BPhO Computational Challenge

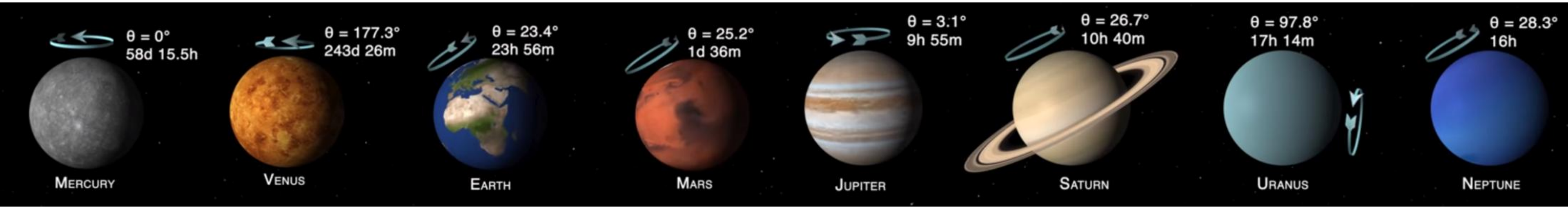
Seminar 08: Dynamic simulations of Gravity using the Verlet method

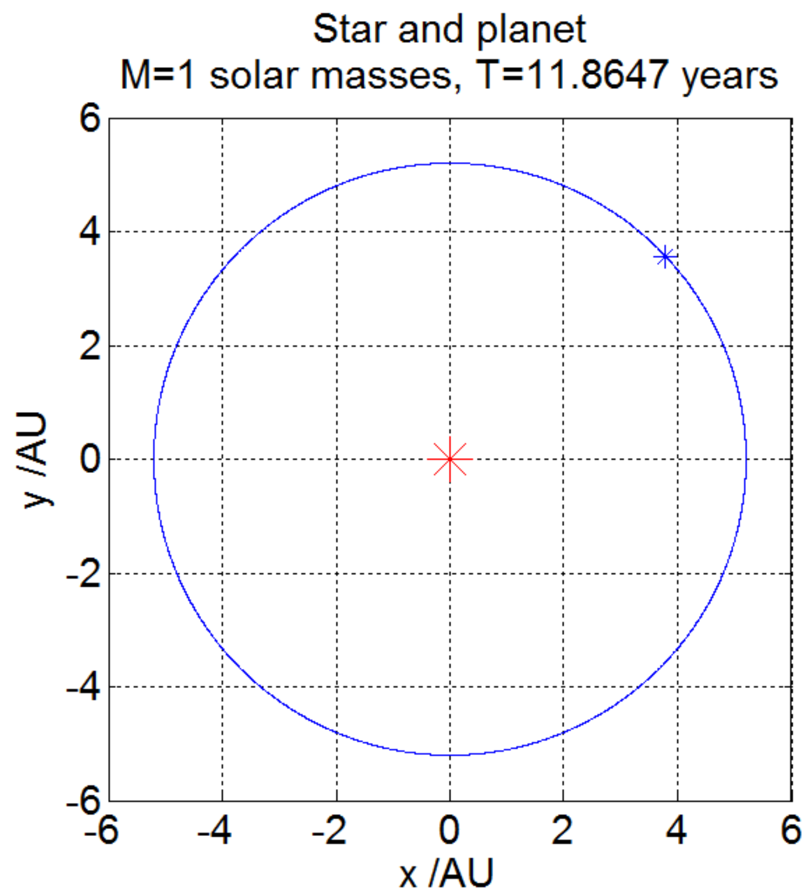
Dr Andrew French.
December 2021.

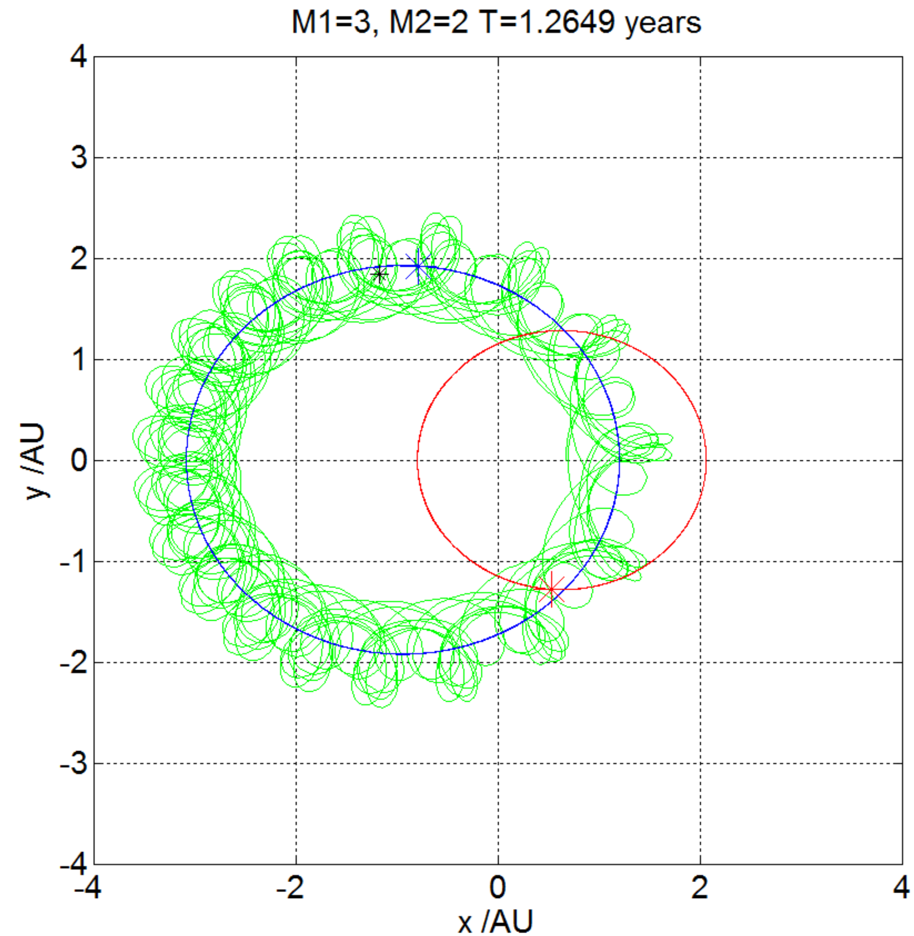
BPhO British Physics Olympiad

The Planets

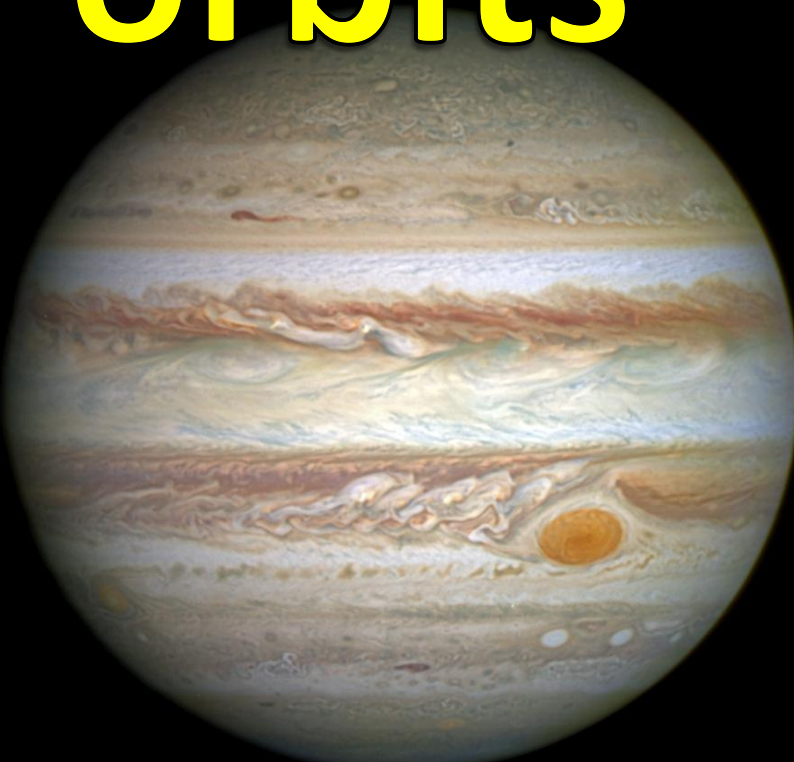
Make a gravity simulation based upon one the solar system planets and its moons (or the Solar System itself). Learn how to make animated plots.

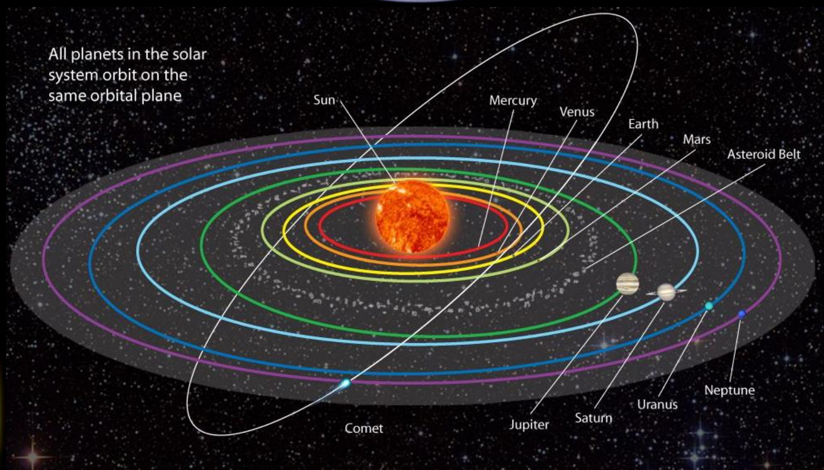


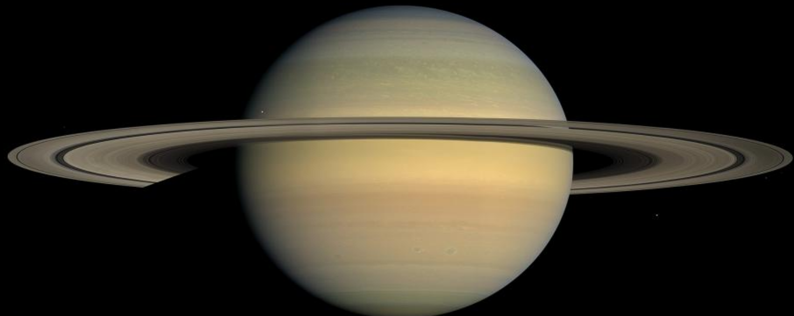




Solar System orbits



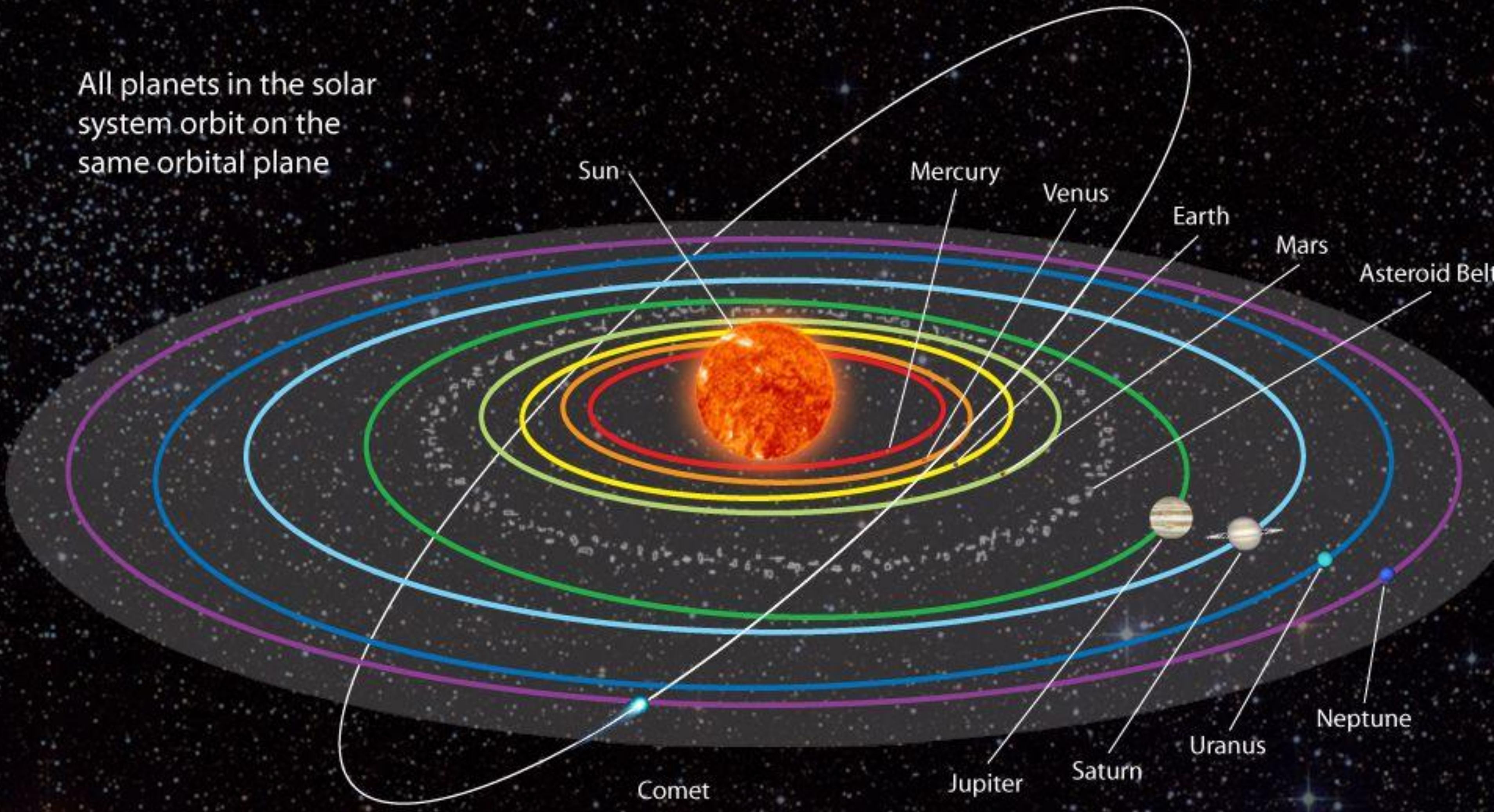




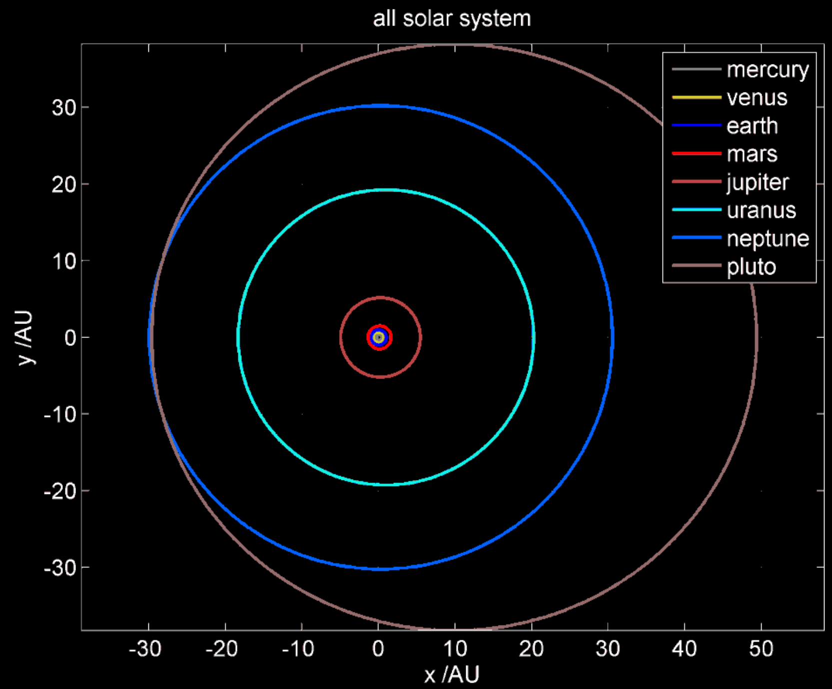
https://www.nationalgeographic.org/media/orbital-plane/

