BPhO Computational Challenge

Seminar 07: Numeric Calculus Methods

Dr Andrew French.
December 2021.

BPhO British Physics Olympiad

Differentiation: i.e. “finding gradients”

$$\frac{dx}{dt}$$

At the time of the London 2012 Olympics (where Bolt won Gold in a time of 9.63s), the 100m world record stood at **9.58s.** This was set at the Berlin World Championships in 2009.

So how *fast* did he go? What indeed does this statement actually mean? Did he pull away from the rest of the field, or slow down? To answer these questions we need to analyse the race using **kinematics*** (literally, the study of motion).

*From the Greek κίνημα, *kinema* (movement, motion)

In **kinematics** we describe motion by a **graph** in *three* ways:

1. Displacement vs time $(t,\boldsymbol{x})$
2. Velocity vs time $(t,\boldsymbol{v})$
3. Acceleration vs time $(t,\boldsymbol{a})$

- Displacement $\boldsymbol{x}$ is the *position vector* from a specified origin.
- Velocity $\boldsymbol{v}$ is the *rate of change of displacement* at any given instant
- Acceleration $\boldsymbol{a}$ is the *rate of change of velocity* at any given instant

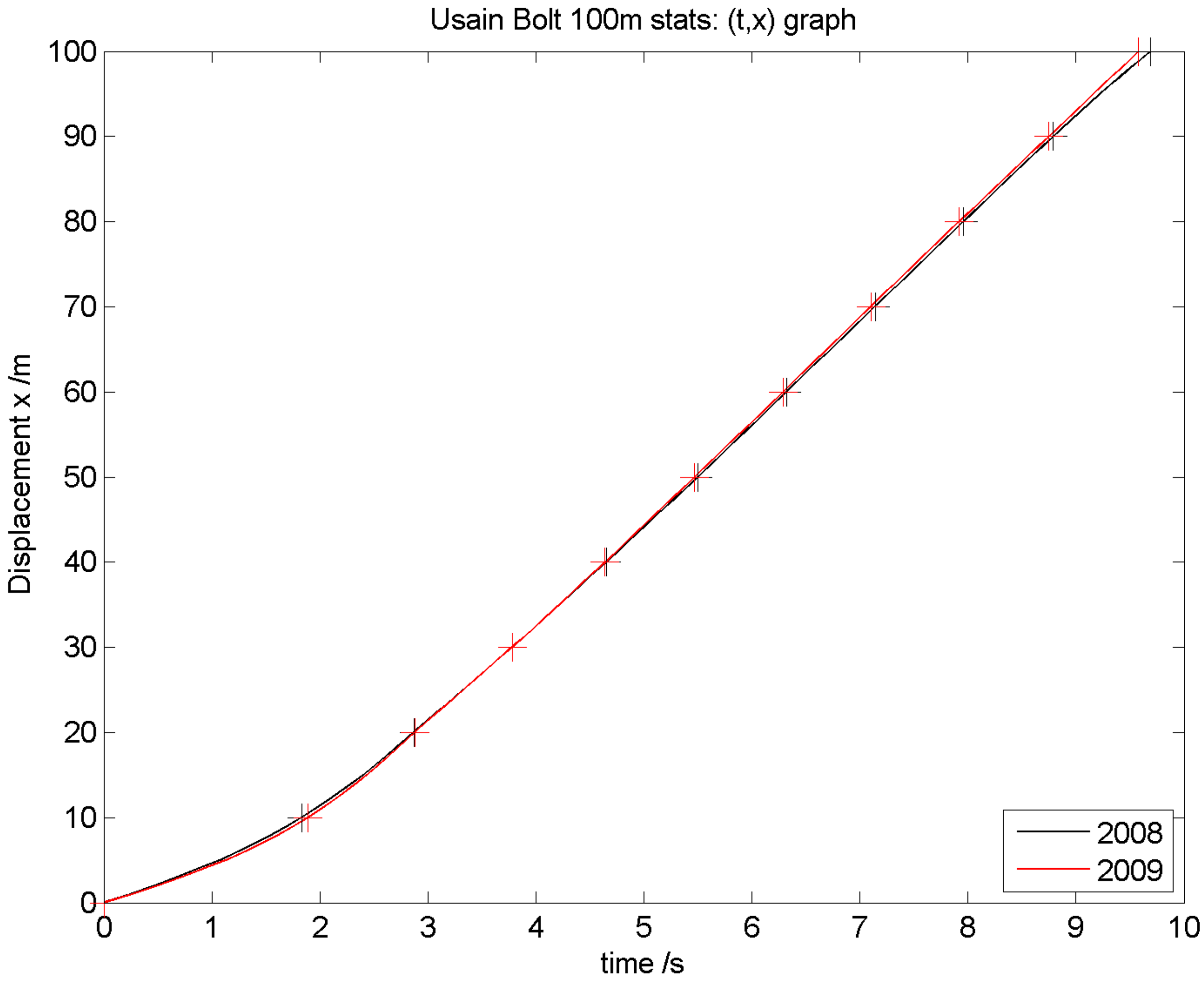
For simplicity at this stage we will consider displacements, velocities and accelerations in a **single direction**, i.e. down the 100m track. Note however that these quantities are actually **vectors** and therefore have both *magnitude* and *direction*.

Let's look at the displacement vs time graph first:

Bolt's 100m races. Time elapsed /s every 10m*

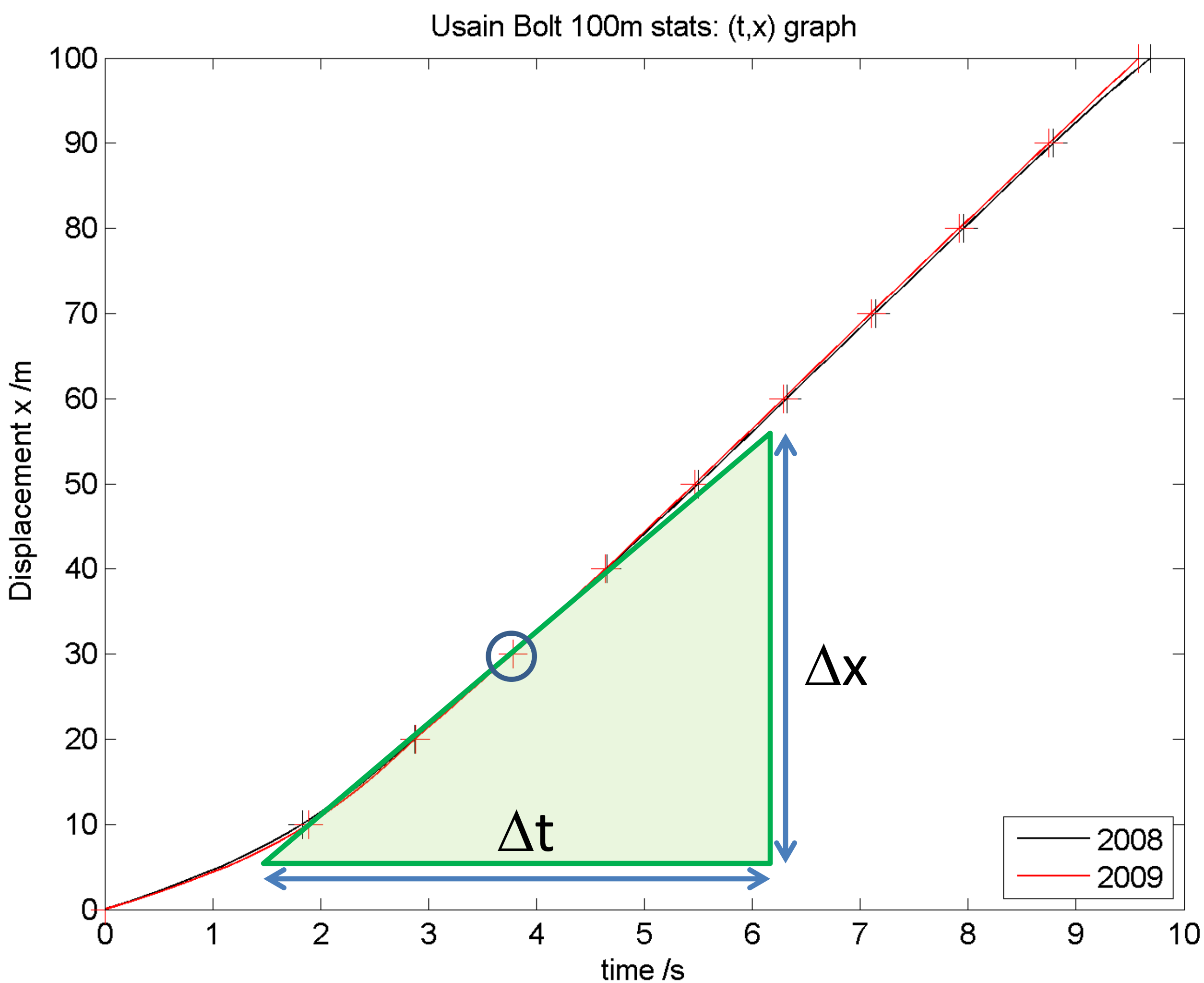
Bolt	10	20	30	40	50	60	70	80	90	100	
2008	1.83	2.87	3.78	4.65	5.5	6.32	7.14	7.96	8.79	9.69	Olympic final, Beijing
2009	1.89	2.88	3.78	4.64	5.47	6.29	7.10	7.92	8.75	9.58	World Champs, Berlin

Photo credit



* http://rcuksportscience.wikispaces.com/file/vie Analysing+men+100m+Nspire.pdf

To find the **time, velocity graph** we could calculate the *gradient* of the (t,x) graph, at *different* times



