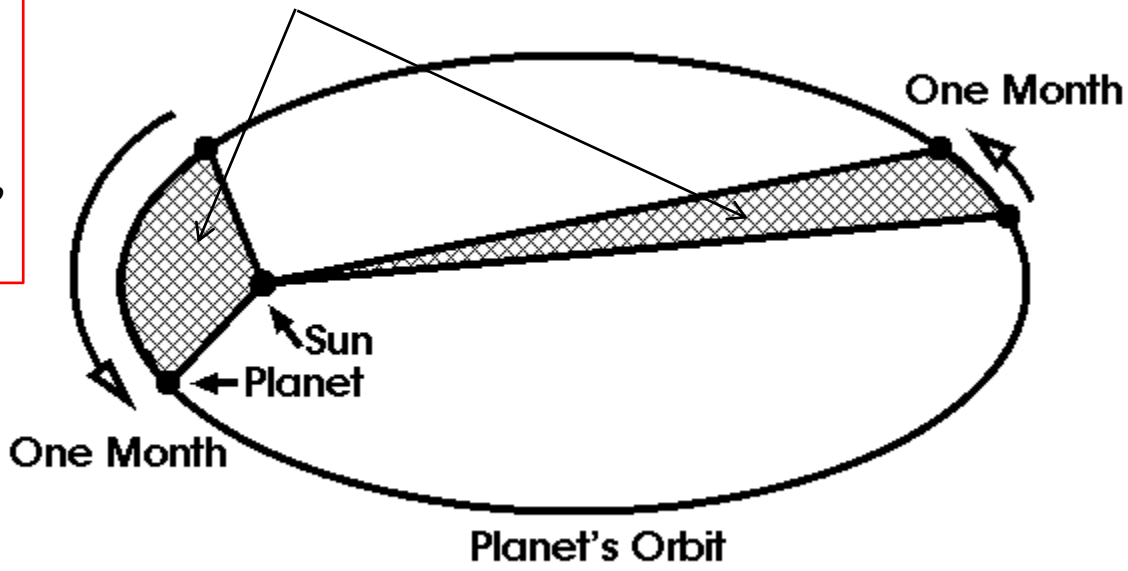
$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$ *Eccentricity* of ellipse

$$P^2 = \frac{4\pi^2}{G(M+M_\odot)}a^3$$ Orbital period *P*

$$\frac{dA}{dt} = \tfrac{1}{2}\sqrt{G(M+M_\odot)\left(1-\varepsilon^2\right)a}$$

This is a constant

Equal areas swept out in equal times



Johannes Kepler
1571-1630

Radii of planets not to scale!

Johannes Kepler
1571-1630

Kepler's
Third Law

$$P \propto a^{\frac{3}{2}}$$

$$\frac{P}{\mathrm{Yr}} \approx \left(\frac{a}{\mathrm{AU}}\right)^{\frac{3}{2}}$$