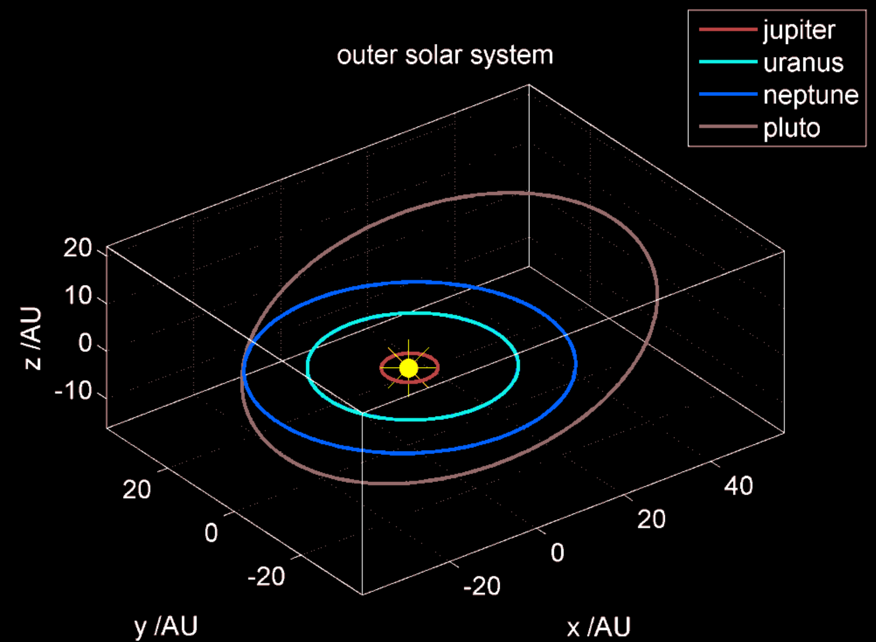
Orbital Plane

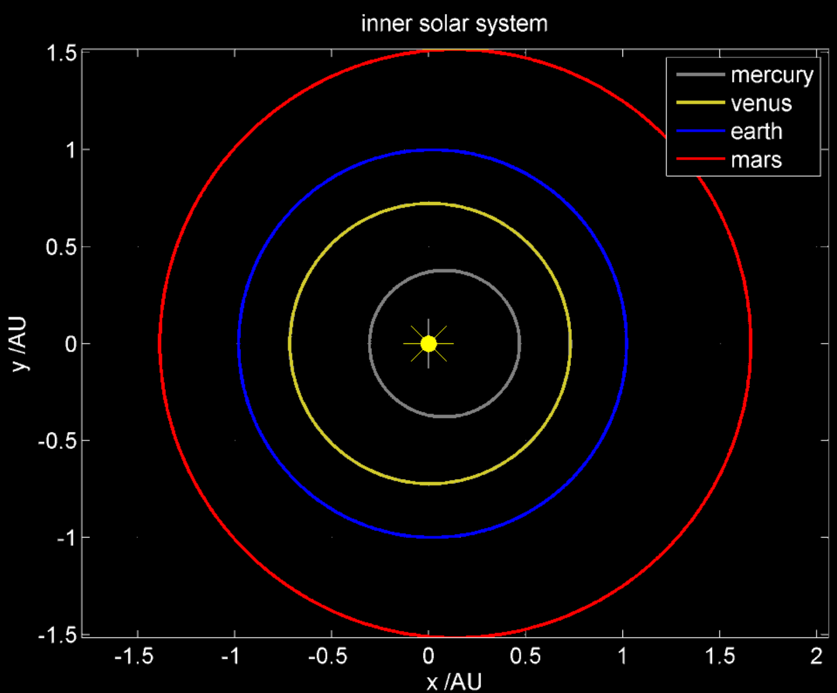
All planets in the solar system orbit on the same orbital plane

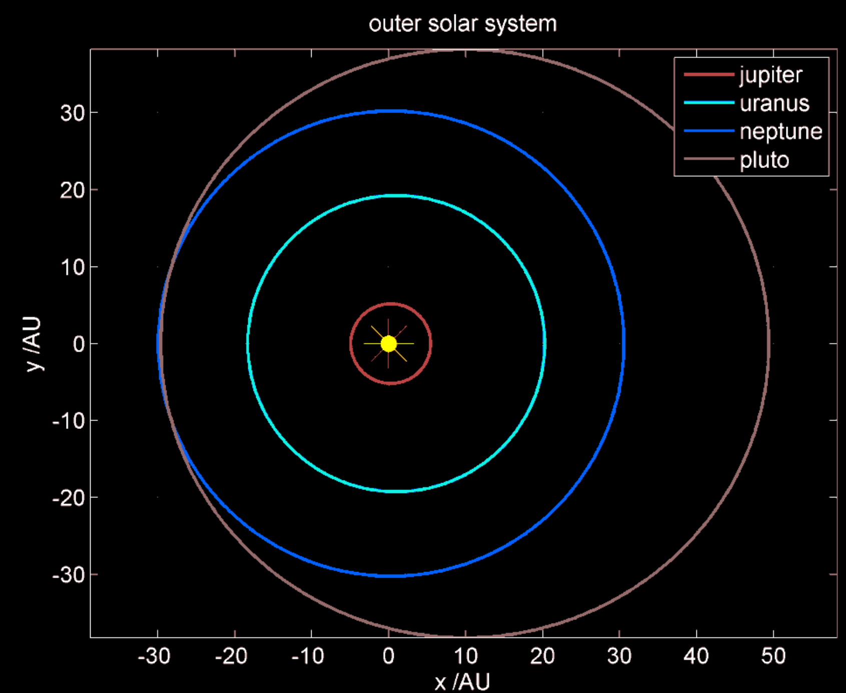


* Many comets exist outside the orbital plane

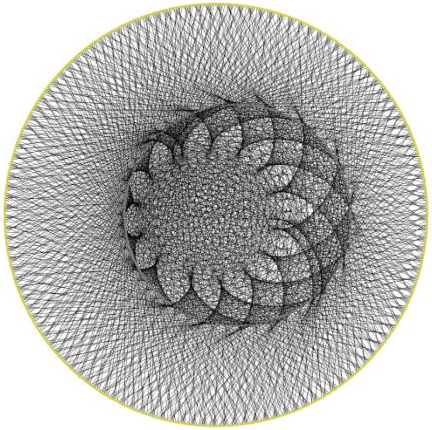

all solar system
mercury
venus
earth
mars
jupiter
uranus
neptune
pluto
x /AU
y /AU