At ○

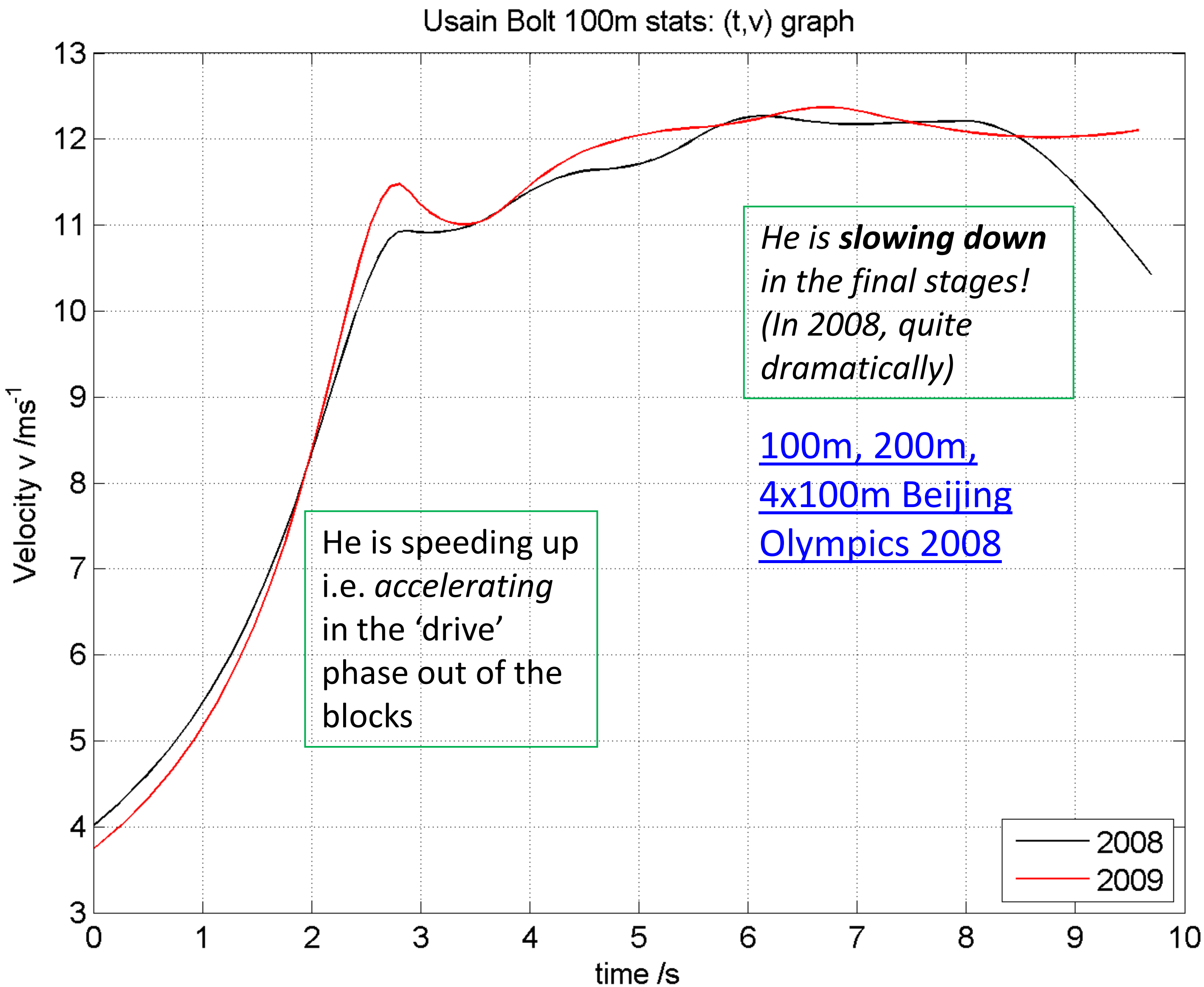
the *local gradient* is

$$v = \frac{\Delta x}{\Delta t}$$

$= 11.2\mathrm{ms}^{-1}$

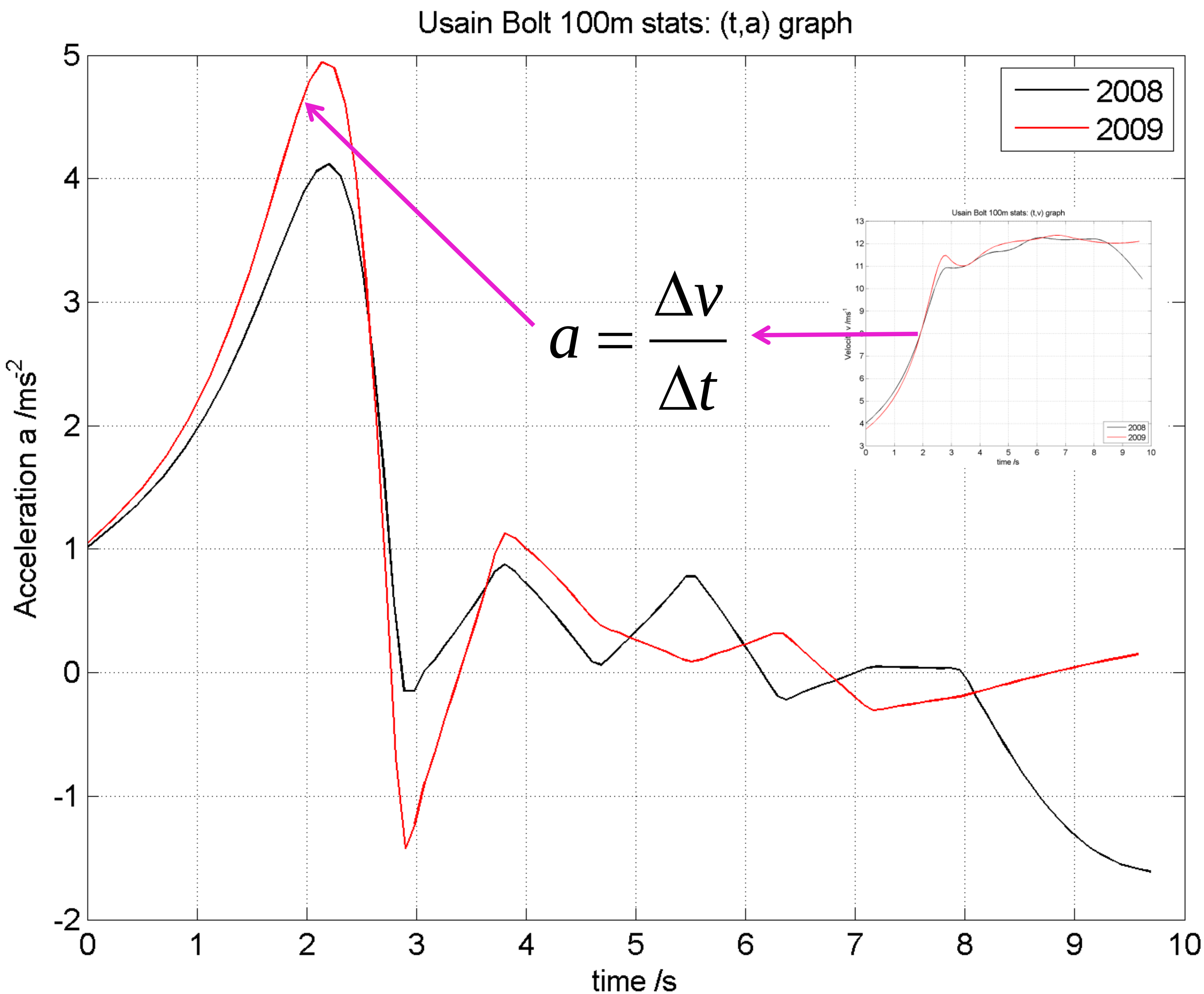
THIS IS DONE BY FIRSTLY FITTING *CUBIC SPLINES* BETWEEN THE DATA POINTS

The graph below has been constructed from the *local gradients* calculated *every second* along a *smooth curve* drawn between the elapsed time data recorded at 10m intervals

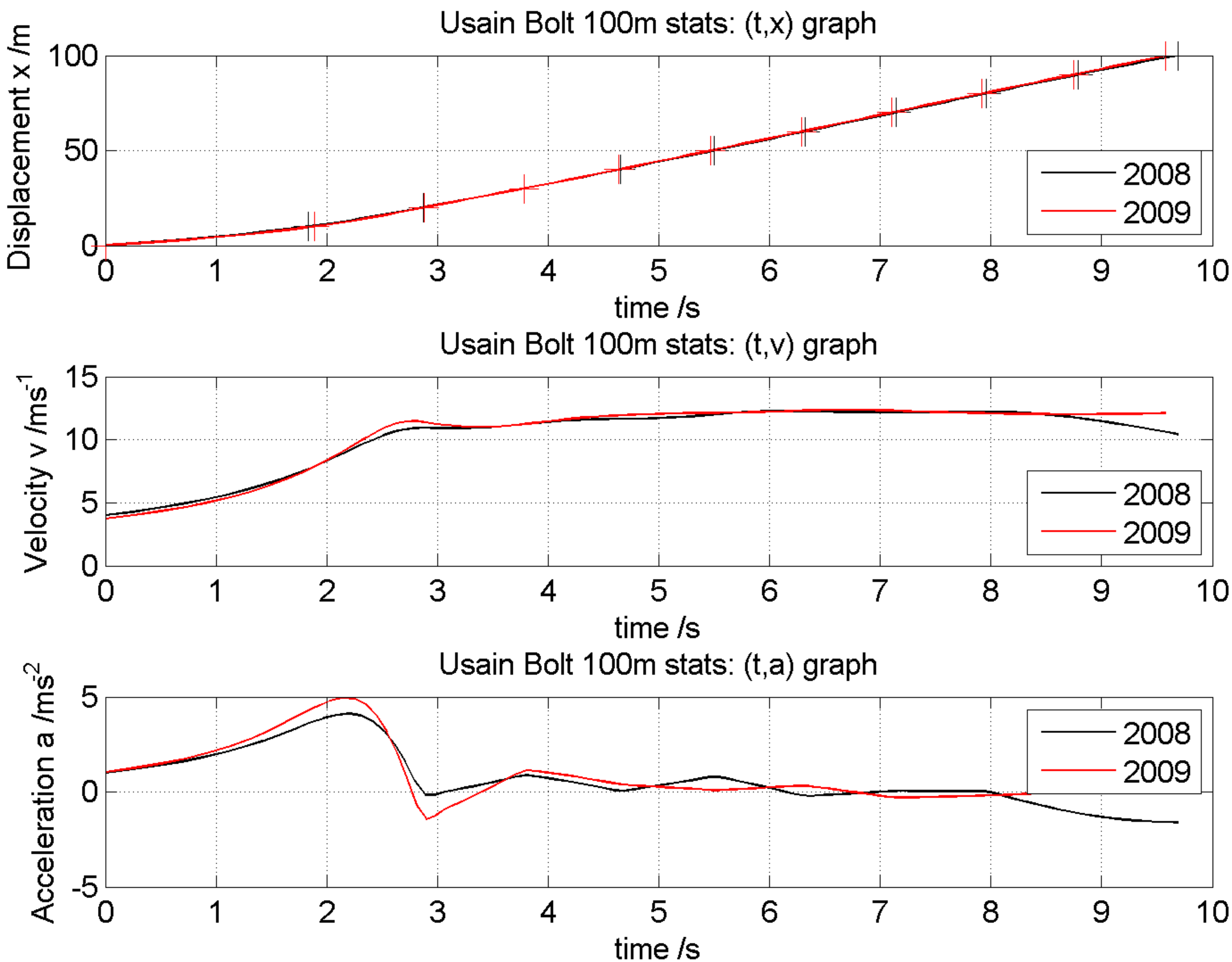


[100m, 200m, 4x100m Beijing Olympics 2008]

We can go one step further and find the graph of *acceleration vs time* by working out the local gradients of the (t,v) graph.



For a complete view we can compare (t,x), (t,v) and (t,a) traces.
Note the time axis must be the *same scale* for each graph.