$$P^2 = \frac{4\pi^2}{G\left(M + M_\odot\right)} a^3$$

Neptune

Uranus

Saturn

Jupiter

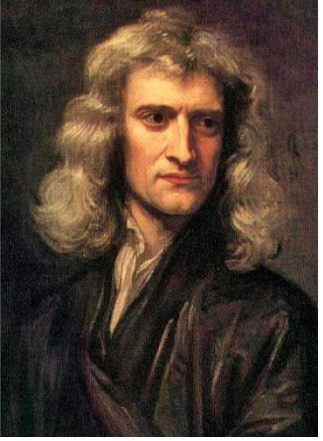
Mercury,
Venus,
Earth,
Mars

Orbital period *P* /years

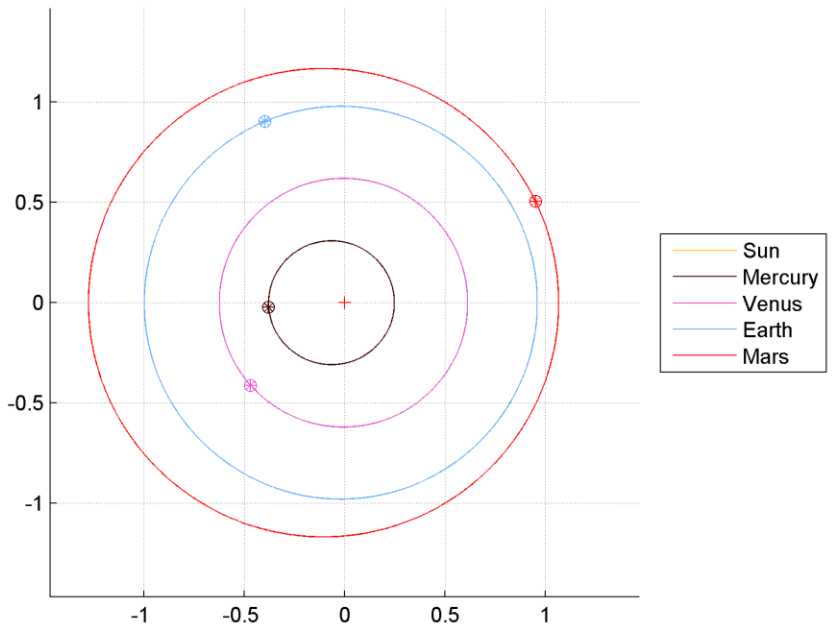
Semi-major axis of orbit *a* / AU

0 20 40 60 80 100 120 140 160 180

0 5 10 15 20 25 30



Isaac Newton
(1642-1727) developed a mathematical model of Gravity which predicted the elliptical orbits proposed by Kepler



Planet and Solar masses

Force of gravity

$$F = \frac{GMM_\odot}{r^2}$$

$$G = 6.67 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$$

$$r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}$$ *Polar* equation of ellipse

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$ *Eccentricity* of ellipse

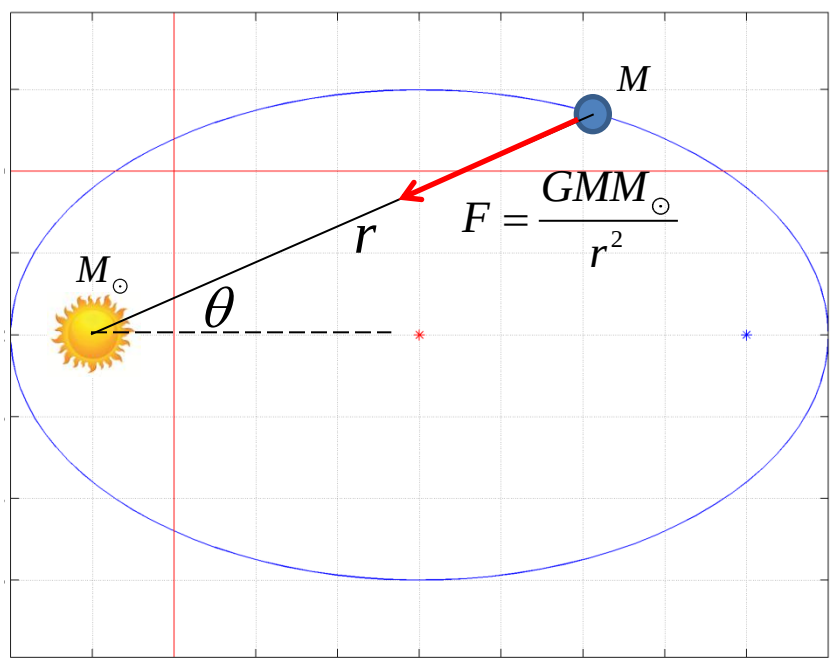
$$P^2 = \frac{4\pi^2}{G(M+M_\odot)}a^3$$ Orbital period *P*

Semi-major axis

$2a$

Semi-minor axis

$2b$



Kepler's Third Law

$$P^2 = \frac{4\pi^2}{G(M+m)}a^3$$

$$\mathrm{Yr}^2 = \frac{4\pi^2}{G(M_\odot + m_\oplus)}\mathrm{AU}^3 \approx \frac{4\pi^2}{GM_\odot}\mathrm{AU}^3$$

$$\therefore \left(\frac{P}{\mathrm{Yr}}\right)^2 = \left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right)^{-1}\left(\frac{a}{\mathrm{AU}}\right)^3$$

$$\therefore 2\log\left(\frac{P}{\mathrm{Yr}}\right) = -\log\left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right) + 3\log\left(\frac{a}{\mathrm{AU}}\right)$$

$$y = 2\log\left(\frac{P}{\mathrm{Yr}}\right) + \log\left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right)$$