outer solar system
jupiter
uranus
neptune
pluto
z /AU
y /AU
x /AU

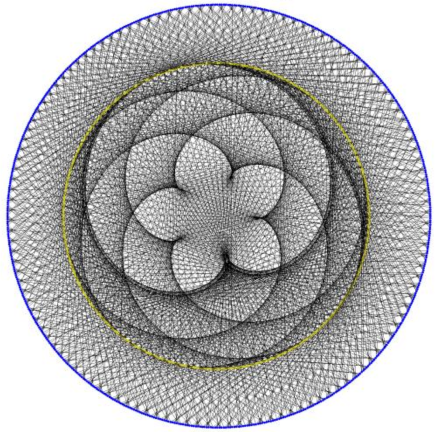

inner solar system
mercury
venus
earth
mars
x /AU
y /AU


outer solar system
jupiter
uranus
neptune
pluto
x /AU
y /AU

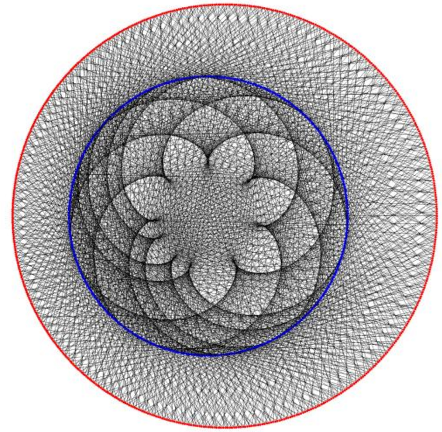
mercury venus spirograph



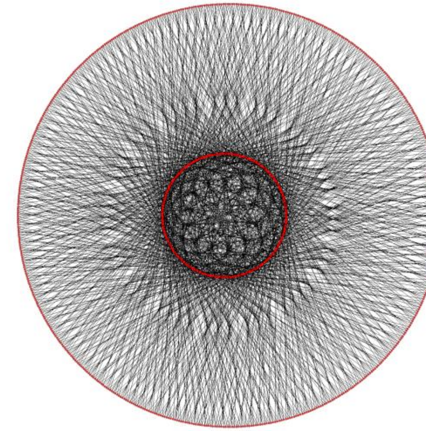
venus earth spirograph



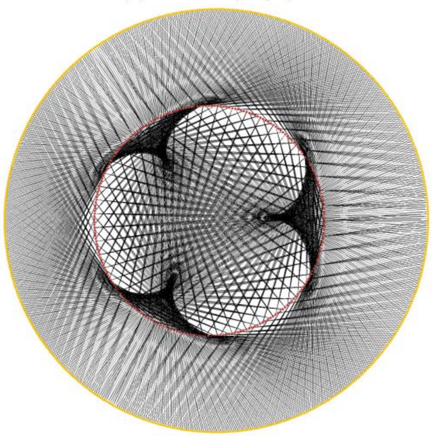
earth mars spirograph



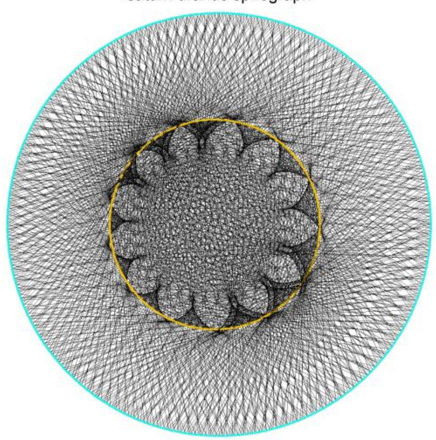
mars jupiter spirograph



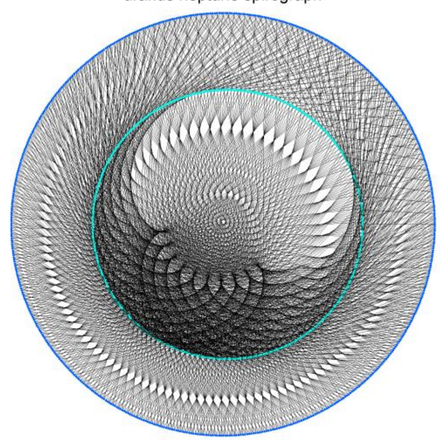
jupiter saturn spirograph



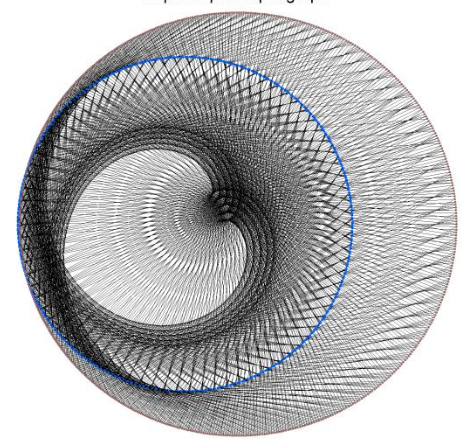
saturn uranus spirograph

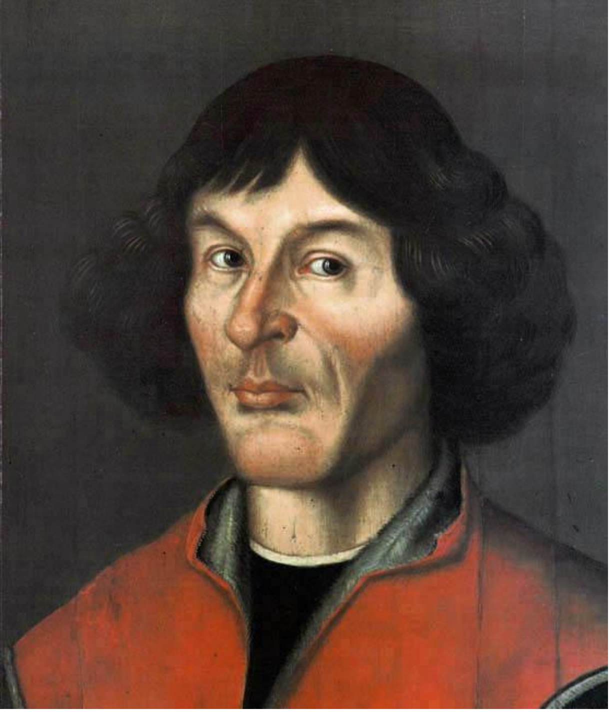


uranus neptune spirograph



neptune pluto spirograph



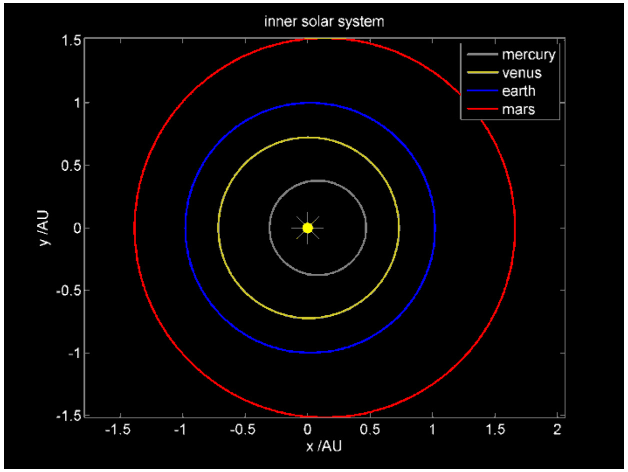


Nicolaus Copernicus
1473-1543

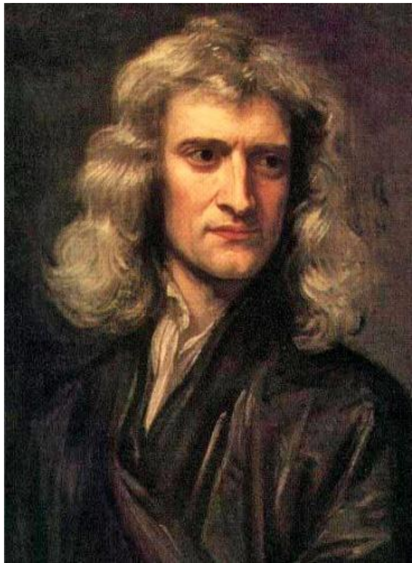
Tycho Brahe
1546-1601

Nose lost in 1566 following a sword duel with third cousin Manderup Parsberg over the legitimacy of a mathematical formula!

Johannes
Kepler
1571-1630



Isaac
Newton
1642-
1727



Kepler's three laws are:

1. *The orbit of every planet in the solar system is an ellipse with the Sun at one of the two foci.*

2. *A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.*

3. *The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.*

The wording of Kepler's laws implies a specific application to the solar system. However, the laws are more generally applicable to any system of two masses whose mutual attraction is an inverse-square law.

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$ *Polar* equation of ellipse

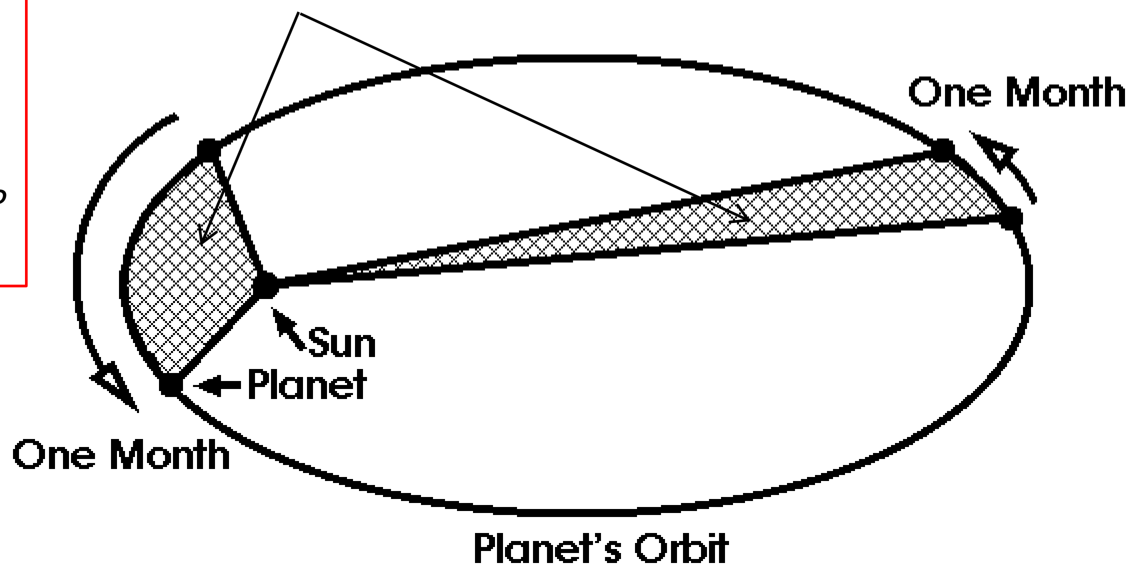
$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$ *Eccentricity* of ellipse

$$P^2 = \frac{4\pi^2}{G(m+M_\odot)}a^3$$ Orbital period *P*

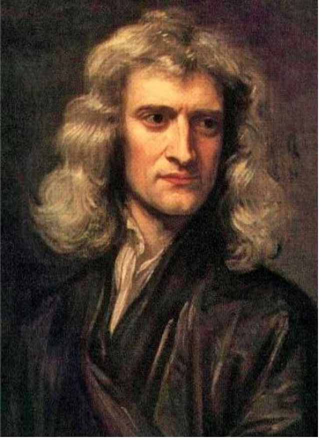
$$\frac{dA}{dt} = \tfrac{1}{2}\sqrt{G(m+M_\odot)\left(1-\varepsilon^2\right)a}$$

This is a constant

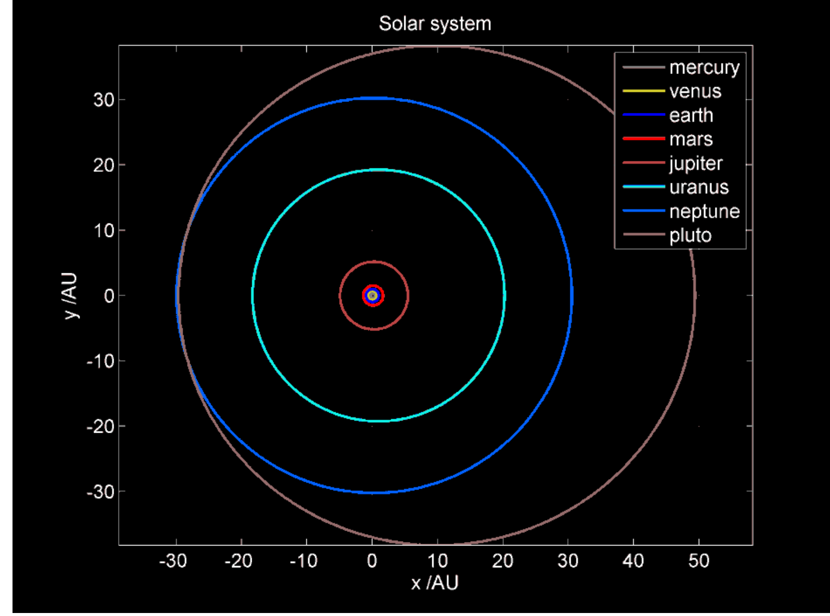
Equal areas swept out in equal times



Johannes Kepler
1571-1630



Isaac Newton (1642-1727) developed a mathematical model of Gravity which predicted the elliptical orbits proposed by Kepler



Planet and Solar masses

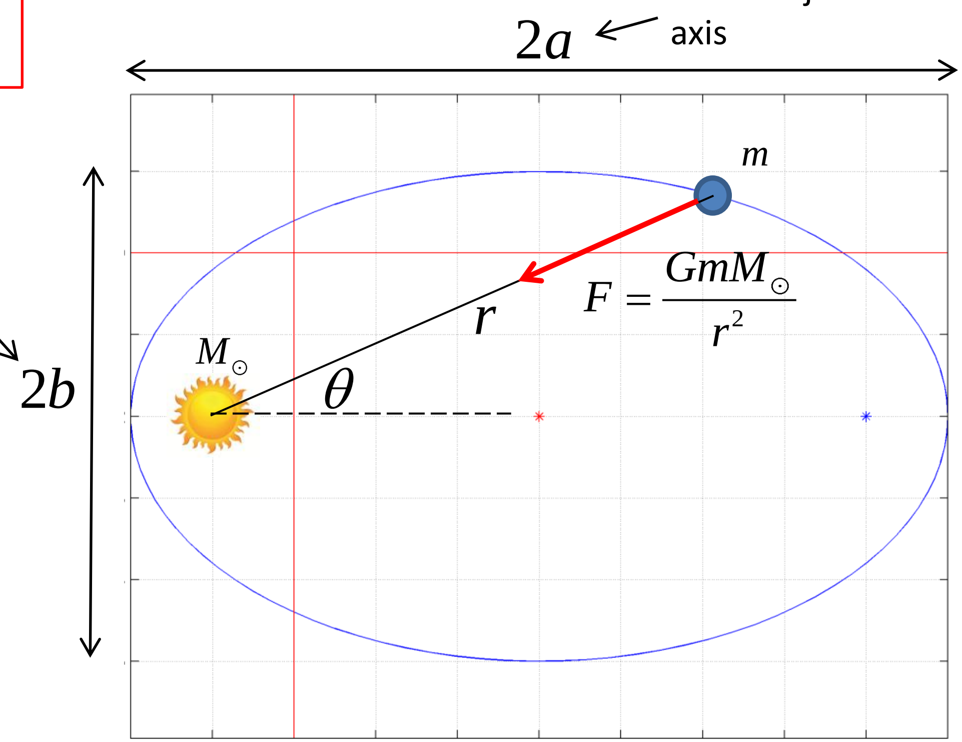
Force of gravity

$$F = \frac{GmM_{\odot}}{r^2}$$

$$G = 6.67 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$$

Semi-major axis

Semi-minor axis



$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

Polar equation of ellipse

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

Eccentricity of ellipse

$$P^2 = \frac{4\pi^2}{G(m+M_{\odot})}a^3$$

Orbital period P

Object	Mass in Earth masses	Distance from Sun in AU	Radius in Earth radii	Rotational period /days	Orbital period /years
Saturn	95.16	9.58	9.45	0.44	29.63
Uranus	14.50	19.29	4.01	0.72	84.75
Jupiter	317.85	5.20	11.21	0.41	11.86
Sun	332,837	-	109.12	-	-
Neptune	17.20	30.25	3.88	0.67	166.34
Pluto	0.00	39.51	0.19	6.39	248.35
Mars	0.107	1.523	0.53	1.03	1.88
Venus	0.815	0.723	0.95	243.02	0.62
Mercury	0.055	0.387	0.38	58.65	0.24
Earth	1.000	1.000	1.00	1.00	1.00

Gravitational field (in terms of g = 9.81 ms^-2)
1.07
0.90
2.53
27.95
1.14
0.09
0.38
0.90
0.37
1.00

For our Solar System:

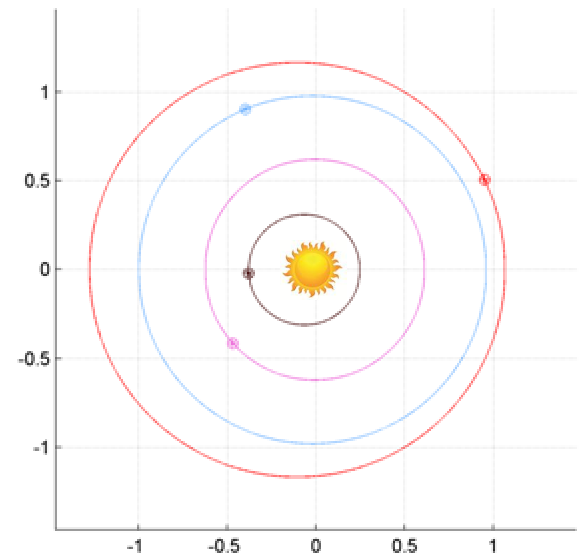
$m \ll M_{\odot}$

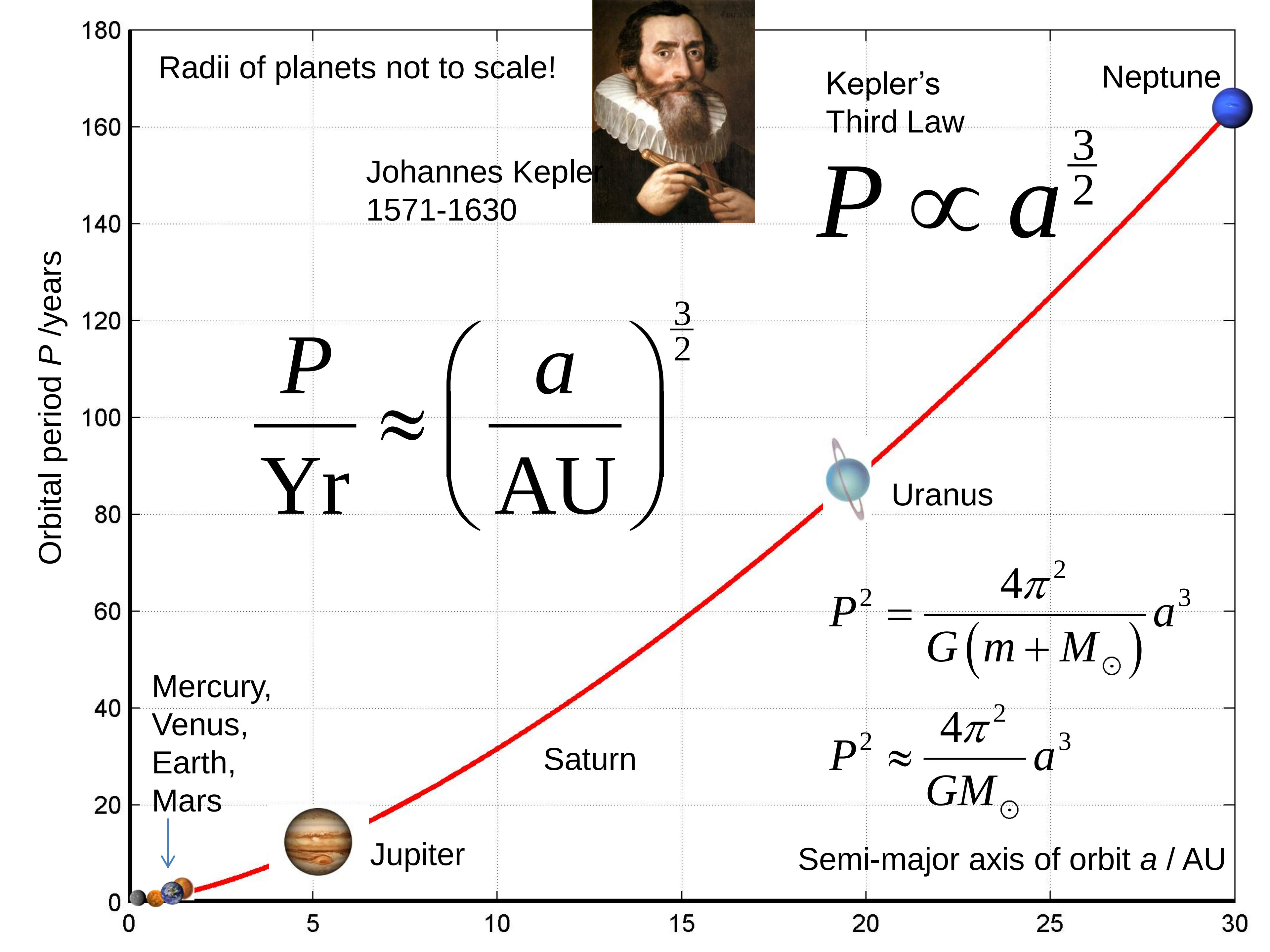
$$P^2 = \frac{4\pi^2}{G\left(m + M_{\odot}\right)} a^3$$