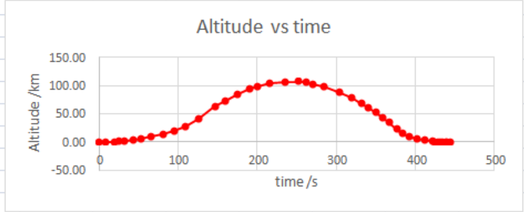


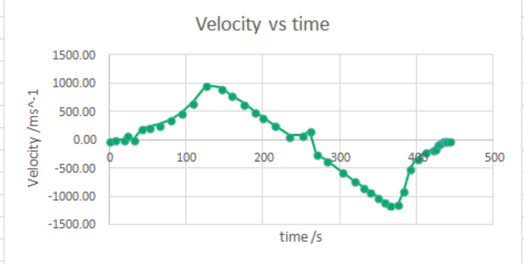
Youtube link　　1 mile = 1609.3m　　1ft = 0.3048m

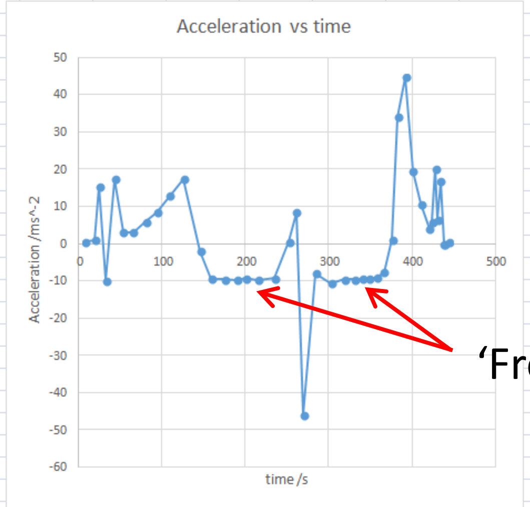
Kinematics of Blue Origin (New Shephard) launch. Data for rocket booster (which separates from capsule).

20/July/2020. Geoff Bezos (Amazon founder) + three other space tourists!

Time min	Time s	t /s	Altitude /ft	Velocity /mph	x /km	v /ms^-1	a /ms^-2
0	0	0	-2	0	0.00	0.00	
0	8	8	6	10	0.00	4.47	0.56
0	19	19	1110	35	0.34	15.65	1.02
0	24	24	2536	208	0.77	92.98	15.47
0	32	32	5647	31	1.72	13.86	-9.89
0	43	43	11707	458	3.57	204.74	17.35
0	53	53	19302	526	5.88	235.14	3.04
1	5	65	28769	609	8.77	272.24	3.09
1	20	80	44494	806	13.56	360.30	5.87
1	34	94	63223	1069	19.27	477.87	8.40
1	49	109	91965	1500	28.03	670.54	12.84
2	6	126	137204	2159	41.82	965.13	17.33
2	26	146	203398	2070	62.00	925.35	-1.99
2	39	159	239985	1794	73.15	801.97	-9.49
2	55	175	278503	1449	84.89	647.74	-9.64
3	10	190	307615	1122	93.76	501.57	-9.75
3	20	200	322077	916	98.17	409.48	-9.21
3	35	215	339091	591	103.35	264.19	-9.69
3	55	235	350085	179	106.71	80.02	-9.21
4	12	252	351210	197	107.05	88.06	0.47
4	21	261	346519	368	105.62	164.51	8.49
4	30	270	336308	-558	102.51	-249.44	-45.99
4	44	284	324430	-808	98.89	-361.20	-7.98
5	3	303	292043	-1261	89.01	-563.70	-10.66
5	19	319	258169	-1606	78.69	-717.93	-9.64
5	31	331	227370	-1865	69.30	-833.71	-9.65
5	40	340	201843	-2054	61.52	-918.20	-9.39
5	49	349	173458	-2245	52.87	-1003.58	-9.49
5	58	358	142876	-2425	43.55	-1084.04	-8.94
6	6	366	113551	-2561	34.61	-1144.84	-7.60
6	16	376	76387	-2537	23.28	-1134.11	1.07
6	23	383	49662	-2004	15.14	-895.84	34.04
6	32	392	30292	-1104	9.23	-493.52	44.70
6	41	401	18569	-711	5.66	-317.84	19.52
6	51	411	10145	-472	3.09	-211.00	10.68
7	1	421	3708	-383	1.13	-171.21	3.98
7	4	424	2092	-343	0.64	-153.33	5.96
7	8	428	539	-162	0.16	-72.42	20.23
7	11	431	313	-118	0.10	-52.75	6.56
7	14	434	66	-5	0.02	-2.24	16.84
7	18	438	33	-5	0.01	-2.24	0.00
7	19	439	12	-5	0.00	-2.24	0.00
7	24	444	0	0	0.00	0.00	0.45







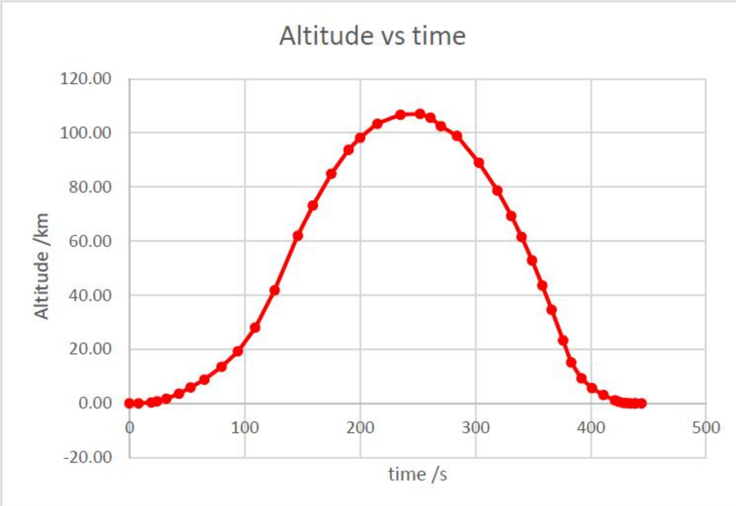
‘Free fall’ at $-9.81 m/s^2$

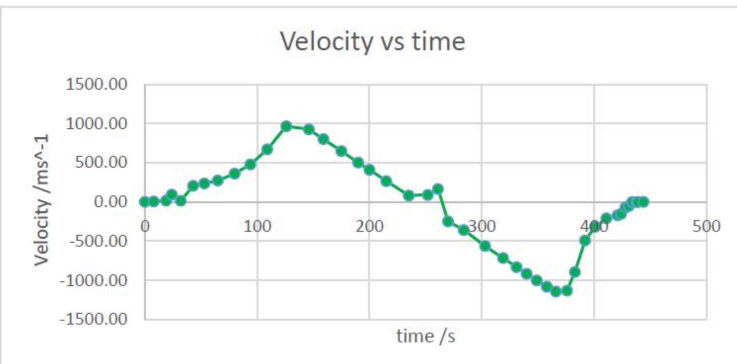
Kinematics of Blue Origin’s *New Shepherd*