$$x = \log\left(\frac{a}{\mathrm{AU}}\right)$$

$M_\odot = 1.98847 \times 10^{30}\,\mathrm{kg}$

$m_J = 1.898 \times 10^{27}\,\mathrm{kg}$

$m_\oplus = 5.972 \times 10^{24}\,\mathrm{kg}$

$G = 6.67 \times 10^{-11}\,\mathrm{Nm^2kg^{-2}}$

$\mathrm{AU} = 1.49597871 \times 10^{11}\mathrm{m}$

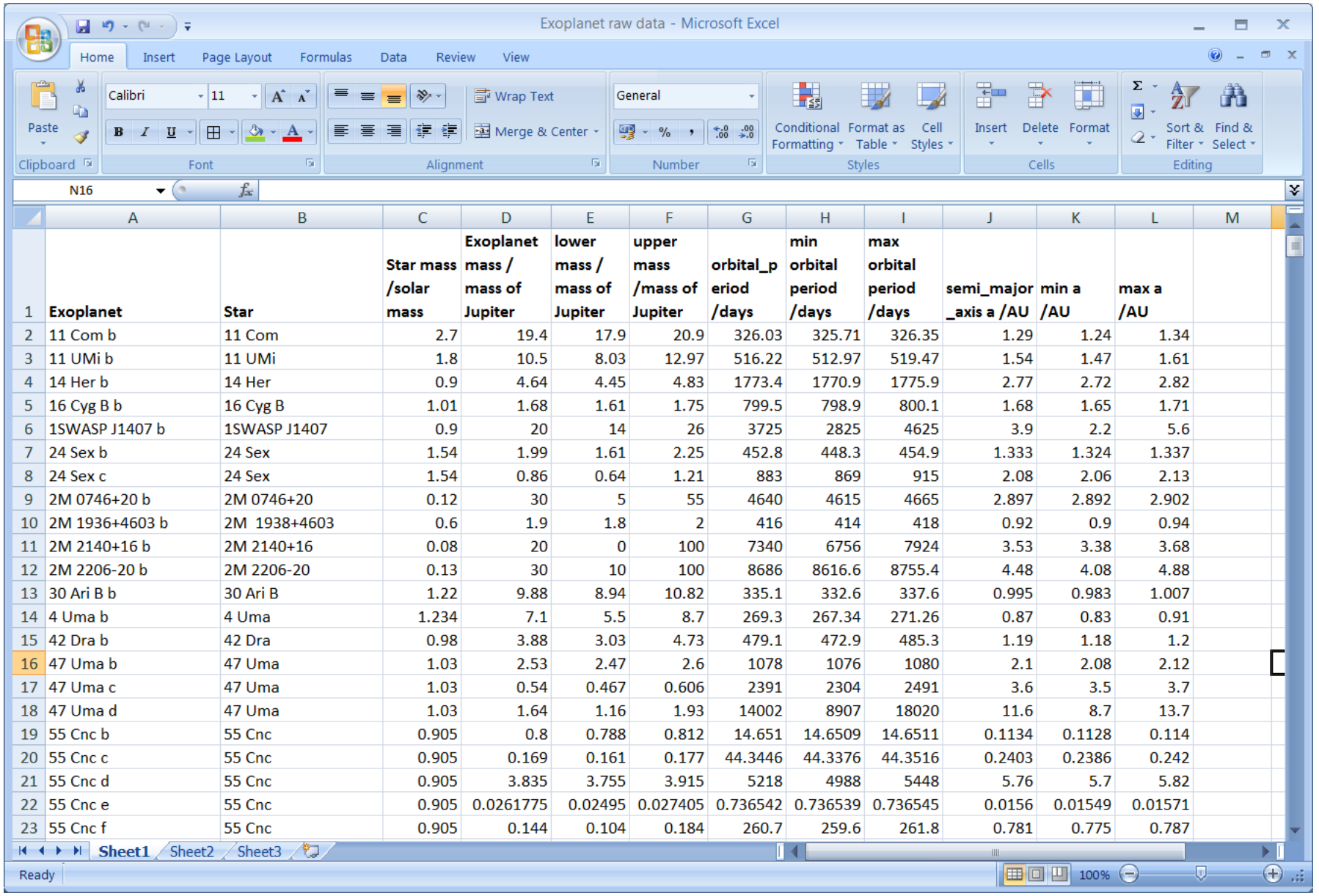
$24 \times 3600\mathrm{s} = 1\mathrm{day}$