$$P^2 \approx \frac{4\pi^2}{GM_{\odot}} a^3$$

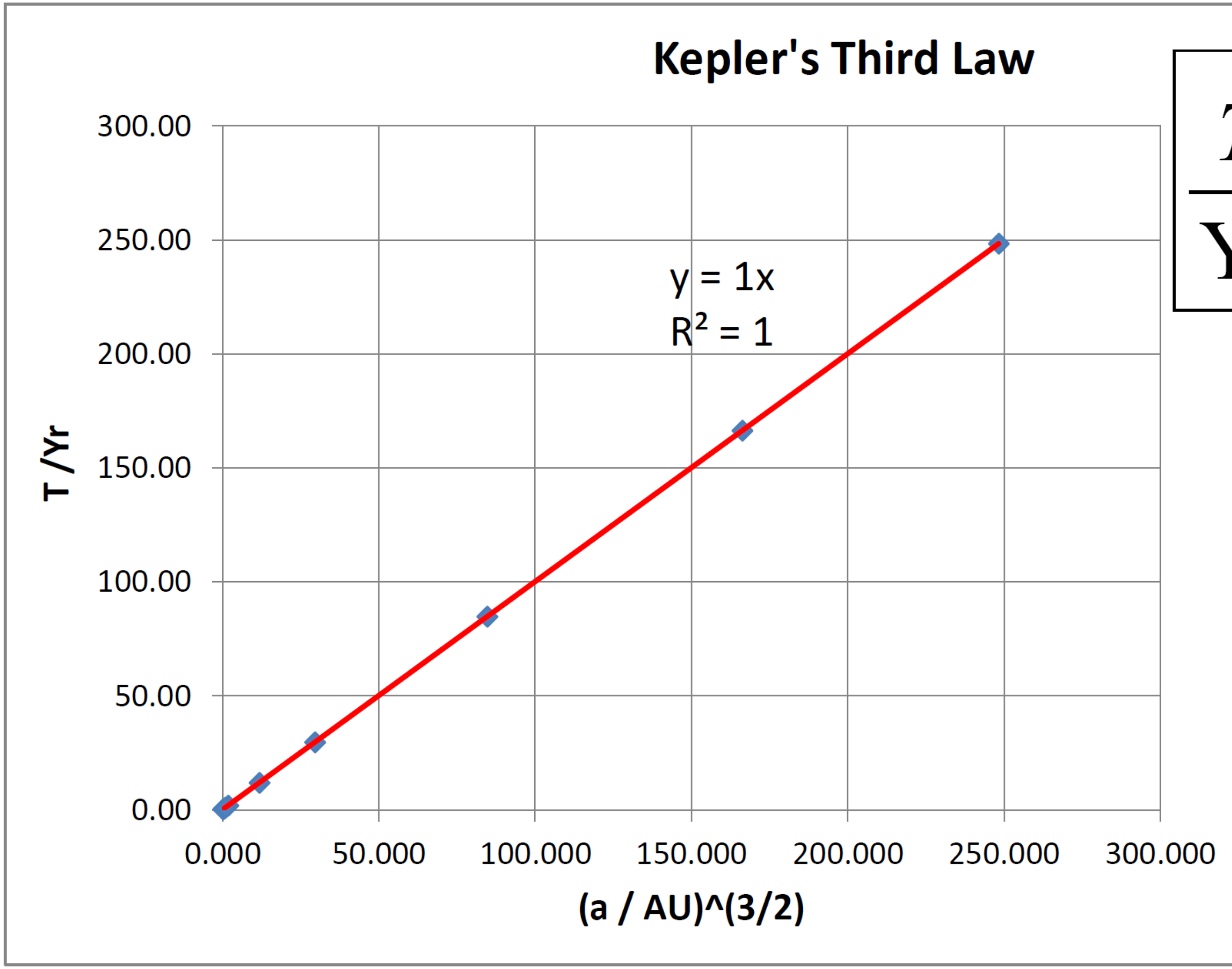
$$\mathrm{Yr}^2 = \frac{4\pi^2}{GM_{\odot}} \mathrm{AU}^3$$

$$\therefore \frac{P}{\mathrm{Yr}} \approx \left(\frac{a}{\mathrm{AU}}\right)^{\frac{3}{2}}$$





Radii of planets not to scale!
Johannes Kepler
1571-1630
Kepler's
Third Law
$P \propto a^{\frac{3}{2}}$
Neptune
$\frac{P}{\mathrm{Yr}} \approx \left(\frac{a}{\mathrm{AU}}\right)^{\frac{3}{2}}$
Uranus
$P^2 = \frac{4\pi^2}{G(m+M_\odot)} a^3$
$P^2 \approx \frac{4\pi^2}{GM_\odot} a^3$
Mercury,
Venus,
Earth,
Mars
Saturn
Jupiter
Semi-major axis of orbit a / AU
Orbital period P /years
0
20
40
60
80
100
120
140
160
180
0
5
10
15
20
25
30



$$\frac{T}{\mathrm{Yr}} = \left(\frac{R}{\mathrm{AU}}\right)^{\frac{3}{2}}$$

A *very strong* correlation of Kepler III to orbital data for planets in our solar system!

Object	$M/M_\oplus$	a /AU	ε	θ_0	β	$R/R_\oplus$	T_{rot} / days	P /Yr
Sun	332,837	-	-	-	-	109.123	-	-
Mercury	0.055	0.387	0.21	*	7.00	0.383	58.646	0.241
Venus†	0.815	0.723	0.01	*	3.39	0.949	243.018	0.615
Earth	1.000	1.000	0.02	*	0.00	1.000	0.997	1.000
Mars	0.107	1.523	0.09	*	1.85	0.533	1.026	1.881
Jupiter	317.85	5.202	0.05	*	1.31	11.209	0.413	11.861
Saturn	95.159	9.576	0.06	*	2.49	9.449	0.444	29.628
Uranus†	14.500	19.293	0.05	*	0.77	4.007	0.718	84.747
Neptune	17.204	30.246	0.01	*	1.77	3.883	0.671	166.344
Pluto†	0.003	39.509	0.25	*	17.5	0.187	6.387	248.348

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$$

$$P^2 = \frac{4\pi^2}{G(m+M_\odot)}a^3$$

β is the orbital inclination /degrees. In all cases the semi-major axis pointing direction is

$$\mathbf{d} = d_x\hat{\mathbf{x}} + d_y\hat{\mathbf{y}} + d_{\mathbf{y}}\hat{\mathbf{z}} = \cos\beta\hat{\mathbf{x}} + \sin\beta\hat{\mathbf{z}}$$

$$\begin{aligned} M_\odot &= 1.9891\times10^{30}\ \text{kg} \\ R_\odot &= 6.960\times10^{8}\ \text{m} \\ M_\oplus &= 5.9742\times10^{24}\ \text{kg} \\ R_\oplus &= 6.37814\times10^{6}\ \text{m} \\ 1\text{AU} &= 1.495979\times10^{11}\ \text{m} \end{aligned}$$

* For the current orbital polar angle θ_0 (and indeed more accurate values for solar system parameters) see the website of the Jet Propulsion Laboratory (JPL) `http://ssd.jpl.nasa.gov/`

†These planets rotate clockwise about their own internal polar axis. ("Retrograde"). All the other planets rotate anti-clockwise about their own internal axis. All the planets orbit the sun in an anticlockwise direction.

Assume all orbits are **ellipses** with the Sun at the (left) focus. Let this sun position be the origin of a Cartesian coordinate system, and assume the sun is stationary.

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta}$$

$$\varepsilon = \sqrt{1-\frac{b^2}{a^2}} \quad \therefore b = a\left(1-\varepsilon^2\right)$$

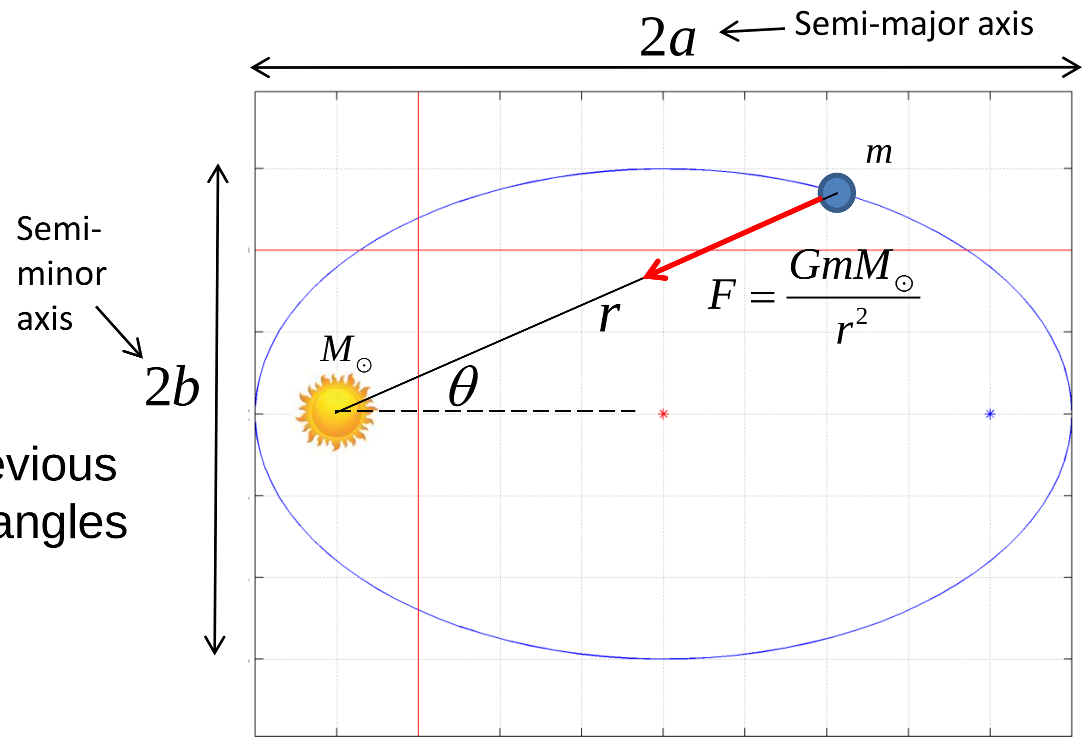
$$P^2 = \frac{4\pi^2}{G(m+M_\odot)}a^3$$

Plot the inner five planets on a separate scale to the outer planets

Ignore the inclination angle β (i.e. set it to be zero for now)

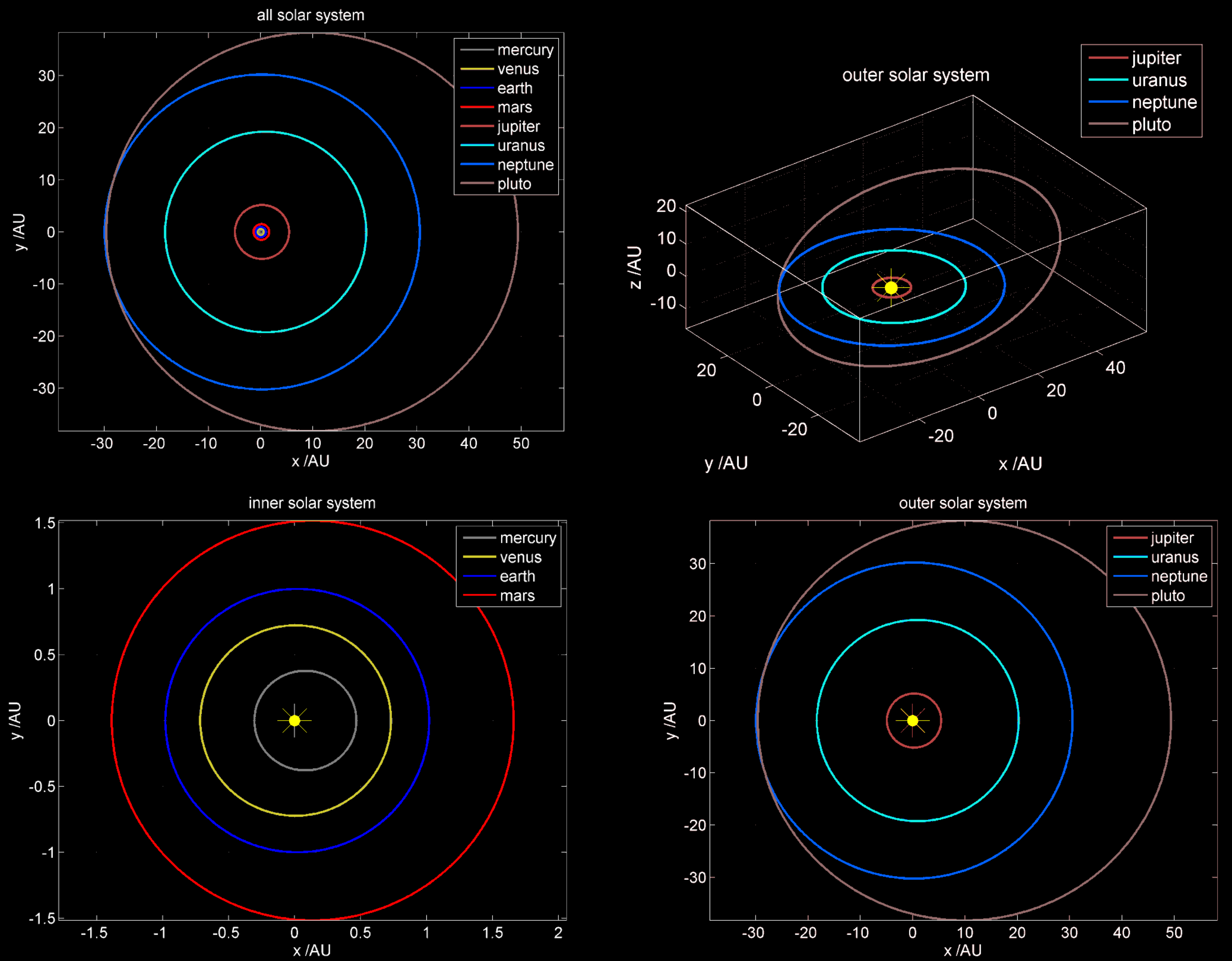
$$x = r\cos\theta, \quad y = r\sin\theta$$

$$\theta = 0....2\pi$$



Use the data in the table on the previous slide. Use a 1,000 linearly spaced angles θ for each orbit.