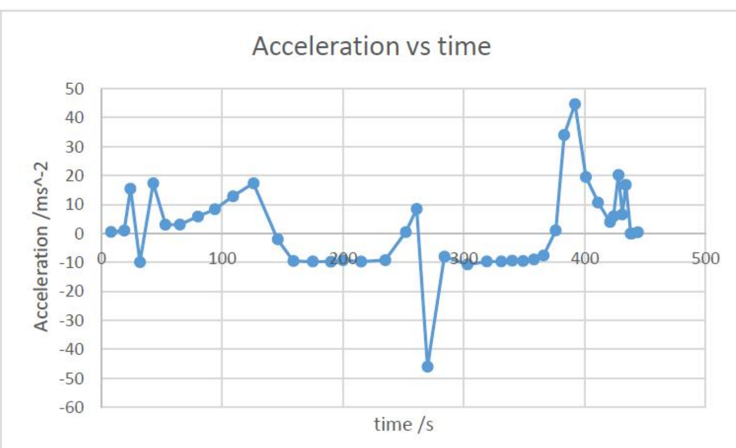
20-July-2021

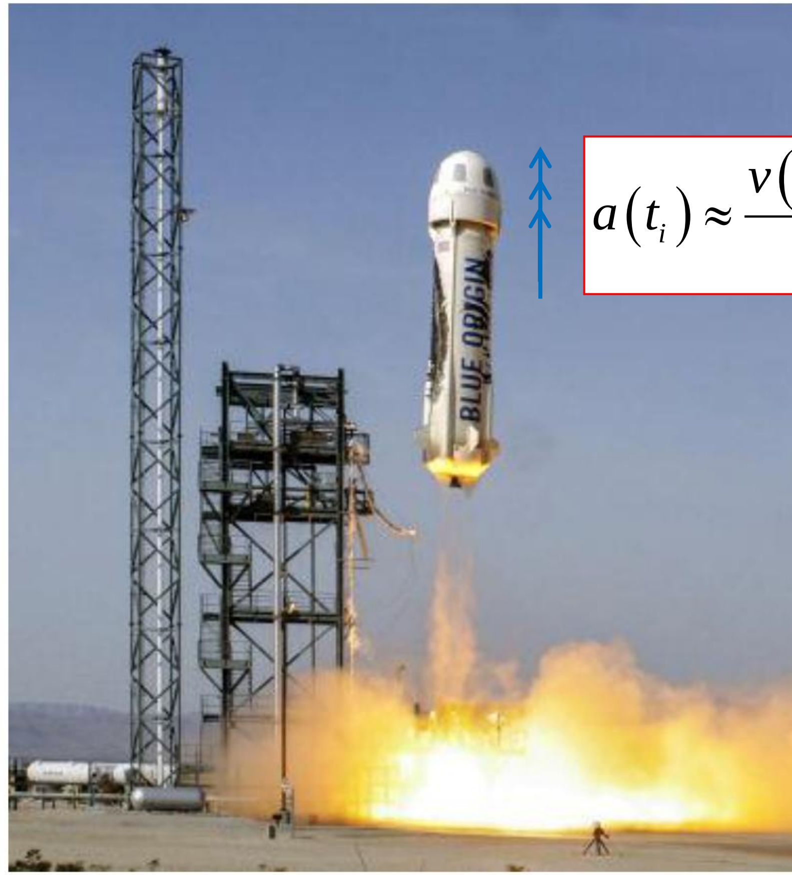
Acceleration estimated from velocity

$$\boxed{a(t_i) \approx \frac{v(t_i) - v(t_{i-1})}{t_i - t_{i-1}}}$$



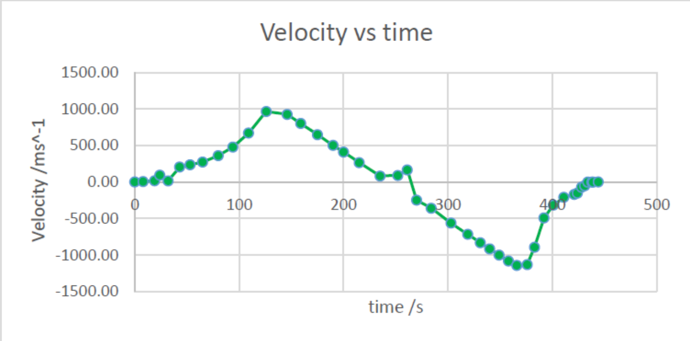




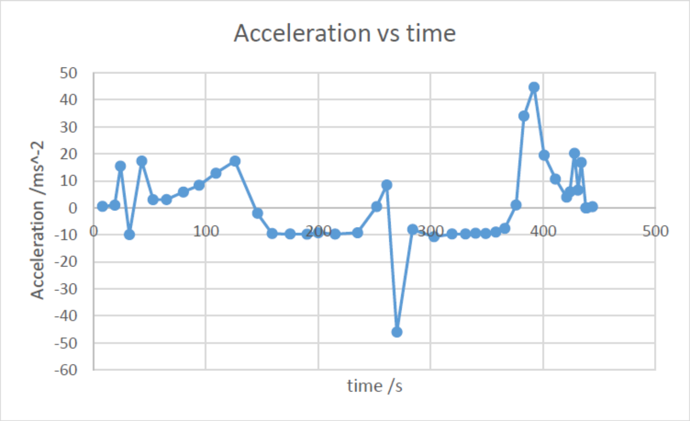


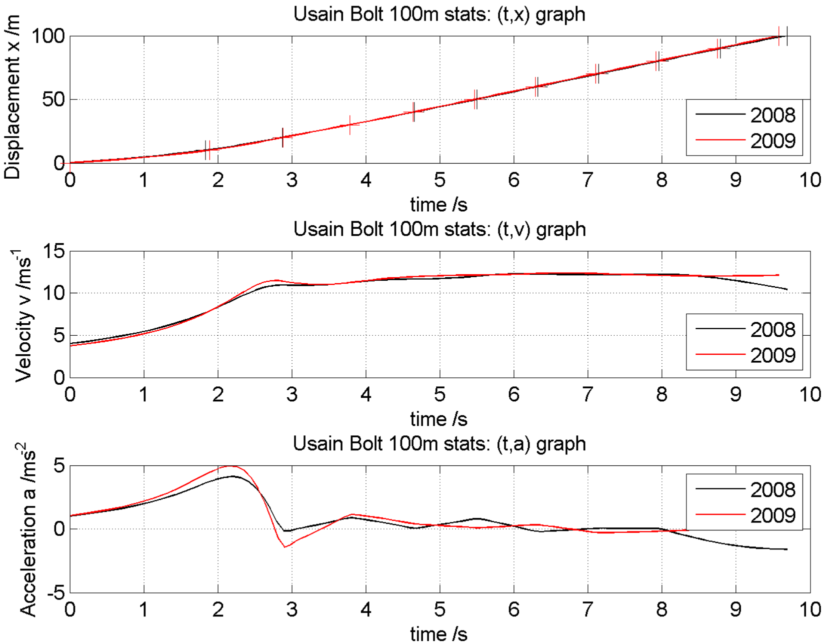
$$a(t_i) \approx \frac{v(t_i) - v(t_{i-1})}{t_i - t_{i-1}}$$

In summary: Kinematics provides a really good reason to wish to know *velocity* or *acceleration* from *displacement* measurements. **i.e. the gradient of the 'underlying curve.'** We may *not know* the functional form of the curve however – we may have to estimate it from the data.

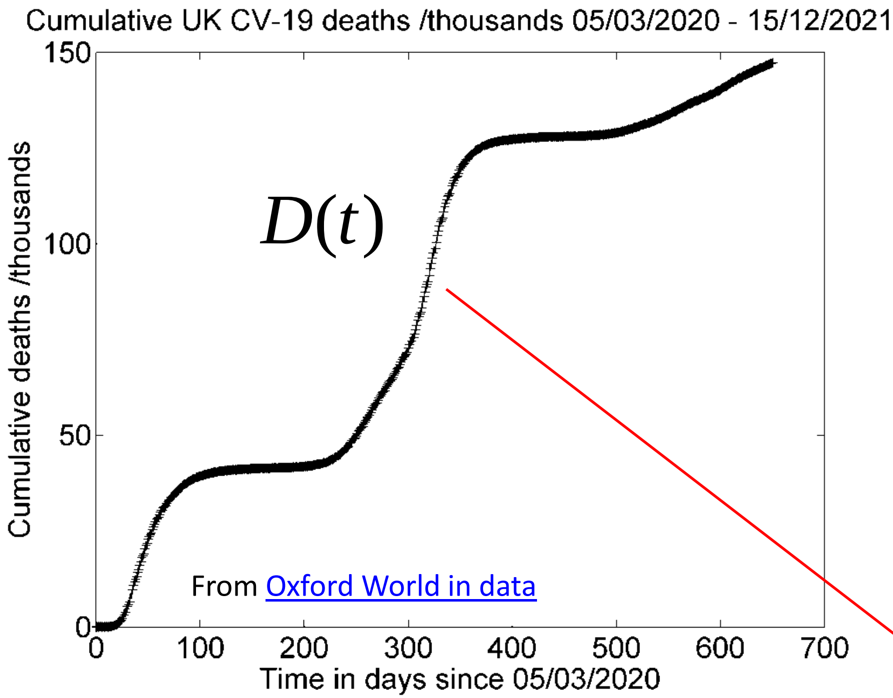


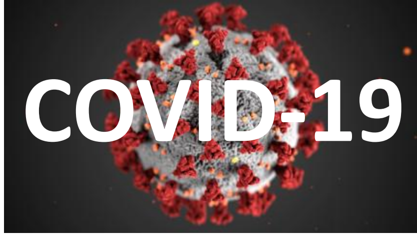
New Shepherd example: Assume a straight line between velocity measurements.





Bolt example: Fit a *cubic spline* between a rolling set of four data points. We can then differentiate the cubic (yielding a quadratic).





One can *estimate* the number of CV-19 **infectives** from the cumulative deaths:

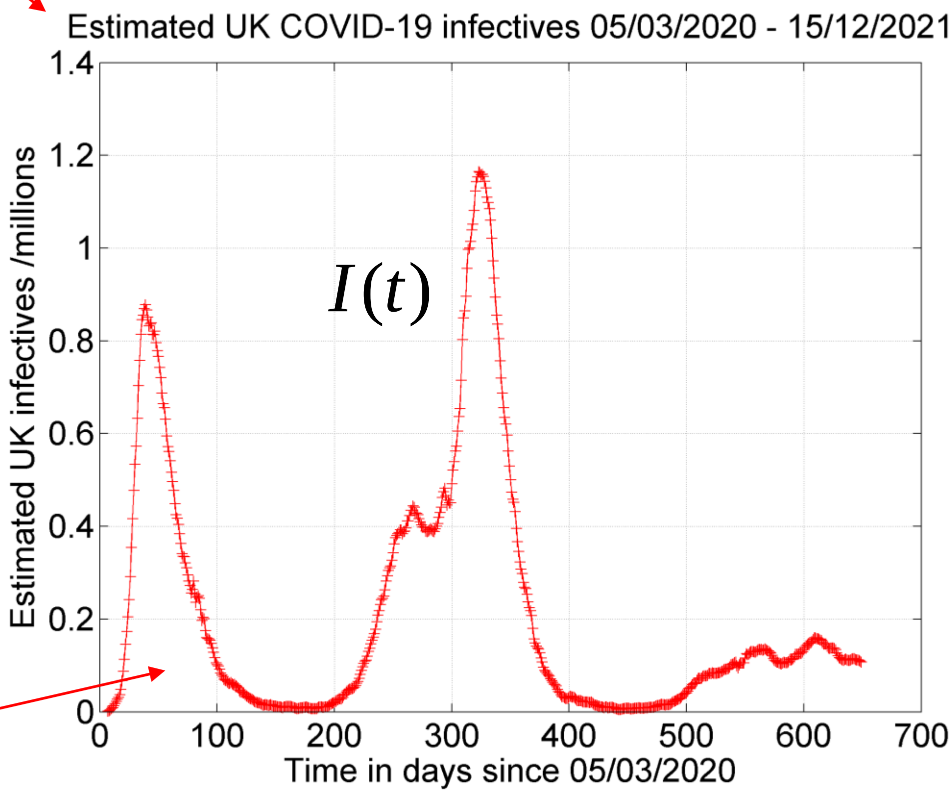
$$I_n = \frac{1}{k\alpha}\frac{dD}{dt} \approx \frac{1}{k\alpha}\frac{D_{n+1} - D_{n-1}}{t_{n+1} - t_{n-1}}$$

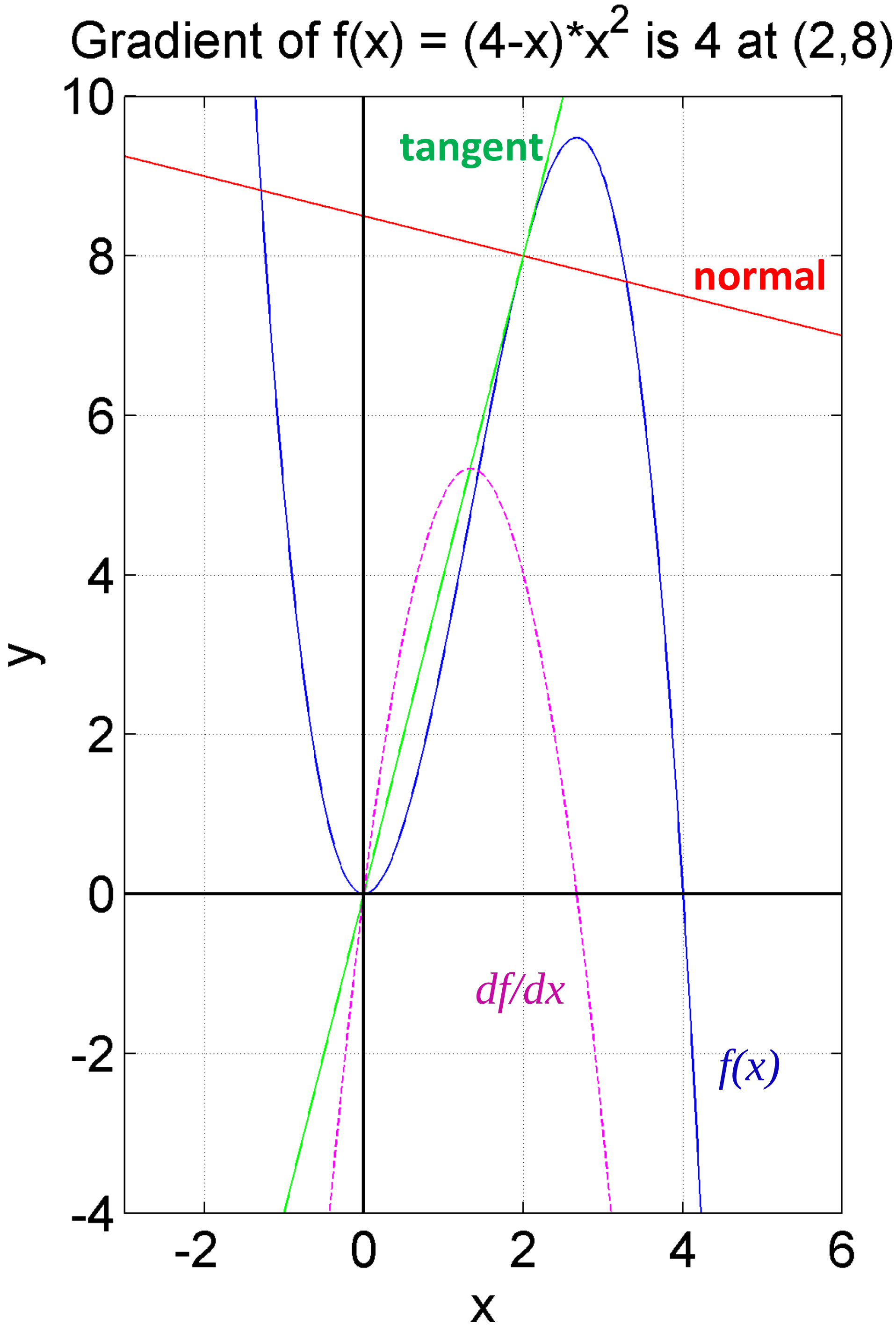
Find the gradient and scale by:

$$k\alpha = 0.01 \times \frac{1}{9.32} \text{ days}^{-1}$$

Note *mortality fraction k* and *disease time constant* α may vary considerably within a population and indeed post-vaccination – so treat with caution!

Note: as per the 'daily death rate' graphs in *World in Data*, we also apply a **seven-day moving average** to smooth the numerical derivative.

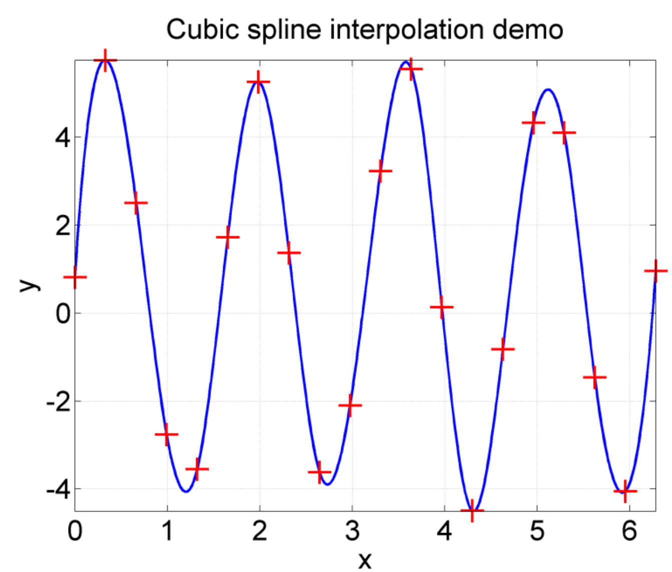




This MATLAB program evaluates a function $f(x)$ over a defined range of x, and then determines a ***cubic-spline*** over the same range.

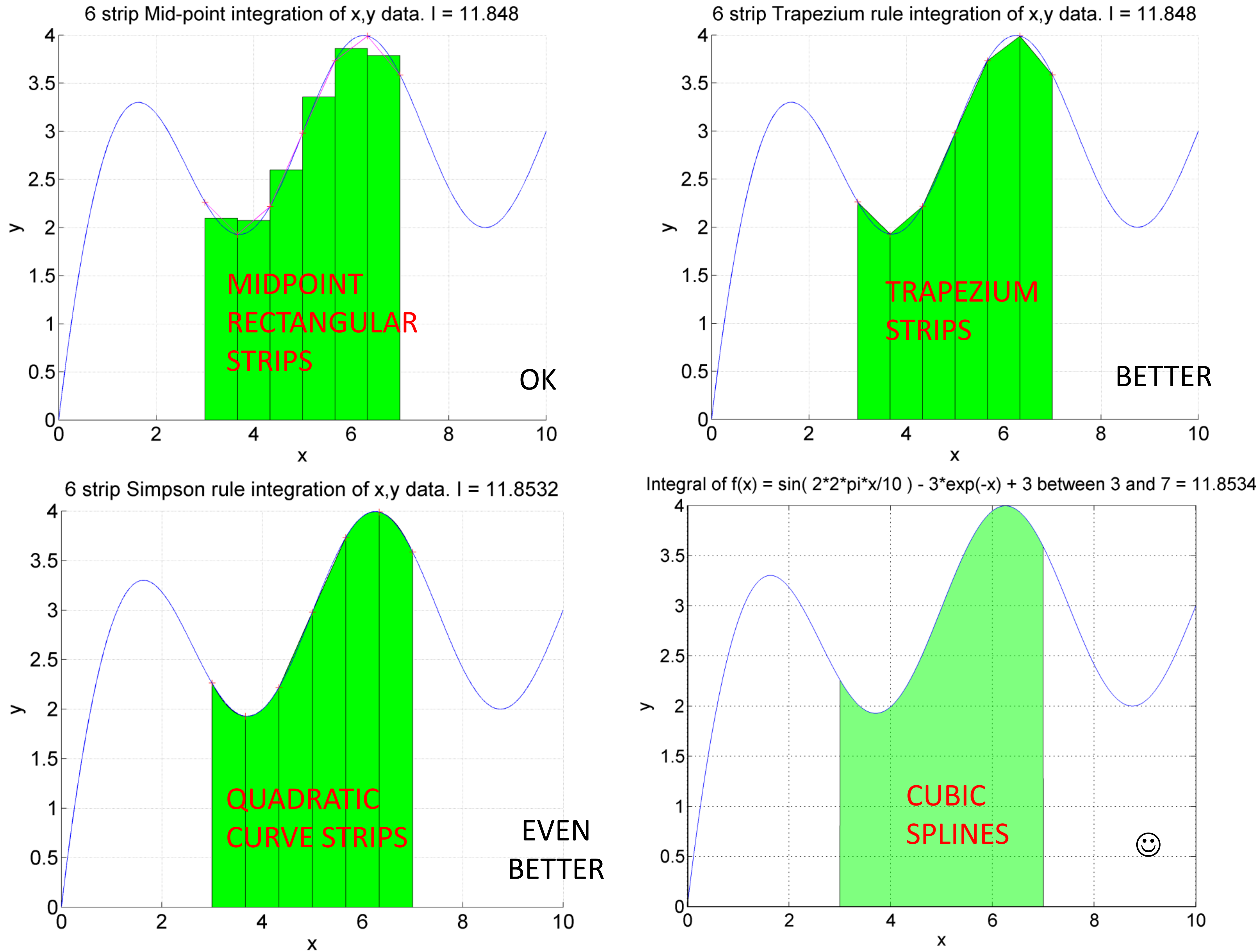
This means the function can be differentiated numerically.

In the example, a **tangent** and **normal** is evaluated at point (2,8) using this system.



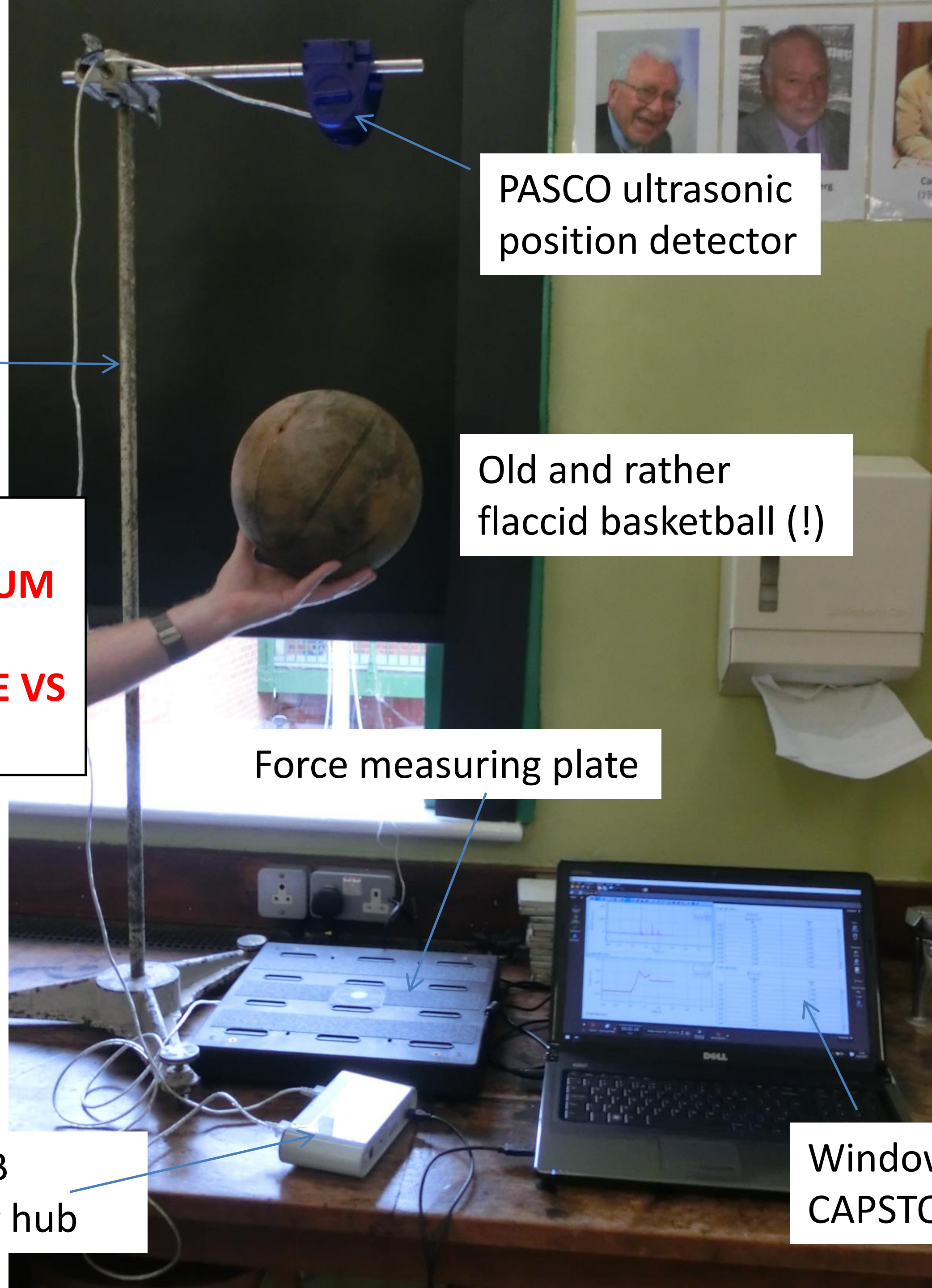
Integration: i.e. “finding areas”

$$\int_a^b f(t)dt$$


6 strip Mid-point integration of x,y data. I = 11.848
MIDPOINT
RECTANGULAR
STRIPS
OK
6 strip Trapezium rule integration of x,y data. I = 11.848
TRAPEZIUM
STRIPS
BETTER
6 strip Simpson rule integration of x,y data. I = 11.8532
QUADRATIC
CURVE STRIPS
EVEN
BETTER
Integral of f(x) = sin(2*2*pi*x/10) - 3*exp(-x) + 3 between 3 and 7 = 11.8534
CUBIC
SPLINES
☺
x
y