$1\mathrm{Yr} = 365.2422\,\mathrm{days}$

So y vs x should be a **straight line** from the origin of **gradient 3**

Raw data (slightly processed!) from www.exoplanet.eu

660 Exoplanets



	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Exoplanet**	**Star**	**Star mass /solar mass**	**Exoplanet mass / mass of Jupiter**	**lower mass / mass of Jupiter**	**upper mass /mass of Jupiter**	**orbital_period /days**	**min orbital period /days**	**max orbital period /days**	**semi_major_axis a /AU**	**min a /AU**	**max a /AU**	
2	11 Com b	11 Com	2.7	19.4	17.9	20.9	326.03	325.71	326.35	1.29	1.24	1.34	
3	11 UMi b	11 UMi	1.8	10.5	8.03	12.97	516.22	512.97	519.47	1.54	1.47	1.61	
4	14 Her b	14 Her	0.9	4.64	4.45	4.83	1773.4	1770.9	1775.9	2.77	2.72	2.82	
5	16 Cyg B b	16 Cyg B	1.01	1.68	1.61	1.75	799.5	798.9	800.1	1.68	1.65	1.71	
6	1SWASP J1407 b	1SWASP J1407	0.9	20	14	26	3725	2825	4625	3.9	2.2	5.6	
7	24 Sex b	24 Sex	1.54	1.99	1.61	2.25	452.8	448.3	454.9	1.333	1.324	1.337	
8	24 Sex c	24 Sex	1.54	0.86	0.64	1.21	883	869	915	2.08	2.06	2.13	
9	2M 0746+20 b	2M 0746+20	0.12	30	5	55	4640	4615	4665	2.897	2.892	2.902	
10	2M 1936+4603 b	2M 1938+4603	0.6	1.9	1.8	2	416	414	418	0.92	0.9	0.94	
11	2M 2140+16 b	2M 2140+16	0.08	20	0	100	7340	6756	7924	3.53	3.38	3.68	
12	2M 2206-20 b	2M 2206-20	0.13	30	10	100	8686	8616.6	8755.4	4.48	4.08	4.88	
13	30 Ari B b	30 Ari B	1.22	9.88	8.94	10.82	335.1	332.6	337.6	0.995	0.983	1.007	
14	4 Uma b	4 Uma	1.234	7.1	5.5	8.7	269.3	267.34	271.26	0.87	0.83	0.91	
15	42 Dra b	42 Dra	0.98	3.88	3.03	4.73	479.1	472.9	485.3	1.19	1.18	1.2	
16	47 Uma b	47 Uma	1.03	2.53	2.47	2.6	1078	1076	1080	2.1	2.08	2.12	
17	47 Uma c	47 Uma	1.03	0.54	0.467	0.606	2391	2304	2491	3.6	3.5	3.7	
18	47 Uma d	47 Uma	1.03	1.64	1.16	1.93	14002	8907	18020	11.6	8.7	13.7	
19	55 Cnc b	55 Cnc	0.905	0.8	0.788	0.812	14.651	14.6509	14.6511	0.1134	0.1128	0.114	
20	55 Cnc c	55 Cnc	0.905	0.169	0.161	0.177	44.3446	44.3376	44.3516	0.2403	0.2386	0.242	
21	55 Cnc d	55 Cnc	0.905	3.835	3.755	3.915	5218	4988	5448	5.76	5.7	5.82	
22	55 Cnc e	55 Cnc	0.905	0.0261775	0.02495	0.027405	0.736542	0.736539	0.736545	0.0156	0.01549	0.01571	
23	55 Cnc f	55 Cnc	0.905	0.144	0.104	0.184	260.7	259.6	261.8	0.781	0.775	0.787	