Use an axis scale of AU



all solar system
mercury
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x /AU
y /AU
outer solar system
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uranus
neptune
pluto
z /AU
y /AU
x /AU
inner solar system
mercury
venus
earth
mars
x /AU
y /AU
outer solar system
jupiter
uranus
neptune
pluto
x /AU
y /AU

Create a 2D animation of the solar system orbits

Use an axis scale of AU

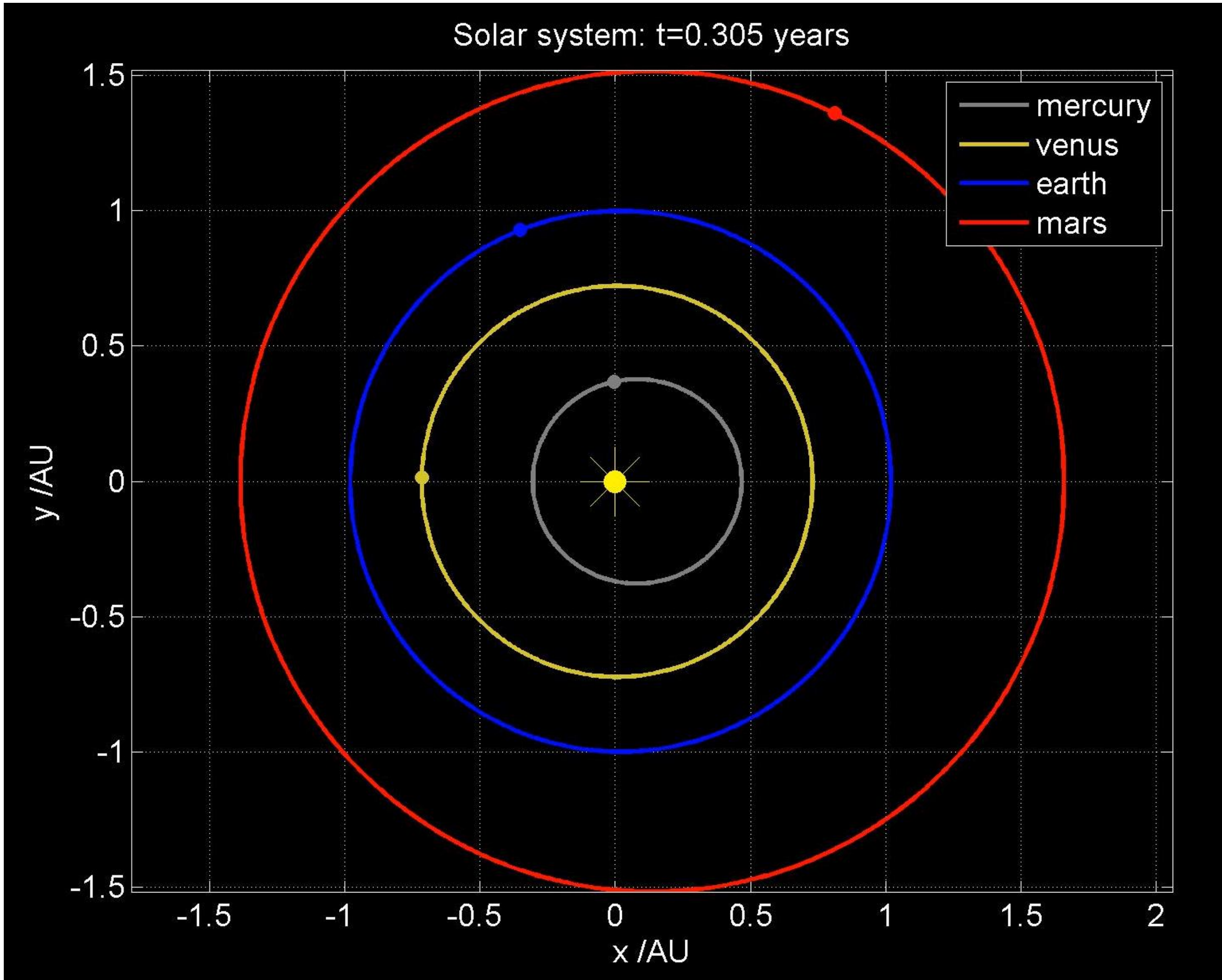
Plot the inner five planets on a separate scale to the outer planets

For the *inner* planets, set a frame rate such that one orbit of the Earth takes a second i.e. **one year is one second.** For the *outer* planets, **set the orbit of *Jupiter* to take one second**.

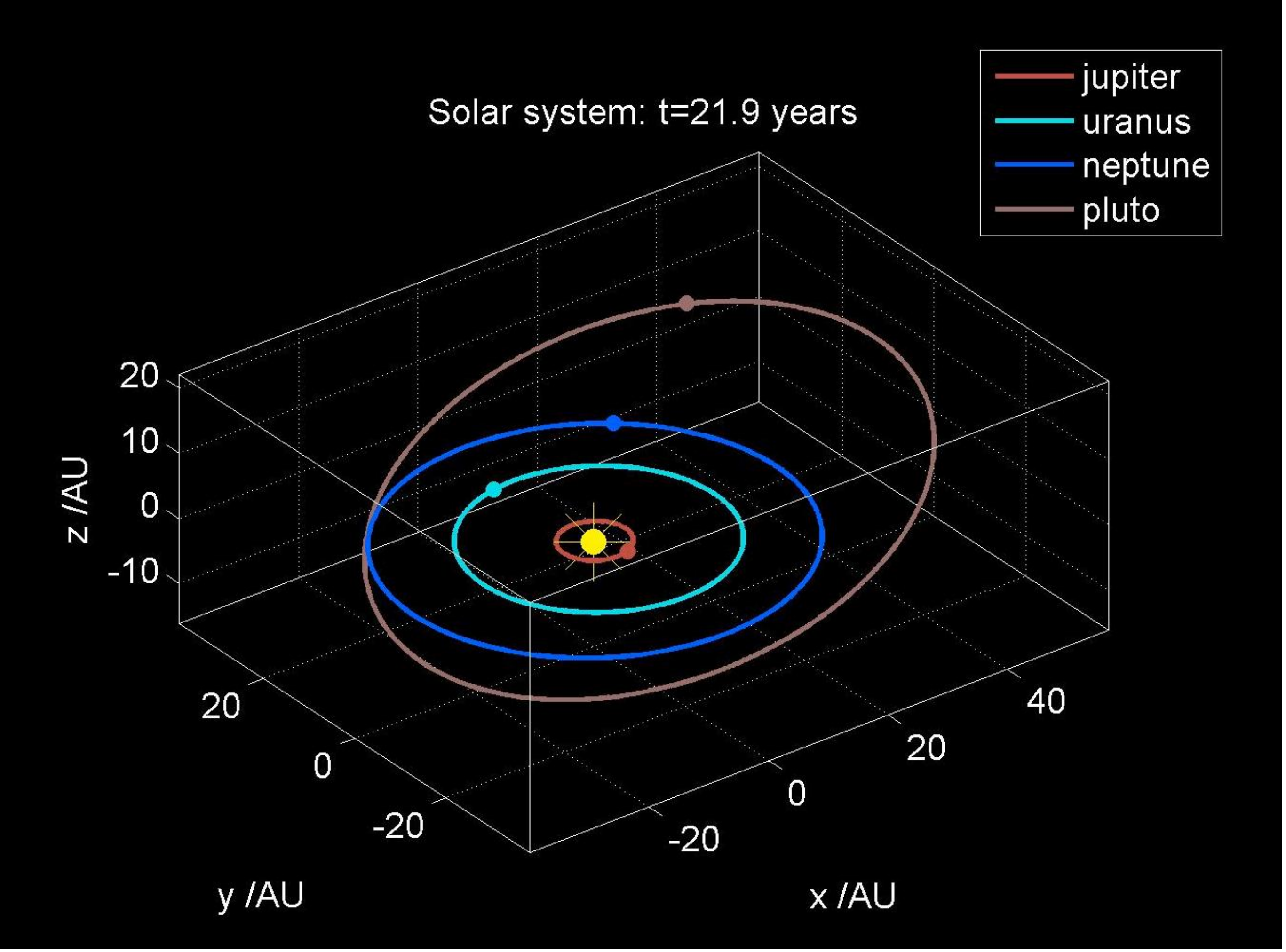
$$x = r\cos\theta, \;\; y = r\sin\theta$$

$$\theta = \frac{2\pi t}{P}$$

Run the simulation till the outermost planet completes at least one orbit.



[YouTube example video]()



Create a 3D animation of the solar system orbits

YouTube example video

Use the elliptical inclination angle β. Most orbits won't change much, but Pluto is the exception! The coordinate change is:

$$x' = x\cos\beta \qquad z' = x\sin\beta \qquad y' = y$$

β is the orbital inclination /degrees. In all cases the semi-major axis pointing direction is

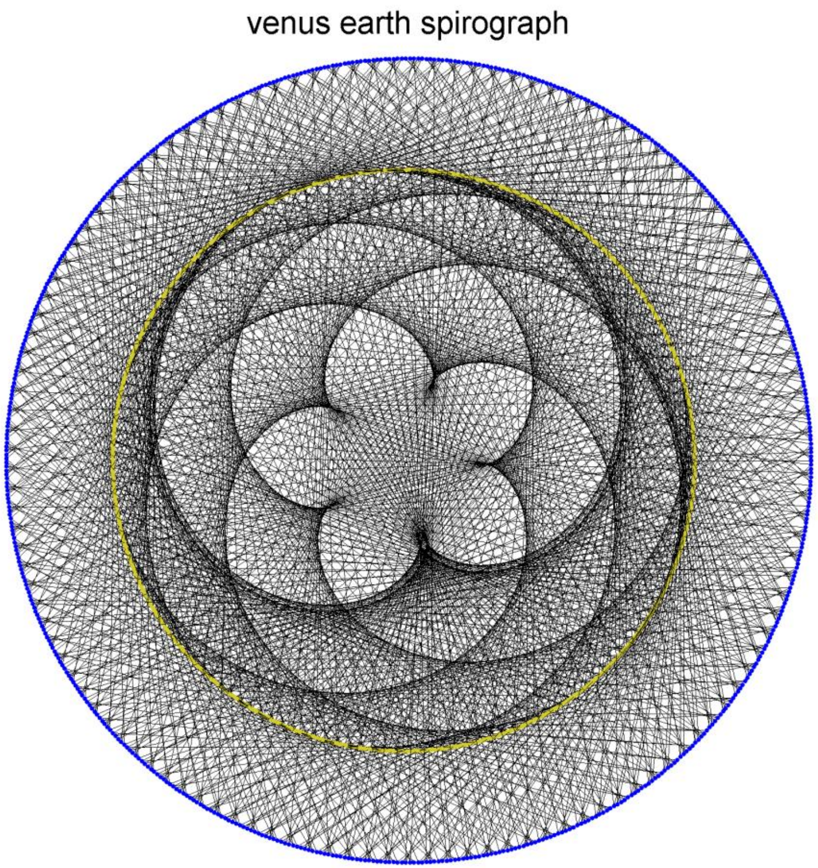
$$\mathbf{d} = d_x\hat{\mathbf{x}} + d_y\hat{\mathbf{y}} + d_{\mathbf{y}}\hat{\mathbf{z}} = \cos\beta\hat{\mathbf{x}} + \sin\beta\hat{\mathbf{z}}$$