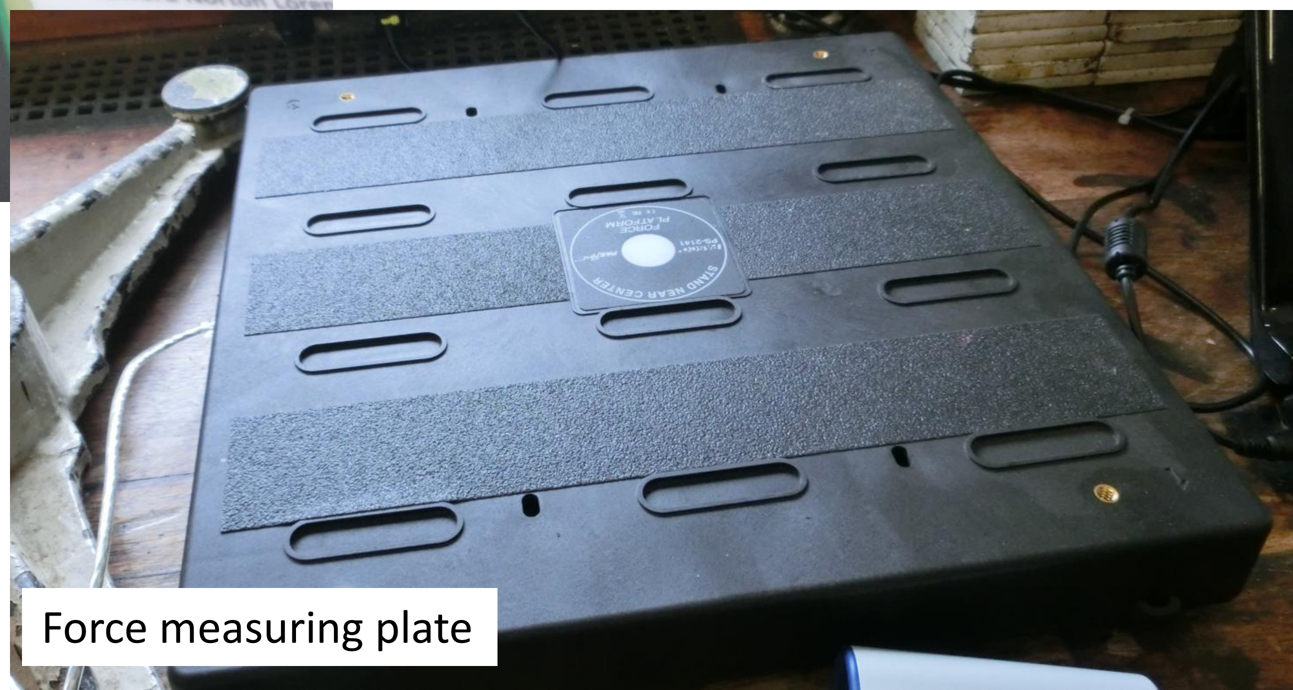
EXPERIMENTALLY VERIFYING MOMENTUM CHANGE IS THE *AREA* UNDER A FORCE VS TIME GRAPH