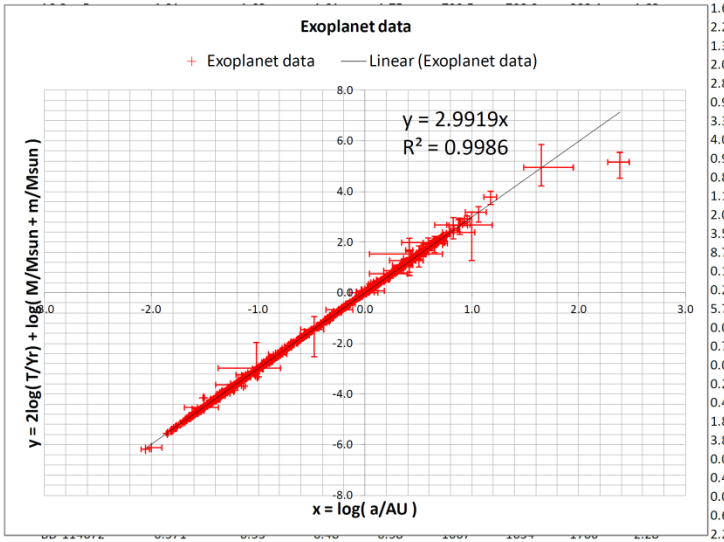
Analysis using Microsoft Excel

Use these to set the error bars

EXOPLANET ANALYSIS
A. FRENCH March 2019

Solar mass /kg	Jupiter mass /kg	Yr in days
1.98847E+30	1.90E+27	365.2422

Exoplanet	Star	Star mass /solar mass	Exoplanet mass / mass of Jupiter	lower mass / mass of Jupiter	upper mass /mass of Jupiter	orbital_period /days	min orbital period /days	max orbital period /days	semi_major_axis a /AU	min a /AU	max a /AU	x - xmin	x = log(a/AU)	xmax - x	2log(Tmin/Yr)	2log(T/Yr)	2log(Tmax/Yr)	lower	log((M+m)/M sun)	upper	y - ymin	y	ymax - y
11 Com b	11 Com	2.7	19.4	17.9	20.9	326.03	325.71	326.35	1.29	1.24	1.34	0.0172	0.1106	0.0165	-0.0995	-0.0986	-0.0978	0.4341	0.4343	0.4346	0.0011	0.3357	0.0011
11 UMi b	11 UMi	1.8	10.5	8.03	12.97	516.22	512.97	519.47	1.54	1.47	1.61	0.0202	0.1875	0.0193	0.2950	0.3005	0.3060	0.2571	0.2577	0.2582	0.0061	0.5582	0.0060
14 Her b	14 Her	0.9	4.64	4.45	4.83	1773.4	1770.9	1775.9	2.77	2.72	2.82	0.0079	0.4425	0.0078	1.3712	1.3725	1.3737	-0.0437	-0.0436	-0.0435	0.0013	1.3288	0.0013
16 Cyg B b										1.65	1.71	0.0078	0.2253	0.0077	0.6798	0.6805	0.6811	0.0050	0.0050	0.0050	0.0007	0.6855	0.0007
1SWASP J1407 b										2.2									-0.0366	-0.0339	0.2429	1.9804	0.1907
24 Sex b										1.324									0.1881	0.1881	0.0088	0.3747	0.0041
24 Sex c										2.06									0.1878	0.1878	0.0139	0.9545	0.0310
2M 0746+20 b										2.892									-0.8279	-0.7632	0.0807	1.3800	0.0693
2M 1936+4603 b										0.9									-0.2205	-0.2205	0.0043	-0.1075	0.0042
2M 2140+16 b										3.38									-1.0040	-0.7558	0.1650	1.6023	0.3146
2M 2206-20 b										4.08									-0.7996	-0.6469	0.0627	1.9529	0.1596
30 Ari B b										0.983									0.0897	0.0900	0.0068	0.0149	0.0068
4 Uma b										0.83									0.0937	0.0942	0.0069	-0.1710	0.0068
42 Dra b										1.18									-0.0071	-0.0068	0.0117	0.2286	0.0115
47 Uma b										2.08									0.0139	0.0139	0.0016	0.9539	0.0016
47 Uma c										3.5									0.0131	0.0131	0.0322	1.6451	0.0356
47 Uma d										8.7									0.0135	0.0136	0.3931	3.1807	0.2192
55 Cnc b										0.1128									-0.0430	-0.0430	0.0000	-2.8364	0.0000
55 Cnc c										0.2386									-0.0433	-0.0433	0.0001	-1.8748	0.0001
55 Cnc d										5.7									-0.0416	-0.0416	0.0392	2.2682	0.0375
55 Cnc e										0.01549									-0.0433	-0.0433	0.0000	-5.4341	0.0000
55 Cnc f										0.775									-0.0433	-0.0433	0.0037	-0.3362	0.0037
61 Vir b										0.050196									-0.0223	-0.0223	0.0001	-3.8978	0.0001
61 Vir c										0.2174									-0.0223	-0.0222	0.0008	-1.9874	0.0008
61 Vir d										0.475									-0.0222	-0.0222	0.0039	-0.9675	0.0039
7 CMa b										1.8									0.1826	0.1827	0.0197	0.8224	0.0193
BD+49 828										3.88									0.1823	0.1823	0.1070	1.8837	0.0584
BD-061339 b										0.0421									-0.1549	-0.1549	0.0001	-4.1040	0.0001
BD-061339 c										0.428									-0.1548	-0.1548	0.0031	-1.0796	0.0030
BD-082823 b										0.054									-0.1307	-0.1307	0.0031	-3.7595	0.0031
BD-082823 c										0.66	0.7	0.0130	-0.1675	0.0126	-0.3790	-0.3735	-0.3680	-0.1306	-0.1306	-0.1306	0.0055	-0.5041	0.0055
BD-114672 b										2.21	2.35	0.0135	0.3579	0.0131	1.3013	1.3187	1.3357	-0.2430	-0.2430	-0.2429	0.0174	1.0757	0.0171
BD-17 63 b	BD-17 63	0.74	5.1	4.98	5.22	655.6	655	656.2	1.34	1.32	1.36	0.0065	0.1271	0.0064	0.5073	0.5081	0.5089	-0.1280	-0.1279	-0.1279	0.0009	0.3802	0.0009
CFBDS 1458 b	CFBDS 1458	0.024	6.5	2	11	14600	5800	23400	2.6	2.4	2.8	0.0348	0.4150	0.0322	2.4017	3.2035	3.6133	-1.5865	-1.5199	-1.4622	0.8685	1.6836	0.4675
CoRoT-1 b	CoRoT-1	0.95	1.03	0.91	1.15	1.508956	1.508949	1.508962	0.0254	0.025	0.0258	0.0069	-1.5952	0.0068	-4.7678	-4.7678	-4.7678	-0.0219	-0.0218	-0.0218	0.0001	-4.7896	0.0001
CoRoT-10 b	CoRoT-10	0.89	2.75	2.61	2.89	13.2406	13.2404	13.2408	0.1055	0.1034	0.1076	0.0087	-0.9767	0.0086	-2.8814	-2.8813	-2.8813	-0.0494	-0.0493	-0.0493	0.0001	-2.9307	0.0001
CoRoT-11 b	CoRoT-11	1.56	2.33	2.06	2.6	2.994325	2.994304	2.994346	0.04351	0.04315	0.04387	0.0036	-1.3614	0.0036	-4.1726	-4.1726	-4.1726	0.1937	0.1937	0.1938	0.0001	-3.9788	0.0001