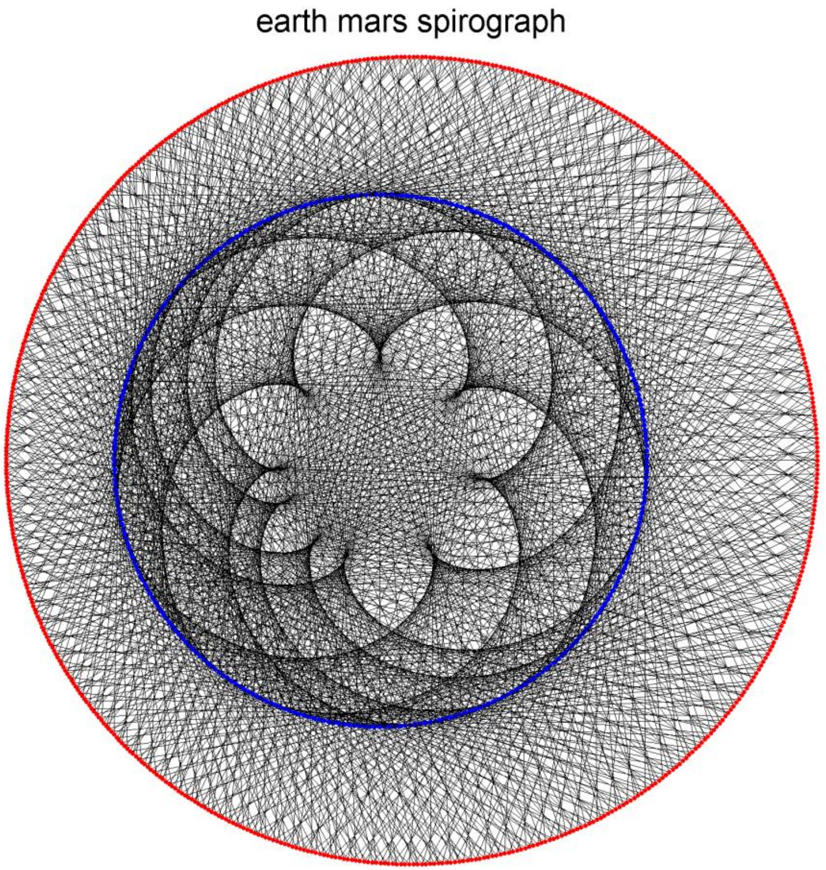
Solar system spirograph!

inspired by: https://engaging-data.com/planetary-spirograph

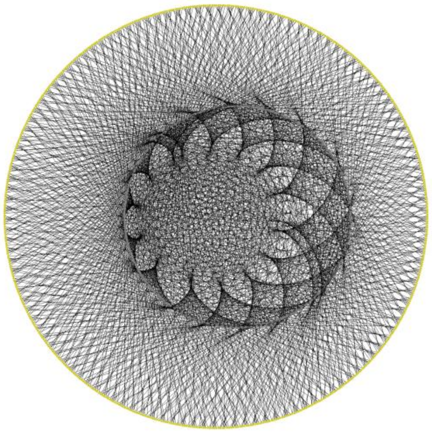
Choose a pair of planets and determine their orbits vs time. At time intervals of Δt, draw a line between the planets and plot this line. Keep going for N orbits of the outermost planet.

N = **10 ,** Δt **=** N x **maximum orbital period /1234**, might be sensible parameters.

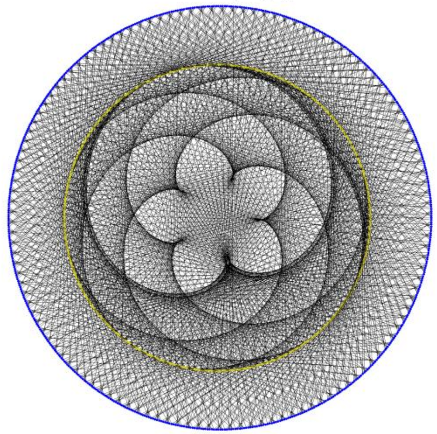




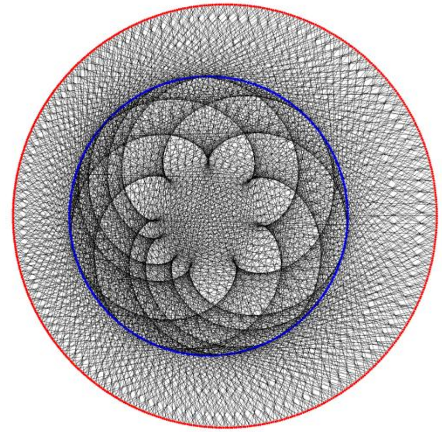
mercury venus spirograph



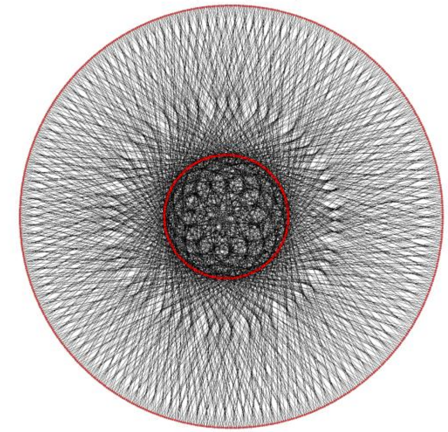
venus earth spirograph



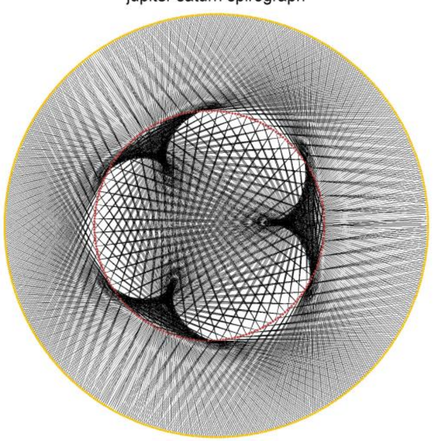
earth mars spirograph



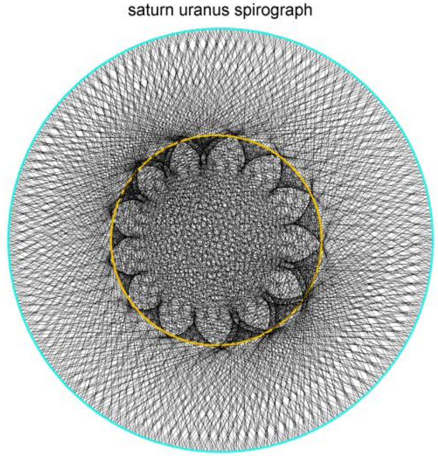
mars jupiter spirograph



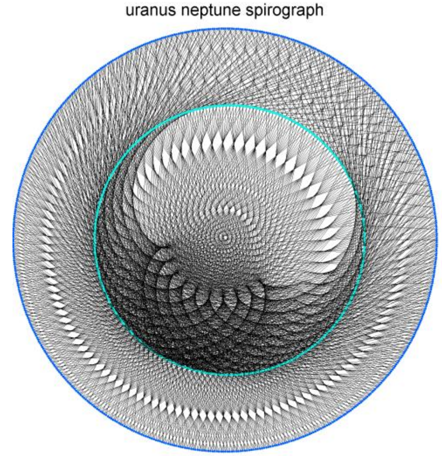
jupiter saturn spirograph



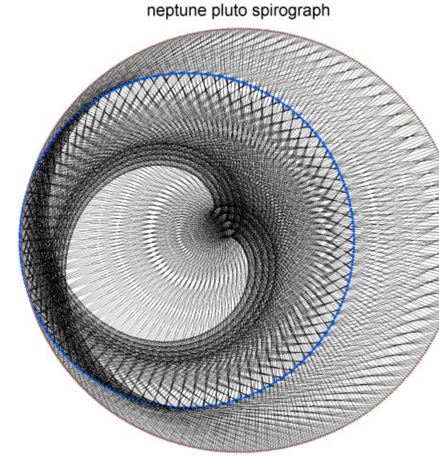
saturn uranus spirograph



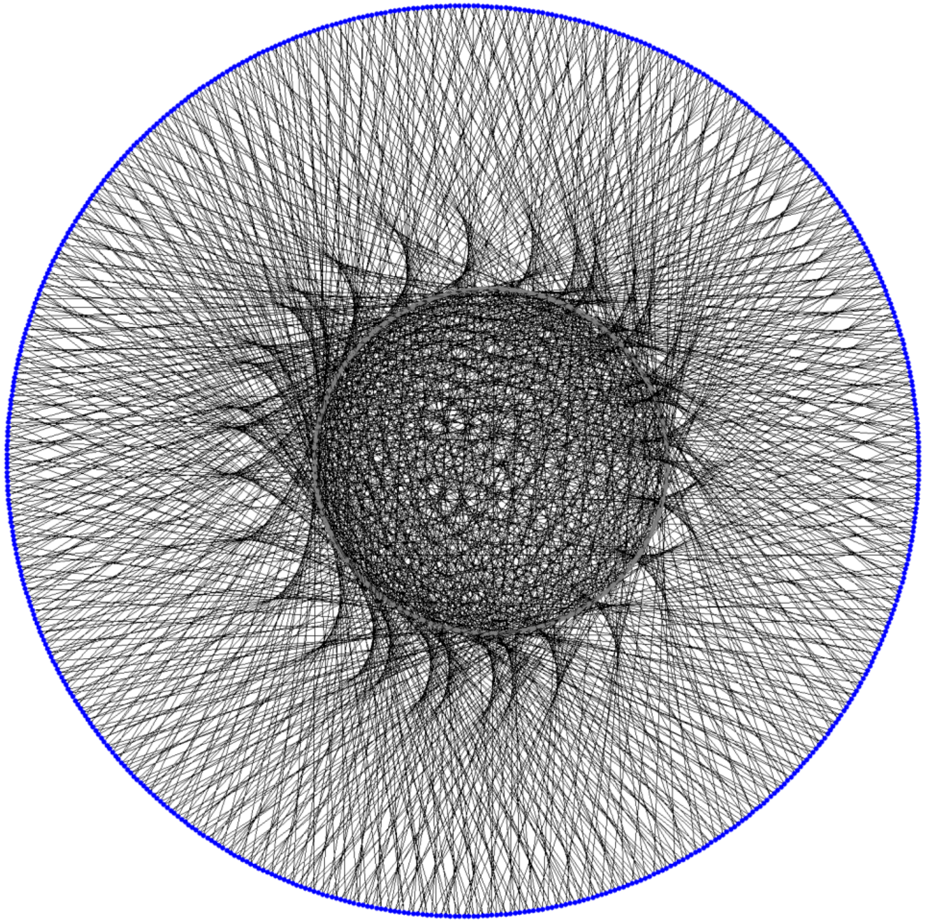
uranus neptune spirograph



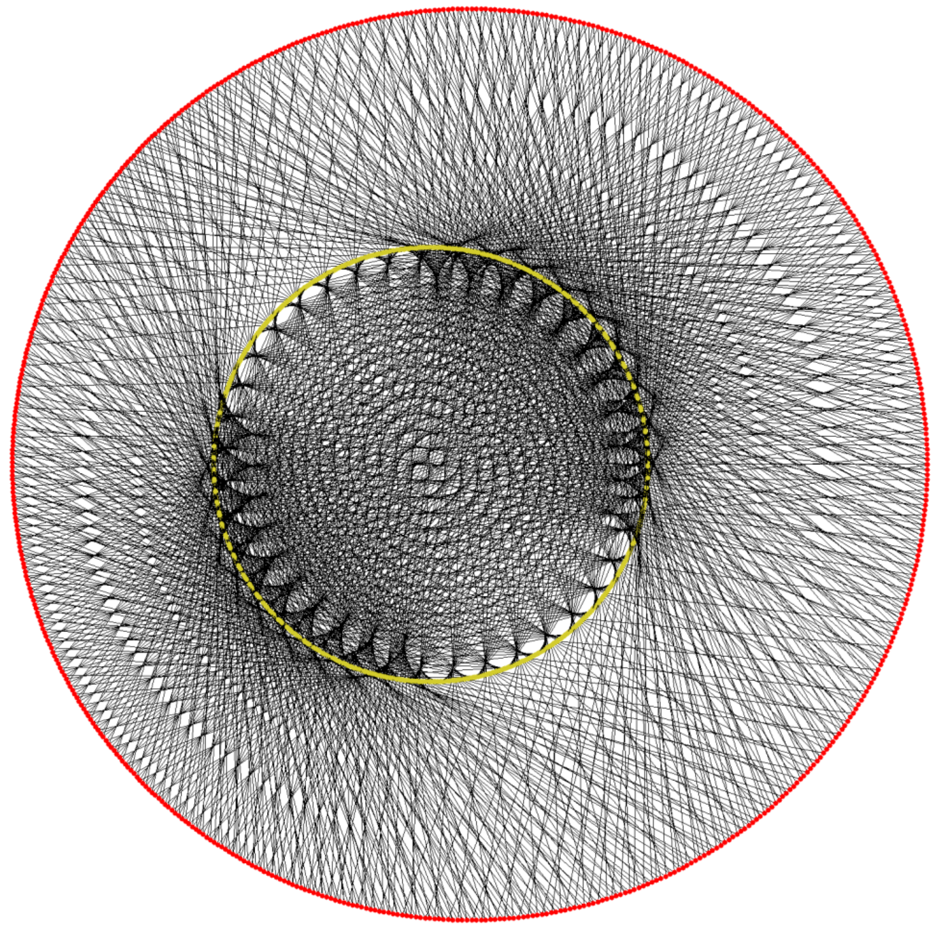
neptune pluto spirograph



mercury earth spirograph



venus mars spirograph



What about systems of *more than* two stars or planets?
We need a numeric method!

The **Verlet Method** implies *constant acceleration motion* between fixed timesteps.

$$\mathbf{a}_n = f(t_n, \mathbf{r}_n, \mathbf{v}_n)$$

Assume we can always calculate acceleration

$$t_{n+1} = t_n + \Delta t$$

Fixed timestep

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{v}_n \Delta t + \tfrac{1}{2}\mathbf{a}_n \Delta t^2$$

Update position based upon constant acceleration motion between timesteps

$$\mathbf{V} = \mathbf{v}_n + \mathbf{a}_n \Delta t$$

$$\mathbf{A} = f(t_{n+1}, \mathbf{r}_{n+1}, \mathbf{V})$$

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \tfrac{1}{2}(\mathbf{a}_n + \mathbf{A})\Delta t$$

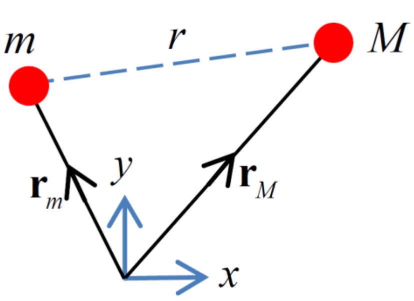
Acceleration may depend upon velocity, so for greater precision we work out an intermediate velocity $\mathbf{V}$, update acceleration ($\mathbf{A}$) and perform an *average* to calculate the velocity update.