$$\Delta p = \int_{t_0}^{t_1} F(t)dt$$



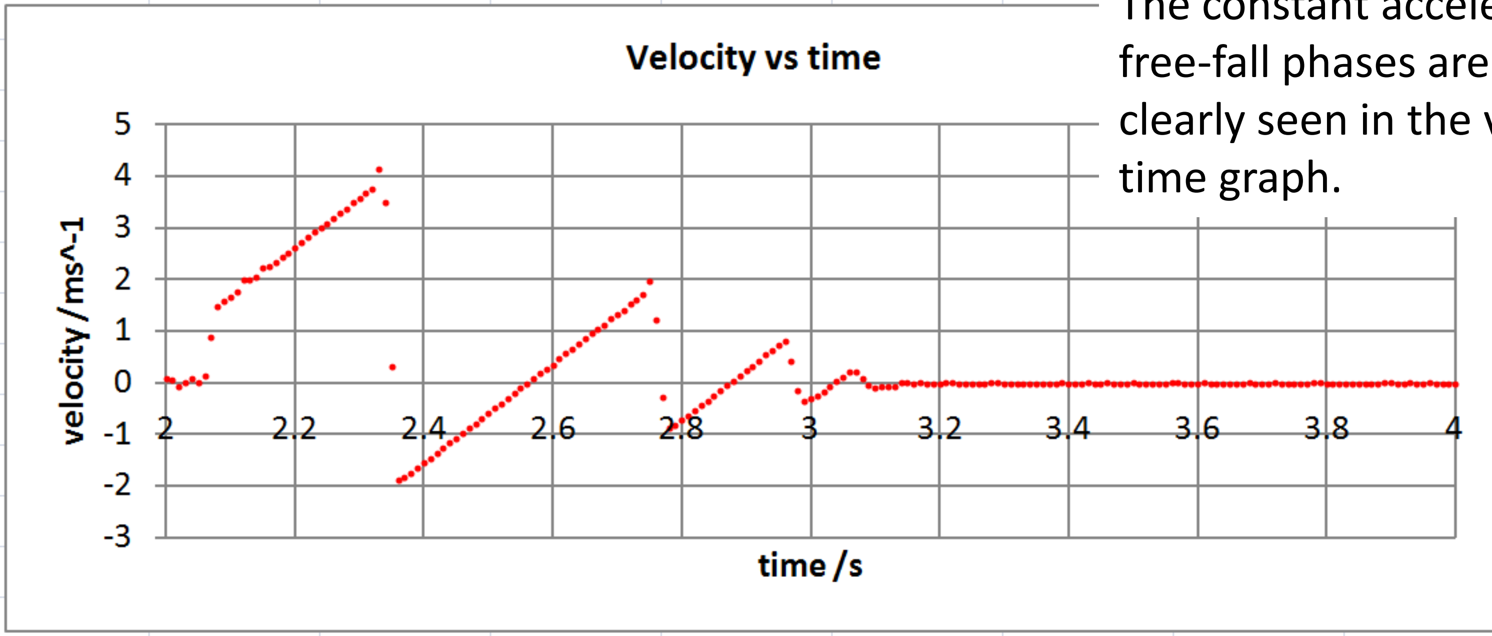
Ultrasonic position sensor

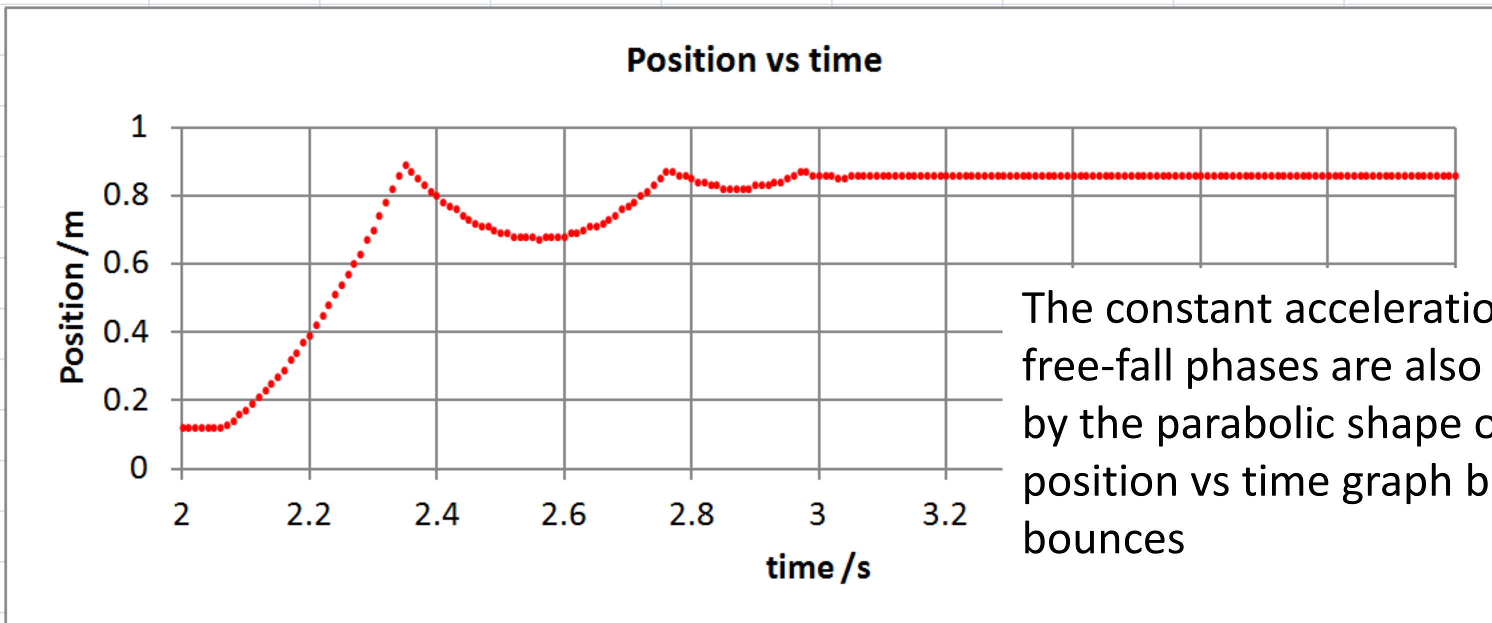
PASCO USB datalogger hub



Force measuring plate

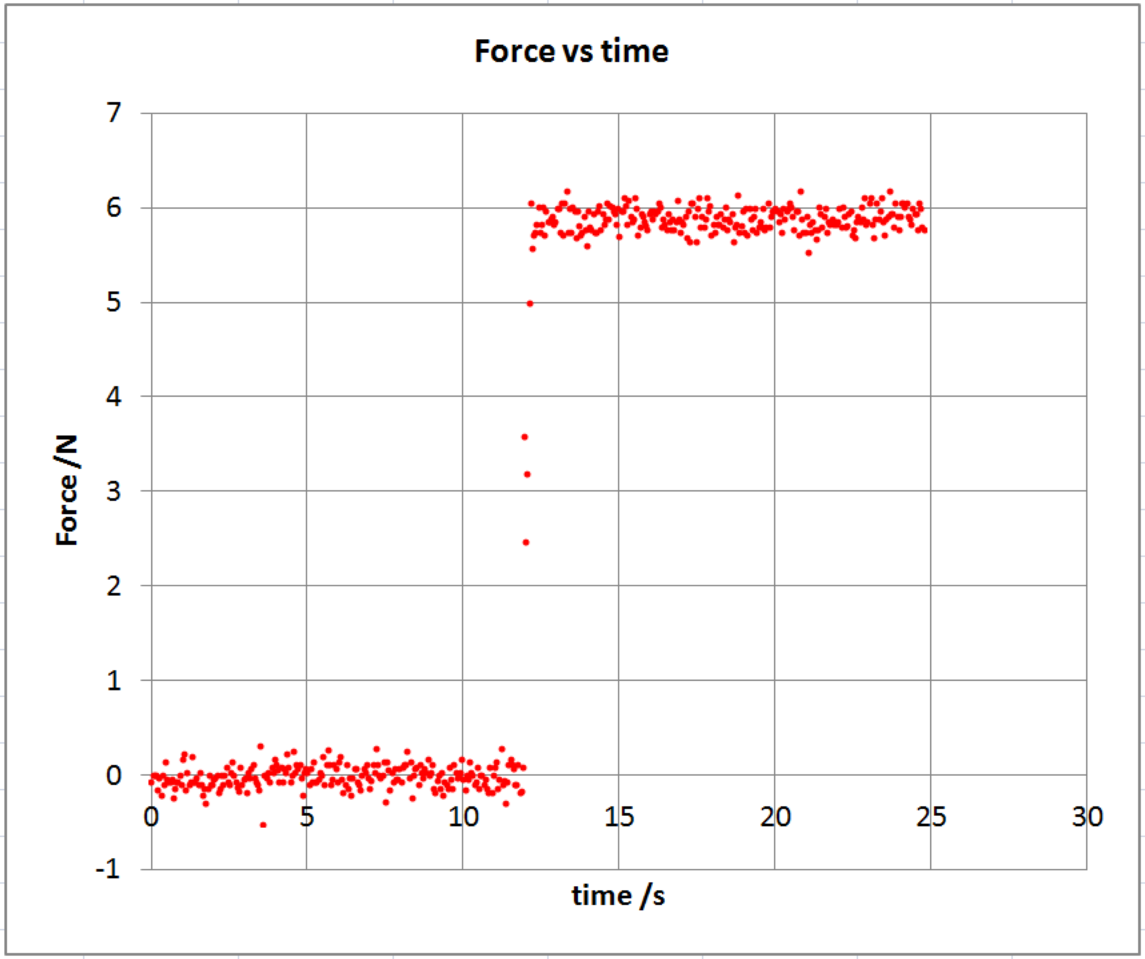
The constant acceleration motion free-fall phases are much more clearly seen in the velocity vs time graph.





The constant acceleration motion free-fall phases are also characterized by the parabolic shape of the position vs time graph between bounces

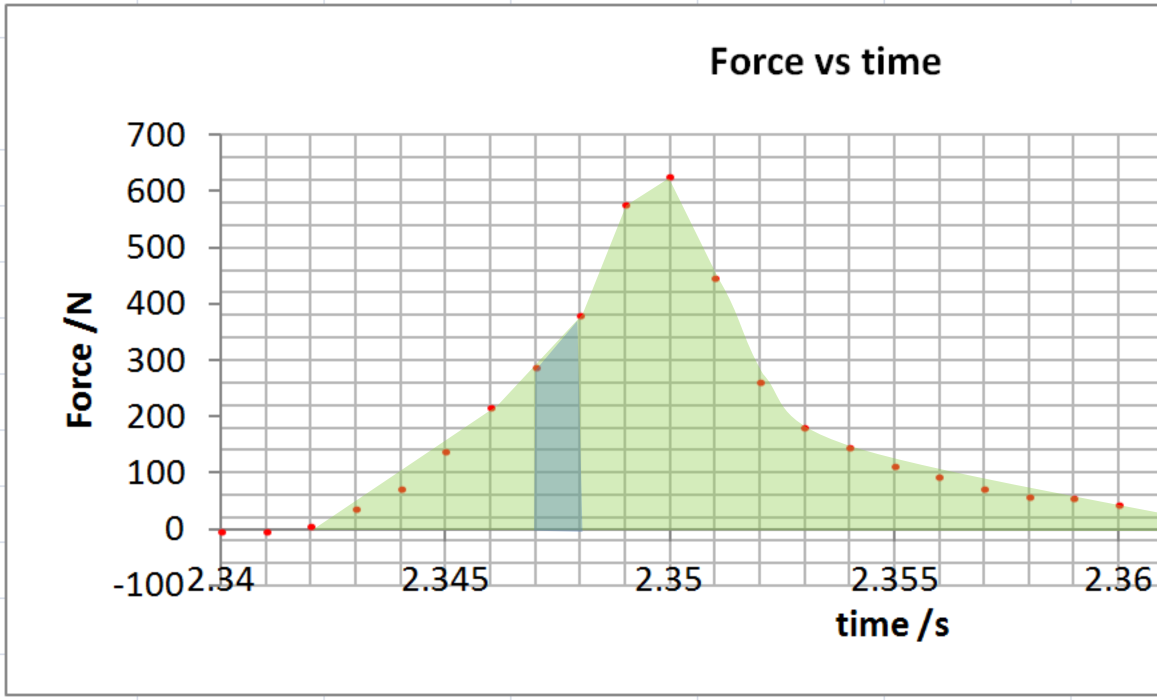
Vertical Force (N)	Time (s)
-0.08	0
0	0.05
0	0.1
2.05E-04	0.15
-0.16	0.2
-0.03	0.25
-0.22	0.3
4.98E-04	0.35
-0.11	0.4
0.14	0.45
-0.05	0.5
-0.08	0.55
-0.08	0.6
-0.05	0.65
-0.25	0.7
-0.14	0.75
-0.08	0.8
-0.08	0.85
0	0.9
-0.11	0.95
0.16	1
0.22	1.05
-0.16	1.1
0.03	1.15
-0.11	1.2
-0.08	1.25
0.19	1.3
-0.08	1.35
-0.08	1.4



Mass measured using mass balance /kg
0.625

Weight /N
6.13

Perhaps a slight under-reading for Force?



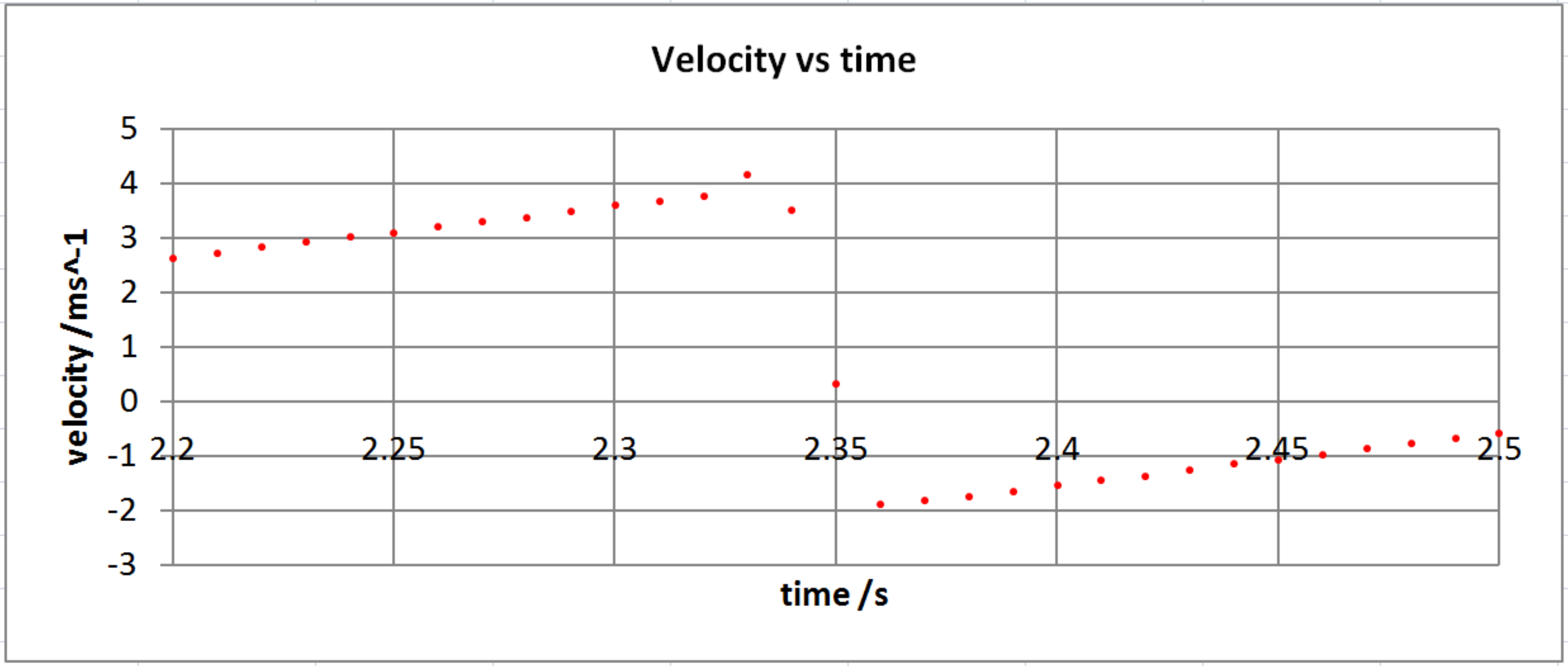
Integration of the area under the force vs time graph during impact should yield the change in momentum of the ball:

$$\Delta p = \int_{2.34}^{2.365} F dt \approx \sum_n \tfrac{1}{2}\left(t_{n+1} - t_n\right)\left(F_{n+1} + F_n\right)$$

impulse force trapezium area

By adding the area of trapeziums the **impulse** is about **3.78Ns**

Time /s	Force /N	Impulse /Ns	
2.34	-4.73		
2.341	-4.35	-0.00454	
2.342	6.33	0.00099	
2.343	35.7	0.021015	
2.344	72.18	0.05394	
2.345	136.89	0.104535	
2.346	217.38	0.177135	
2.347	286.87	0.252125	
2.348	380.53	0.3337	
2.349	576.45	0.47849	
2.35	627.62	0.602035	
2.351	446.66	0.53714	
2.352	260.15	0.353405	
2.353	179.69	0.21992	
2.354	144.34	0.162015	
2.355	112.11	0.128225	
2.356	94.1	0.103105	
2.357	70.93	0.082515	
2.358	57.7	0.064315	
2.359	55.12	0.05641	
2.36	43.09	0.049105	
2.361	21.7	0.032395	
2.362	6.9	0.0143	
2.363	-7.08	-9E-05	
2.364	-18.42	-0.01275	
2.365	-30.85	-0.024635	
		3.78	**Sum total impulse**



For the first bounce, the velocity change was about **6.1ms^{-1}.**

So if the mass was 0.625kg, this means an impulse of 0.625 x 6.1 = **3.8Ns**, which is in agreement to the area under the force vs time graph for the duration of the bounce.