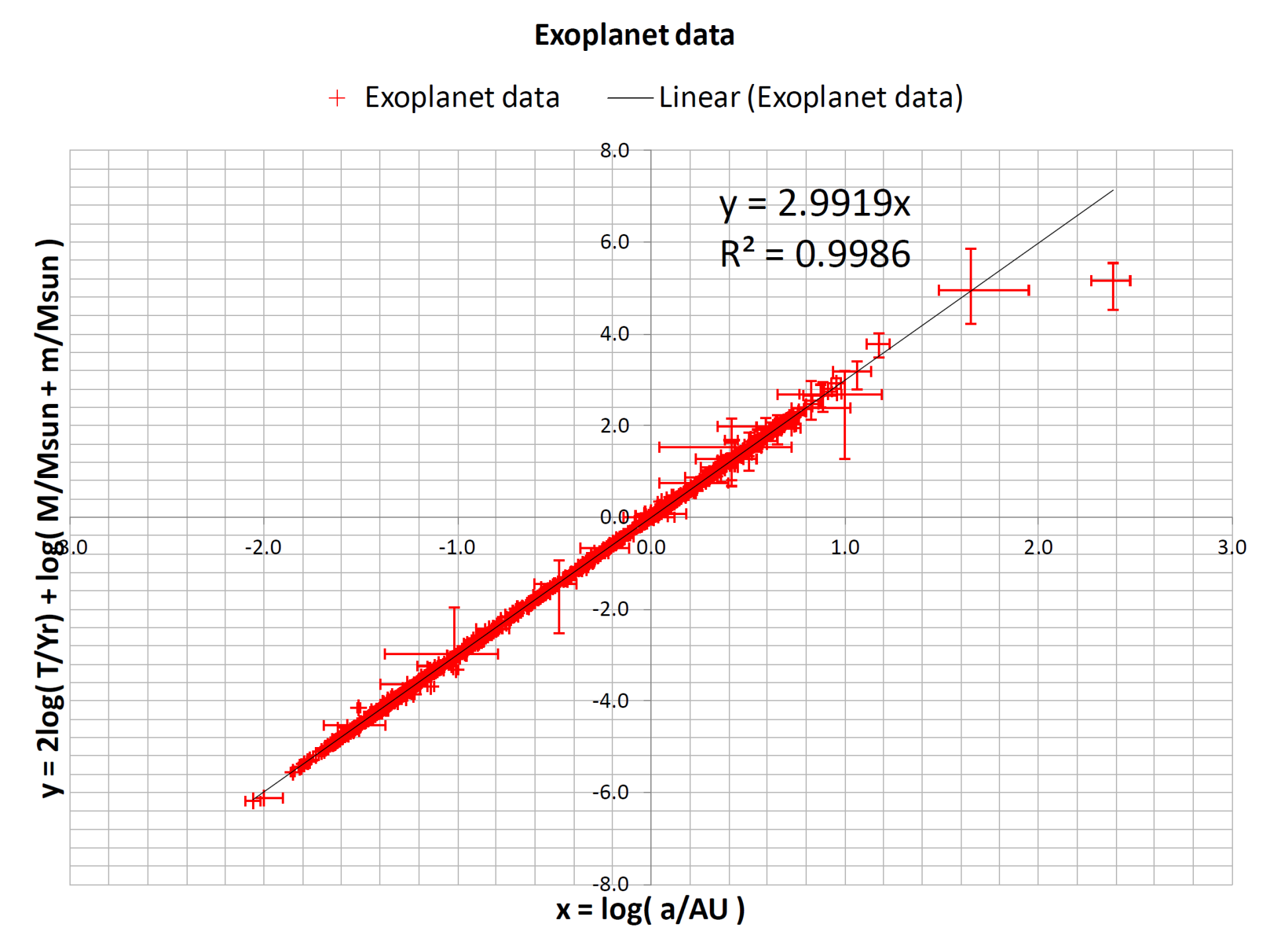
$$\left(\frac{T}{\mathrm{Yr}}\right)^2 = \left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right)^{-1} \left(\frac{a}{\mathrm{AU}}\right)^3$$