Verlet method

$$\mathbf{a}_n = f(t_n, \mathbf{r}_n, \mathbf{v}_n)$$
$$t_{n+1} = t_n + \Delta t$$
$$\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{v}_n \Delta t + \tfrac{1}{2}\mathbf{a}_n \Delta t^2$$
$$\mathbf{V} = \mathbf{v}_n + \mathbf{a}_n \Delta t$$
$$\mathbf{A} = f(t_{n+1}, \mathbf{r}_{n+1}, \mathbf{V})$$
$$\mathbf{v}_{n+1} = \mathbf{v}_n + \tfrac{1}{2}(\mathbf{a}_n + \mathbf{A})\Delta t$$

Newton's Law of Gravitation

$$\mathbf{a}_{n,i} = -G\sum_{j\neq i}^{N} M_j \frac{\mathbf{r}_i - \mathbf{r}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3}$$

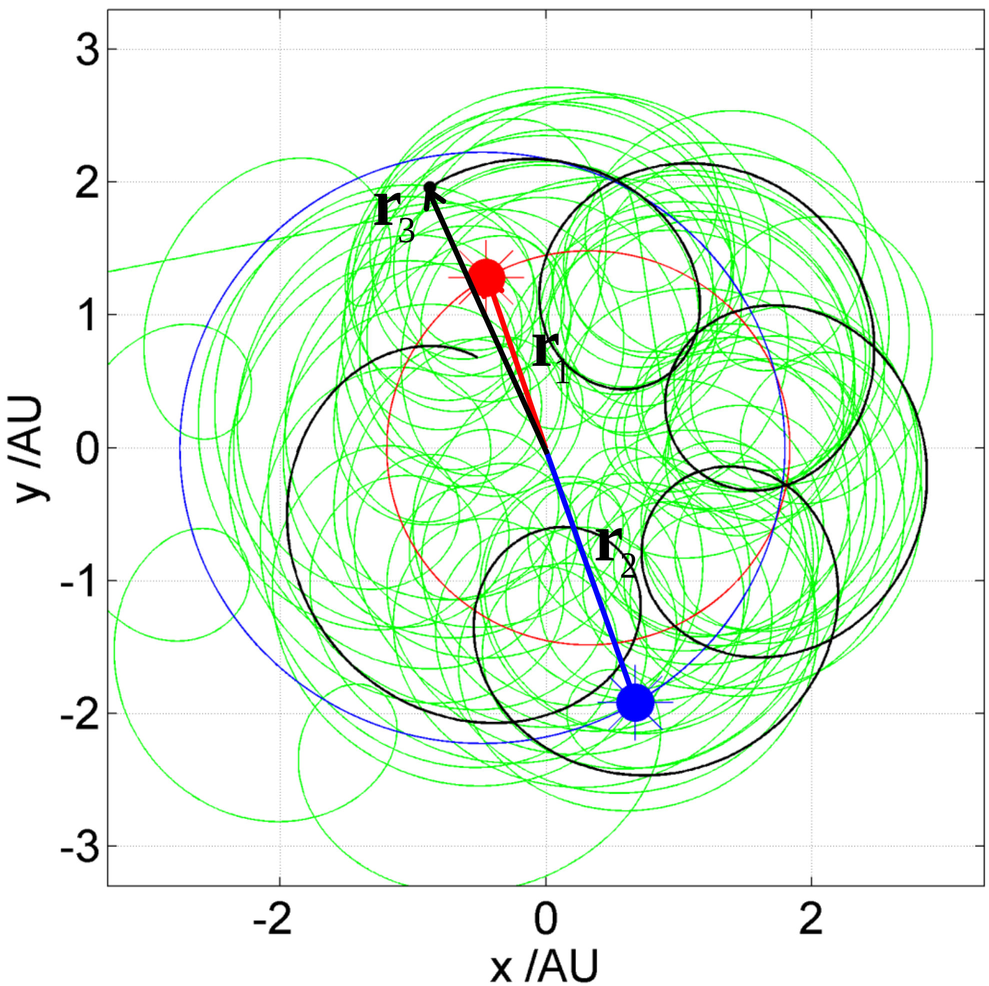


$$\mathbf{r} = \mathbf{r}_M - \mathbf{r}_m$$
$$r = |\mathbf{r}|$$

In this simulation:

$$M_1 = 3M_\odot$$
$$M_2 = 2M_\odot$$
$$M_3 \ll M_\odot$$

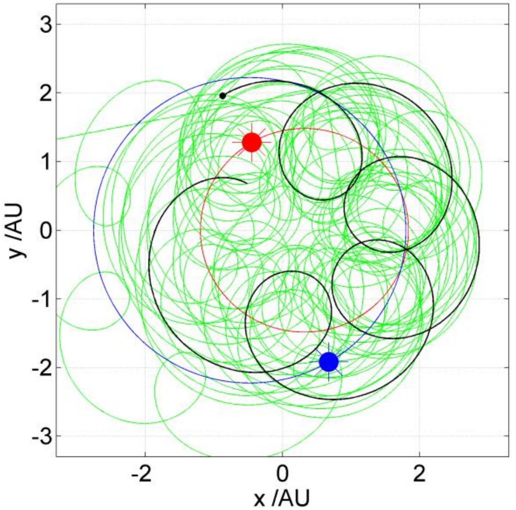
M1=3, M2=2 T=2.32 years, a=3AU, k=1.1, a_p=0.965AU.



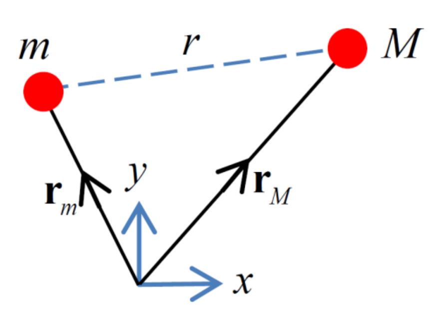
$$\mathbf{a}_n = f(t_n, \mathbf{r}_n, \mathbf{v}_n)$$
$$t_{n+1} = t_n + \Delta t$$
$$\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{v}_n \Delta t + \tfrac{1}{2}\mathbf{a}_n \Delta t^2$$
$$\mathbf{V} = \mathbf{v}_n + \mathbf{a}_n \Delta t$$
$$\mathbf{A} = f(t_{n+1}, \mathbf{r}_{n+1}, \mathbf{V})$$
$$\mathbf{v}_{n+1} = \mathbf{v}_n + \tfrac{1}{2}(\mathbf{a}_n + \mathbf{A})\Delta t$$

$$\mathbf{a}_{n,i} = -G\sum_{j \neq i}^{N} M_j \frac{\mathbf{r}_i - \mathbf{r}_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^3}$$

M1=3, M2=2 T=2.32 years, a=3AU, k=1.1, a_p=0.965AU.

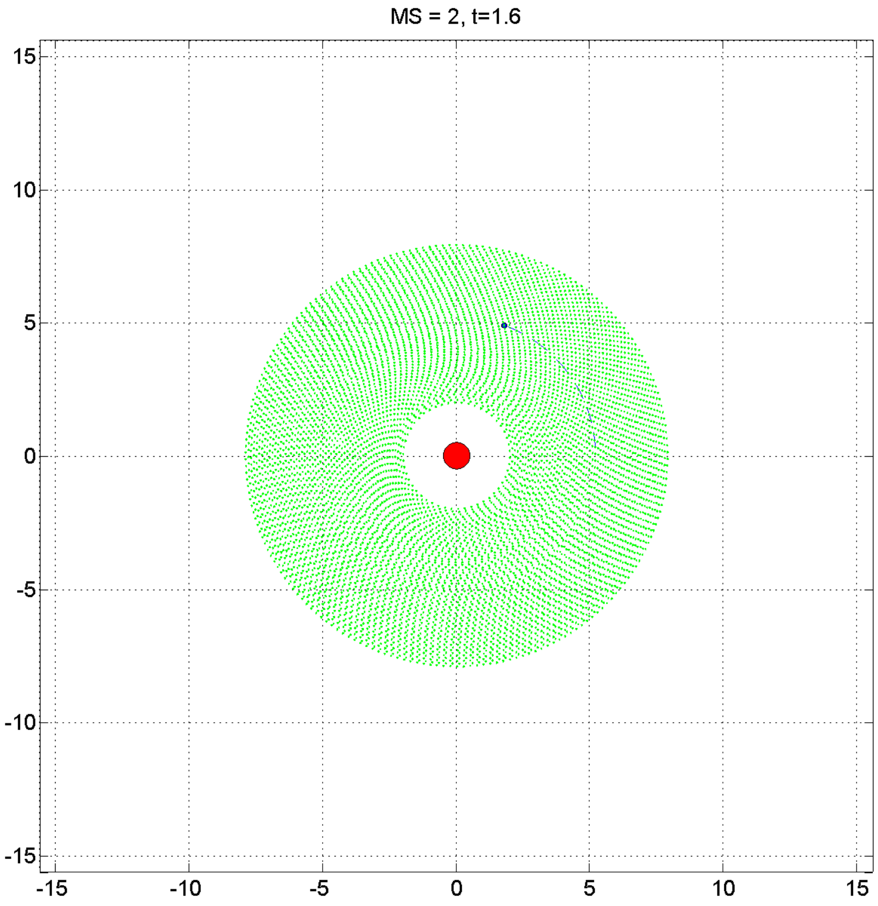


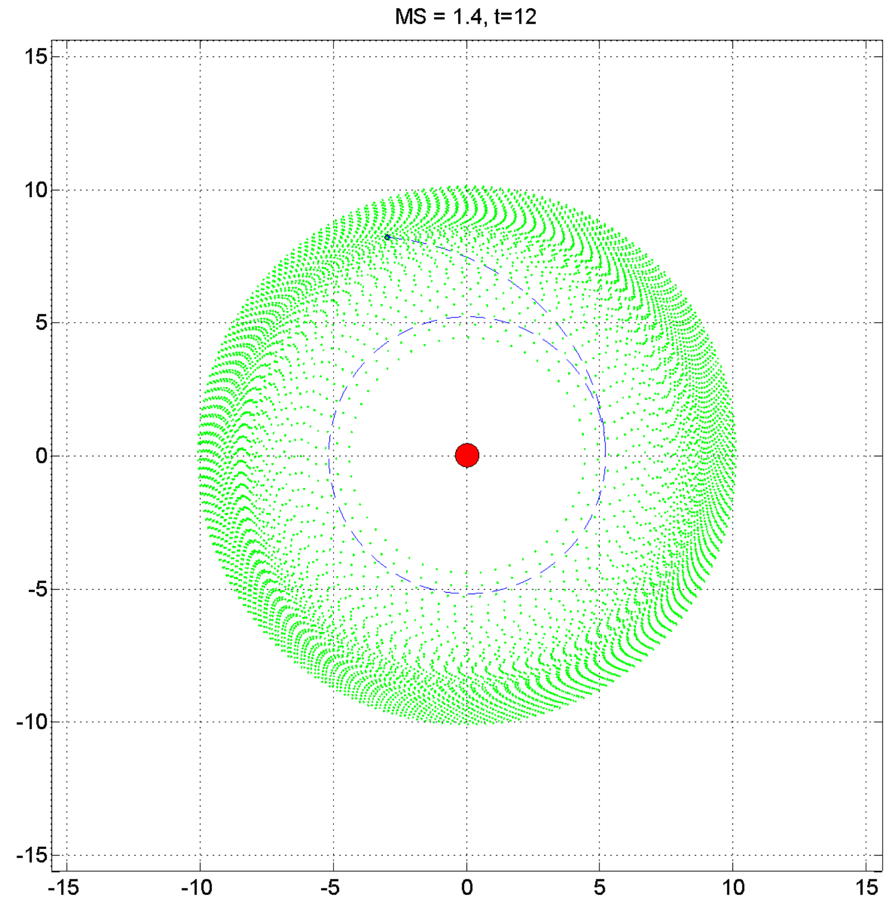

```
function gravity_sim_2_binary_stars_and_planet

%% INPUTS %%

%Semi-major axis of mutual star orbit in AU
a = 3;

%Planet (initial) circular orbit radius about star 1
ap = a/3.11;

%Initial angle from x axis (anticlockwise) of planet /radians
theta0 = pi/4;

%Masses of stars in solar masses
M1 = 3; M2 = 2;

%Initial vy velocity multiplier from mutually circular of stars
k = 1.1;

%Number of orbital periods
num_periods = 50;

%Timestep in years
dt = 0.001;

%Fontsize
fsize = 18;

%Axes limits
limit = 1.1*a;

%Starting period for plot
Pstart = 1.23;
```