... By adding the area of trapeziums the **impulse** is about **3.78Ns**

Investigating the Planck spectrum of radiation from the Sun, and Stefan's law

$$\boxed{I = \sigma T^4}$$

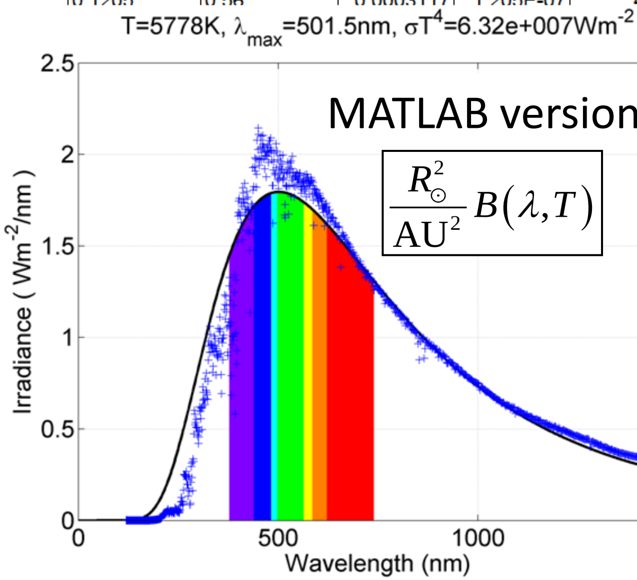
ESTIMATING SOLAR TEMPERATURE FROM THE AREA UNDER THE RADIATION SPECTRUM

A. French. Dec 2021

https://www.nrel.gov/grid/solar-resource/spectra.html

This is scaled by a strange factor ....

Wavelength, microns	E-490 W/m2/micron	Trapezium area element (W/m^2)	Wavelength /m	2*h*c^2/(lamda^5)	hc/(lambda* k*T)	B*(Rsun/AU)^2 /Wm^-2 /micron
0.1195	0.06		1.195E-07	4.89066E+18	20.847939	0.093543664
0.1205	0.56	0.0003117	1.205E-07	4.69107E+18	20.674927	0.106673563
					20.504763	0.121342219
					20.337377	0.137691654
					20.172702	0.155872955
					20.010672	0.176046509
					19.851225	0.198382223
					19.694298	0.223059716
					19.539833	0.250268494
					19.387772	0.28020811
0.1435	0.05	4.895E-05	1.435E-07			
0.1445	0.05	0.000051	1.445E-07			
0.1455	0.06	5.315E-05	1.455E-07			
0.1465	0.07	6.315E-05	1.465E-07			
0.1475	0.08	7.79E-05	1.475E-07			
0.1485	0.08	8.345E-05	1.485E-07			
0.1495	0.08	8.08E-05	1.495E-07			
0.1505	0.09	8.33E-05	1.505E-07			
0.1515	0.09	8.985E-05	1.515E-07			
0.1525	0.12	0.0001045	1.525E-07	1.44497E+18	16.336581	2.516299148
0.1535	0.13	0.0001231	1.535E-07	1.39851E+18	16.230154	2.708883683
0.1545	0.21	0.0001679	1.545E-07	1.35383E+18	16.125105	2.912810814
0.1555	0.21	0.0002102	1.555E-07	1.31086E+18	16.021406	3.128516052
0.1565	0.18	0.0001996	1.565E-07	1.26951E+18	15.919033	3.356439066
0.1575	0.17	0.0001782	1.575E-07	1.22971E+18	15.81796	3.597023176
0.1585	0.17	0.0001696	1.585E-07	1.19141E+18	15.718162	3.850714827
0.1595	0.18	0.0001715	1.595E-07	1.15453E+18	15.619616	4.117963074
0.1605	0.19	0.0001844	1.605E-07	1.119E+18	15.522297	4.399219037



$$I = \int_0^\infty B(\lambda,T)\,d\lambda = \sigma T^4$$

$$\sigma = \frac{2\pi^5 k_B^4}{15c^2h^3}$$

$$B(\lambda,T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}$$

$$\Phi = \frac{4\pi R_\odot^2 \sigma T_\odot^4}{4\pi \mathrm{AU}^2}$$

$$T_\odot = \left(\frac{\mathrm{AU}^2}{R_\odot^2 \sigma}\Phi\right)^{\frac{1}{4}}$$

Φ

Area under spectrum (W/m^2)	This is the average solar radiation power incident upon Earth (W/m^2)
1366.09	

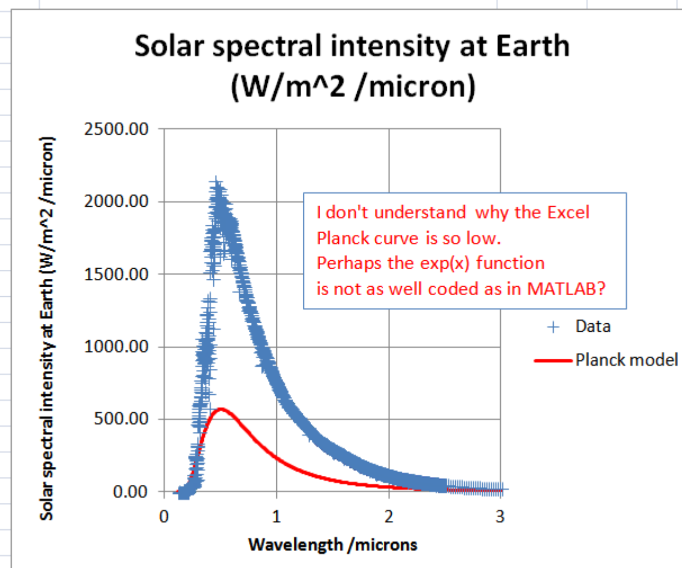
Stefan's constant (Wm^-2 K^-4)
5.67E-08

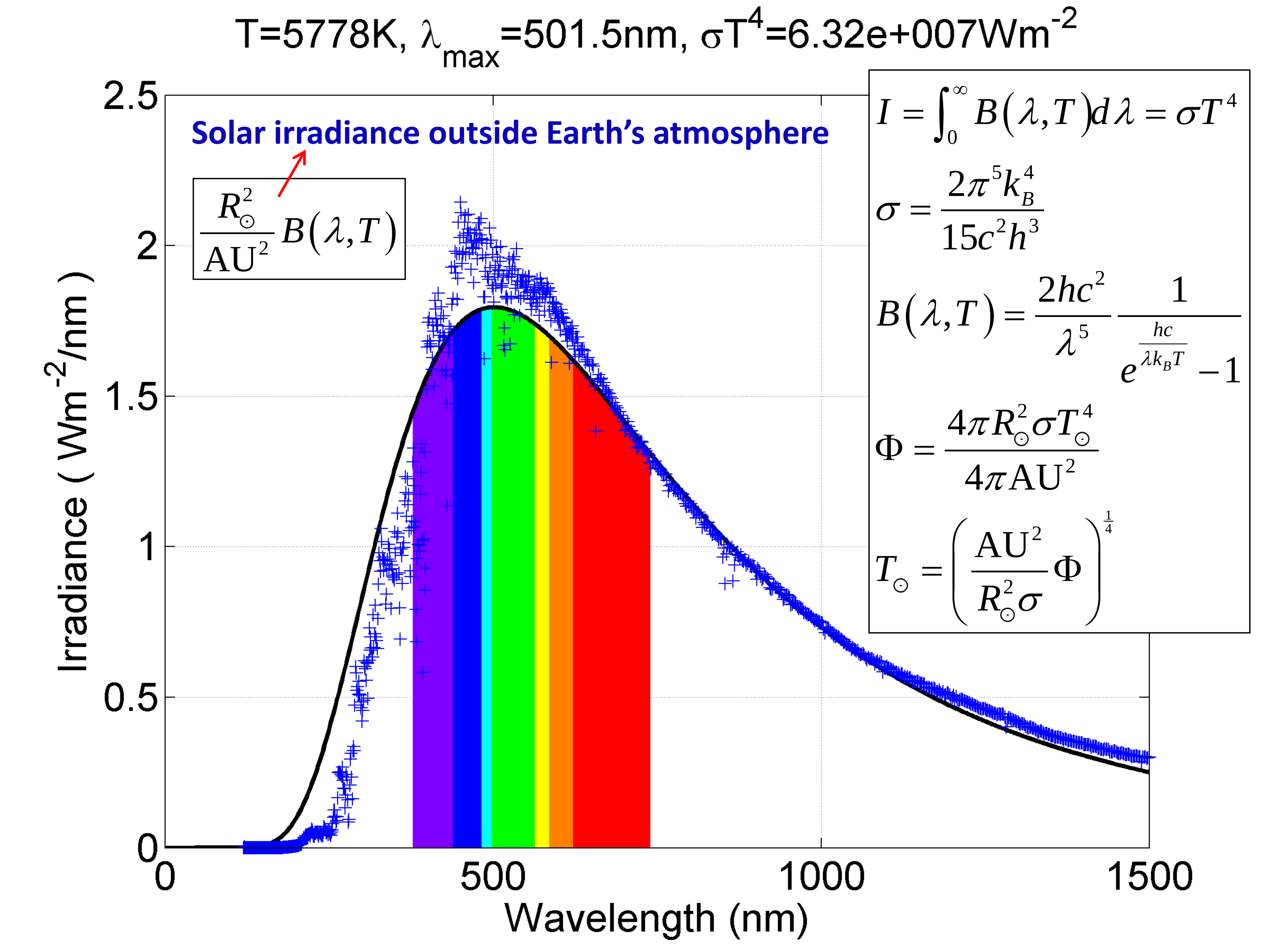
Solar radius /m
696,340,000

Earth to Sun distance /m
1.49598E+11

Estimated solar surface temperature /K	i.e. a pretty close agreement with the official figure!
5,775	

Official figure for Solar surface temperature /K
5,778




T=5778K, λ_{max}=501.5nm, σT^4=6.32e+007Wm^{-2}
Solar irradiance outside Earth's atmosphere
$\frac{R_\odot^2}{\mathrm{AU}^2} B(\lambda, T)$
$I = \int_0^\infty B(\lambda, T) d\lambda = \sigma T^4$
$\sigma = \frac{2\pi^5 k_B^4}{15 c^2 h^3}$
$B(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}$
$\Phi = \frac{4\pi R_\odot^2 \sigma T_\odot^4}{4\pi \mathrm{AU}^2}$
$T_\odot = \left(\frac{\mathrm{AU}^2}{R_\odot^2 \sigma} \Phi \right)^{\frac{1}{4}}$
Irradiance (Wm^{-2}/nm)
Wavelength (nm)
0
0.5
1
1.5
2
2.5
500
1000
1500

BPhO Computational Challenge

- Suggested homework
- Q&A