$$\therefore 2\log\left(\frac{T}{\mathrm{Yr}}\right) = -\log\left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right) + 3\log\left(\frac{a}{\mathrm{AU}}\right)$$

$$y = 2\log\left(\frac{T}{\mathrm{Yr}}\right) + \log\left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right)$$

$$x = \log\left(\frac{a}{\mathrm{AU}}\right)$$

$$y = (2.992 \pm 0.004)\,x$$


Exoplanet data
Exoplanet data
Linear (Exoplanet data)
y = 2.9919x
R² = 0.9986
8.0
6.0
4.0
2.0
0.0
-2.0
-4.0
-6.0
-8.0
-3.0
-2.0
-1.0
0.0
1.0
2.0
3.0
y = 2log(T/Yr) + log(M/Msun + m/Msun)
x = log(a/AU)

LINE OF BEST FIT CALCULATOR y = mx

Dr Andy French. March 2019

paste as values x,y data here

x	y	x^2	y^2	xy	xfit	yfit	(y-fit)^2	ylower	yupper
0.111	0.336	0.012	0.113	0.037	0.111	0.331	0.000	0.330	0.331
0.188	0.558	0.035	0.312	0.105	0.188	0.561	0.000	0.560	0.562
0.442	1.329	0.196	1.766	0.588	0.442	1.324	0.000	1.322	1.326
0.225	0.685	0.051	0.470	0.154	0.225	0.674	0.000	0.673	0.675
0.591	1.980	0.349	3.922	1.171	0.591	1.768	0.045	1.766	1.771
0.125	0.375	0.016	0.140	0.047	0.125	0.373	0.000	0.373	0.374
0.318	0.955	0.101	0.911	0.304	0.318	0.952	0.000	0.950	0.953
0.462	1.380	0.213	1.904	0.637	0.462	1.382	0.000	1.380	1.384
-0.036	-0.108	0.001	0.012	0.004	-0.036	-0.108	0.000	-0.108	-0.109
0.548	1.602	0.300	2.567	0.878	0.548	1.639	0.001	1.637	1.641
0.651	1.953	0.424	3.814	1.272	0.651	1.949	0.000	1.946	1.951
-0.002	0.015	0.000	0.000	0.000	-0.002	-0.007	0.000	-0.007	-0.007
-0.060	-0.171	0.004	0.029	0.010	-0.060	-0.181	0.000	-0.181	-0.181
0.076	0.229	0.006	0.052	0.017	0.076	0.226	0.000	0.226	0.226
0.322	0.954	0.104	0.910	0.307	0.322	0.964	0.000	0.963	0.965
0.556	1.645	0.309	2.706	0.915	0.556	1.664	0.000	1.662	1.667
1.064	3.181	1.133	10.117	3.386	1.064	3.185	0.000	3.180	3.189
-0.945	-2.836	0.894	8.045	2.682	-0.945	-2.829	0.000	-2.824	-2.833
-0.619	-1.875	0.383	3.515	1.161	-0.619	-1.853	0.000	-1.850	-1.855
0.760	2.268	0.578	5.145	1.725	0.760	2.275	0.000	2.272	2.278
-1.807	-5.434	3.265	29.530	9.819	-1.807	-5.406	0.001	-5.398	-5.414
-0.107	-0.336	0.012	0.113	0.036	-0.107	-0.321	0.000	-0.321	-0.322
-1.299	-3.898	1.688	15.193	5.064	-1.299	-3.887	0.000	-3.882	-3.893
-0.663	-1.987	0.439	3.950	1.317	-0.663	-1.982	0.000	-1.979	-1.985
-0.322	-0.968	0.104	0.936	0.312	-0.322	-0.965	0.000	-0.963	-0.966
0.279	0.822	0.078	0.676	0.229	0.279	0.834	0.000	0.833	0.835
0.623	1.884	0.388	3.548	1.174	0.623	1.865	0.000	1.862	1.867
-1.369	-4.104	1.873	16.843	5.617	-1.369	-4.095	0.000	-4.089	-4.101
-0.362	-1.080	0.131	1.166	0.390	-0.362	-1.082	0.000	-1.080	-1.083
-1.252	-3.760	1.567	14.134	4.706	-1.252	-3.745	0.000	-3.740	-3.751
-0.167	-0.504	0.028	0.254	0.084	-0.167	-0.501	0.000	-0.500	-0.502
0.358	1.076	0.128	1.157	0.385	0.358	1.071	0.000	1.069	1.072
0.127	0.380	0.016	0.145	0.048	0.127	0.380	0.000	0.380	0.381
0.415	1.684	0.172	2.835	0.699	0.415	1.242	0.195	1.240	1.243
-1.595	-4.790	2.545	22.941	7.640	-1.595	-4.773	0.000	-4.766	-4.780
-0.977	-2.931	0.954	8.589	2.863	-0.977	-2.922	0.000	-2.918	-2.927
-1.361	-3.979	1.853	15.831	5.417	-1.361	-4.073	0.009	-4.067	-4.079
-1.396	-4.189	1.949	17.550	5.849	-1.396	-4.177	0.000	-4.171	-4.183
-1.292	-3.876	1.670	15.020	5.009	-1.292	-3.867	0.000	-3.861	-3.873
-1.569	-4.710	2.461	22.185	7.388	-1.569	-4.693	0.000	-4.686	-4.700
-1.209	-3.627	1.462	13.157	4.385	-1.209	-3.617	0.000	-3.612	-3.623
-1.336	-3.955	1.786	15.641	5.285	-1.336	-3.998	0.002	-3.992	-4.004
-1.530	-4.588	2.341	21.053	7.021	-1.530	-4.578	0.000	-4.572	-4.585
-1.286	-3.861	1.653	14.904	4.963	-1.286	-3.847	0.000	-3.841	-3.852
-1.551	-4.654	2.407	21.663	7.220	-1.551	-4.641	0.000	-4.635	-4.648

N	660

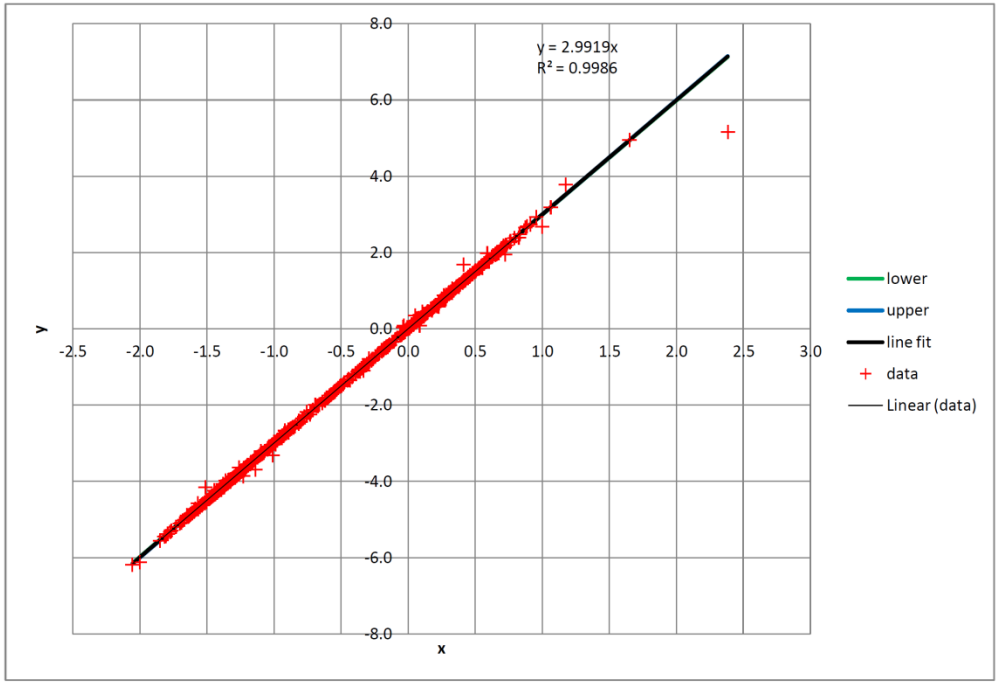
xbar	ybar	x^2 bar	y^2 bar	xy bar
-0.572	-1.716	0.950	8.508	2.841

Vx	Vy	Cov[x,y]	s
0.623	5.563	1.860	0.089

r	r^2
0.999	0.999

m
2.992

dm
0.004



Note bug in old versions fo Excel (<2003), that will give an incorrect R^2 value for the built-in trend line function when 'set intercept at 0,0' is chosen

$$y = 2\log\left(\frac{T}{\mathrm{Yr}}\right) + \log\left(\frac{M}{M_\odot} + \frac{m}{M_\odot}\right), \quad x = \log\left(\frac{a}{\mathrm{AU}}\right)$$

$$y = (2.992 \pm 0.004)\,x$$