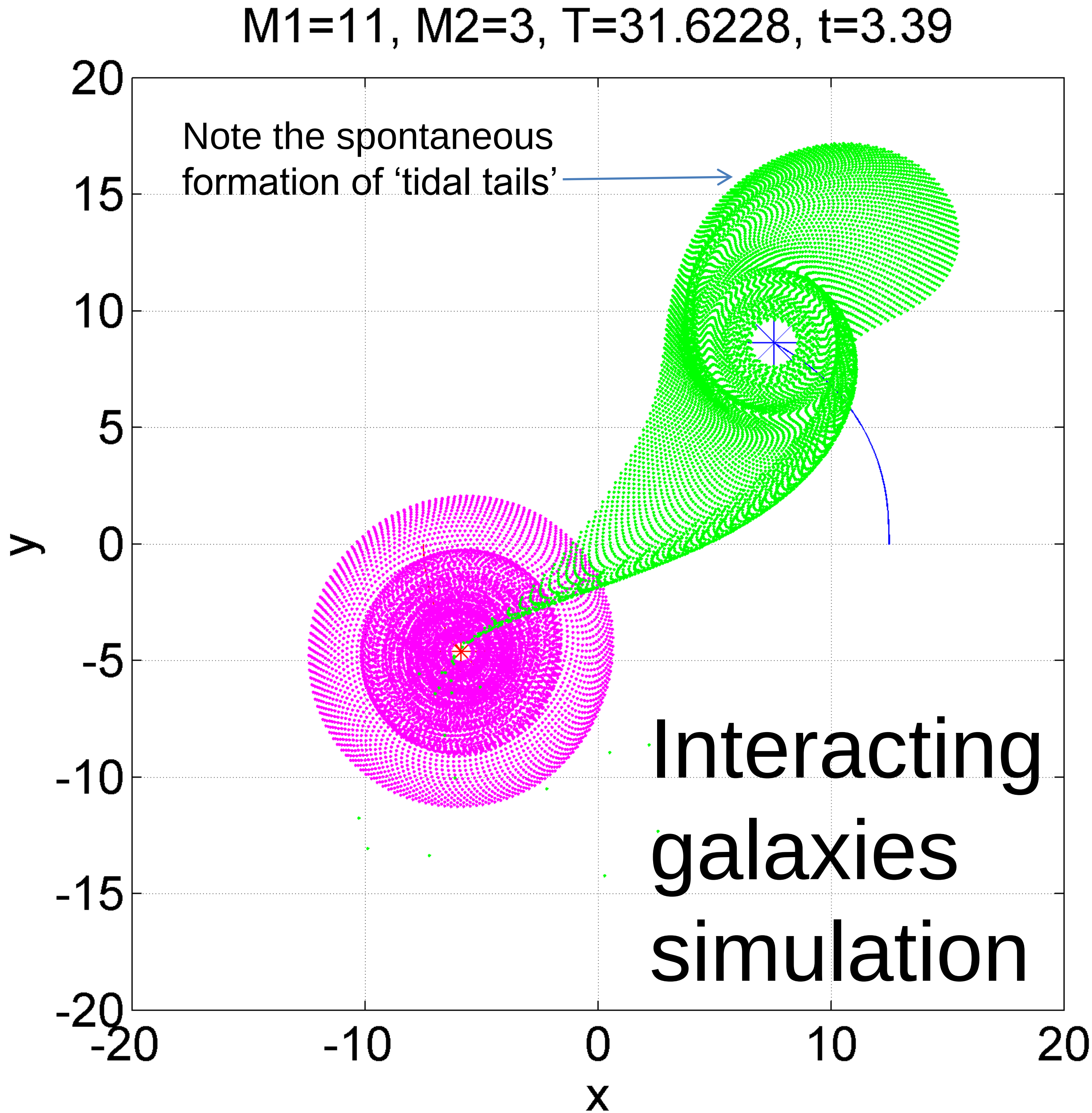


$$\mathbf{r} = \mathbf{r}_M - \mathbf{r}_m$$
$$r = |\mathbf{r}|$$

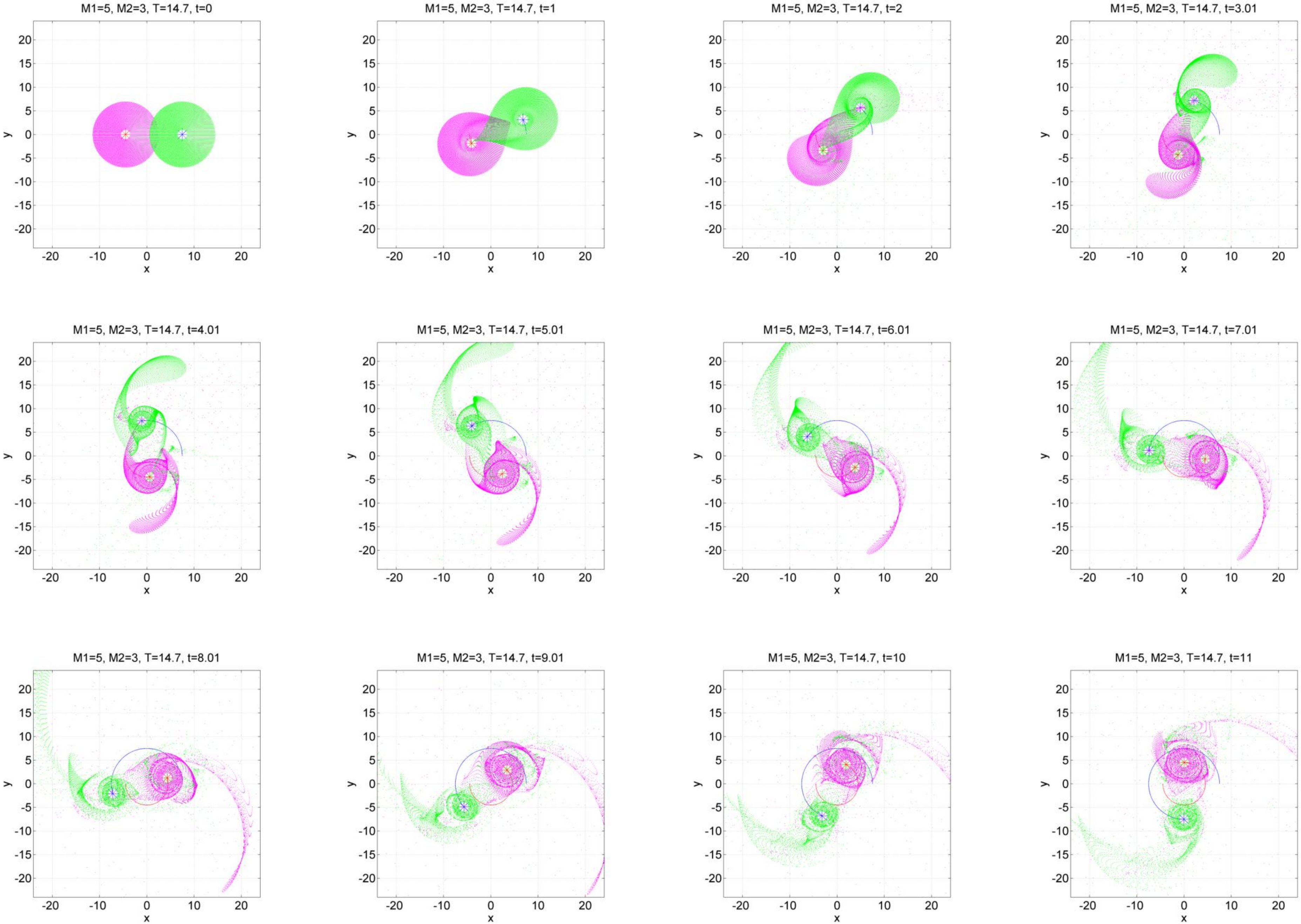
```
% Gravity simulation which begins with a single Jupiter-like planet
% orbiting a sun-like star, plus concentric circles of 'masslets' that act
% like an accretion disc or dust cloud around the star. The planet and
% masslets don't interact, and the star mass is assumed to be much larger
% than then mass of the planet, even after it has shed mass.
%
% After N planet rotations, the star loses fraction f of it's mass. The simulation
% uses Verlet integration to determine the subsequent dynamics for another
% M planet periods, before resetting.
```





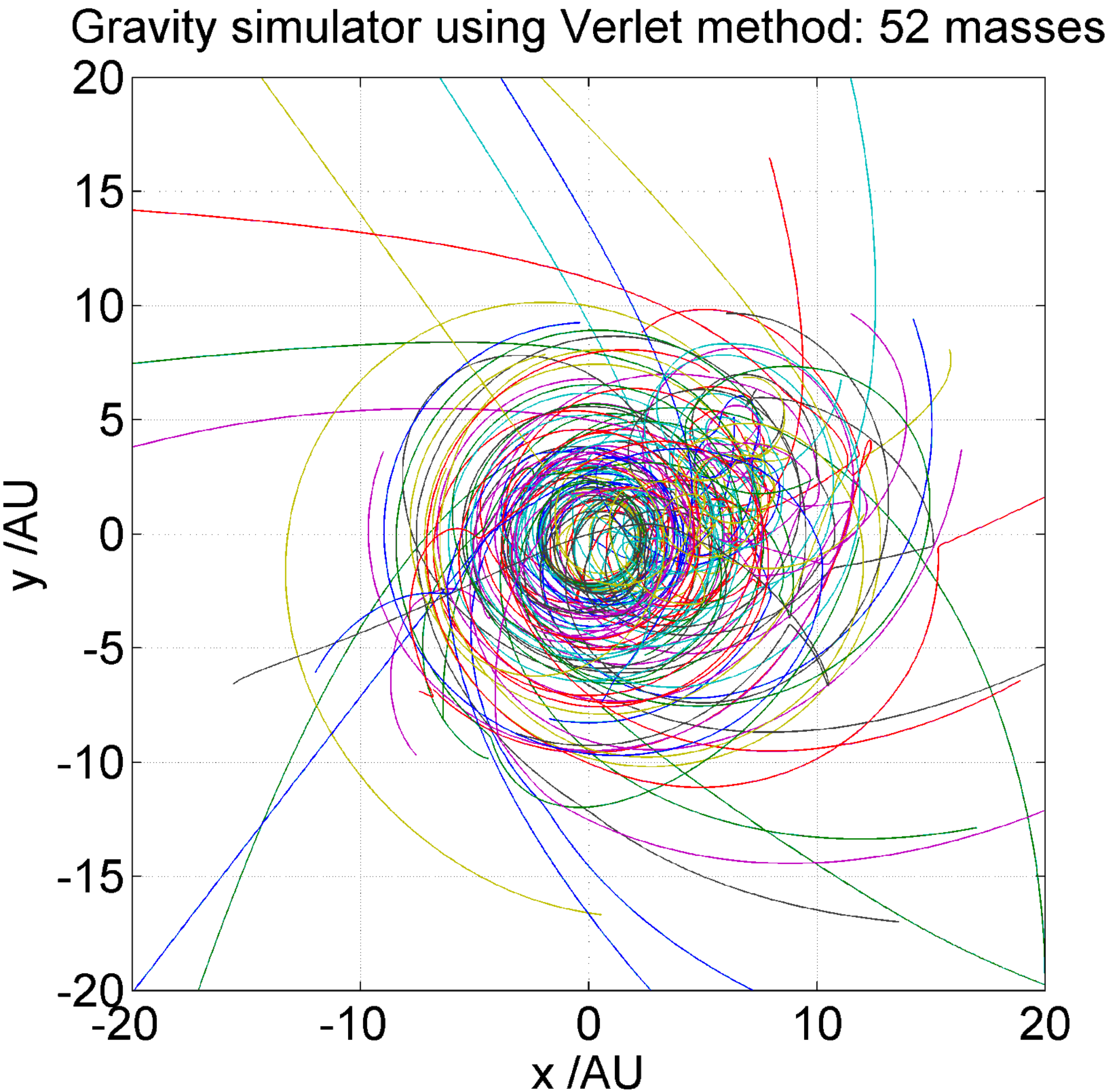


M1=11, M2=3, T=31.6228, t=3.39
Note the spontaneous
formation of 'tidal tails'
Interacting
galaxies
simulation
20
15
10
5
0
-5
-10
-15
-20
-20
-10
0
10
20
y
x



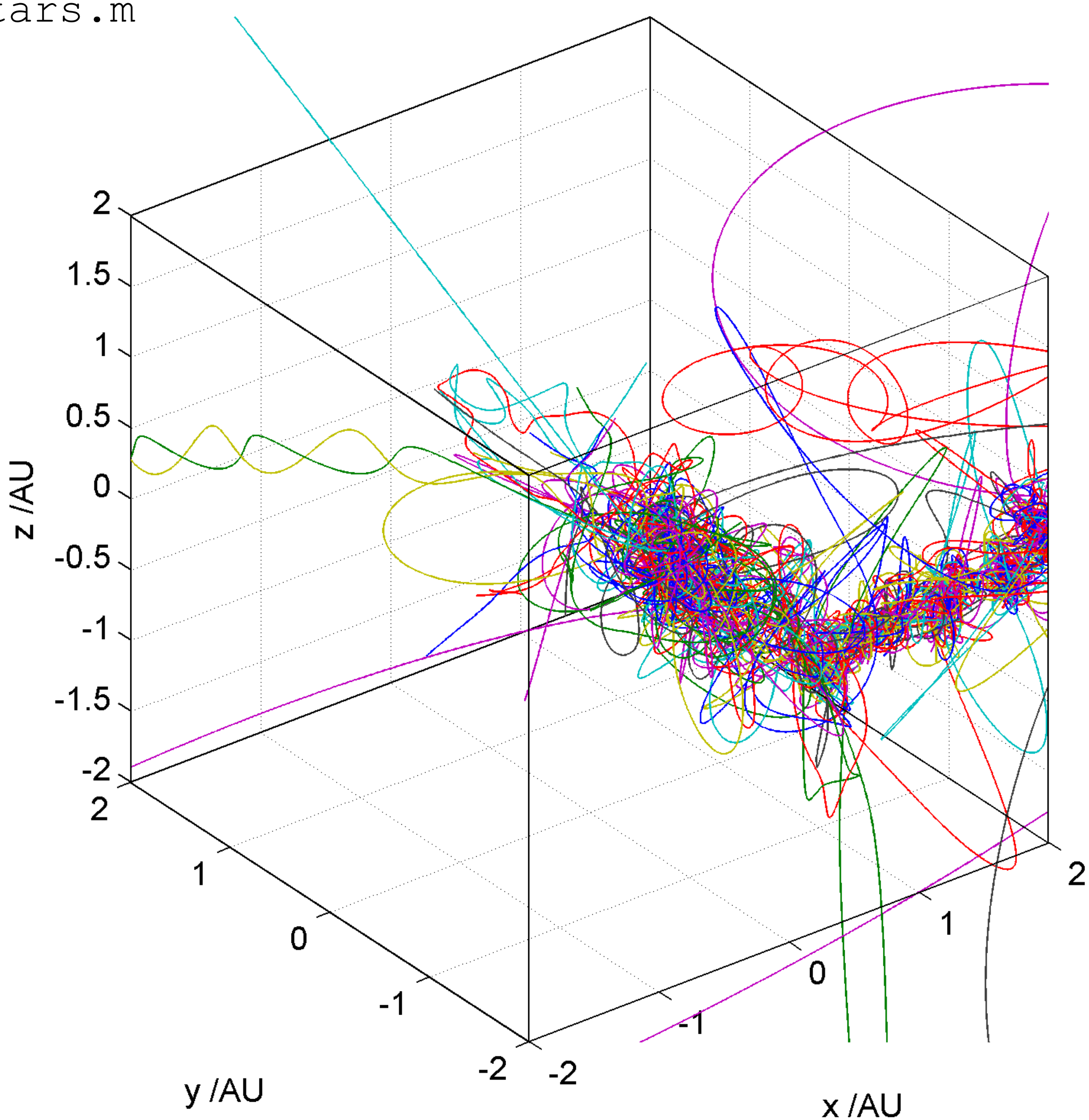
A possible explanation for common spiral galactic forms

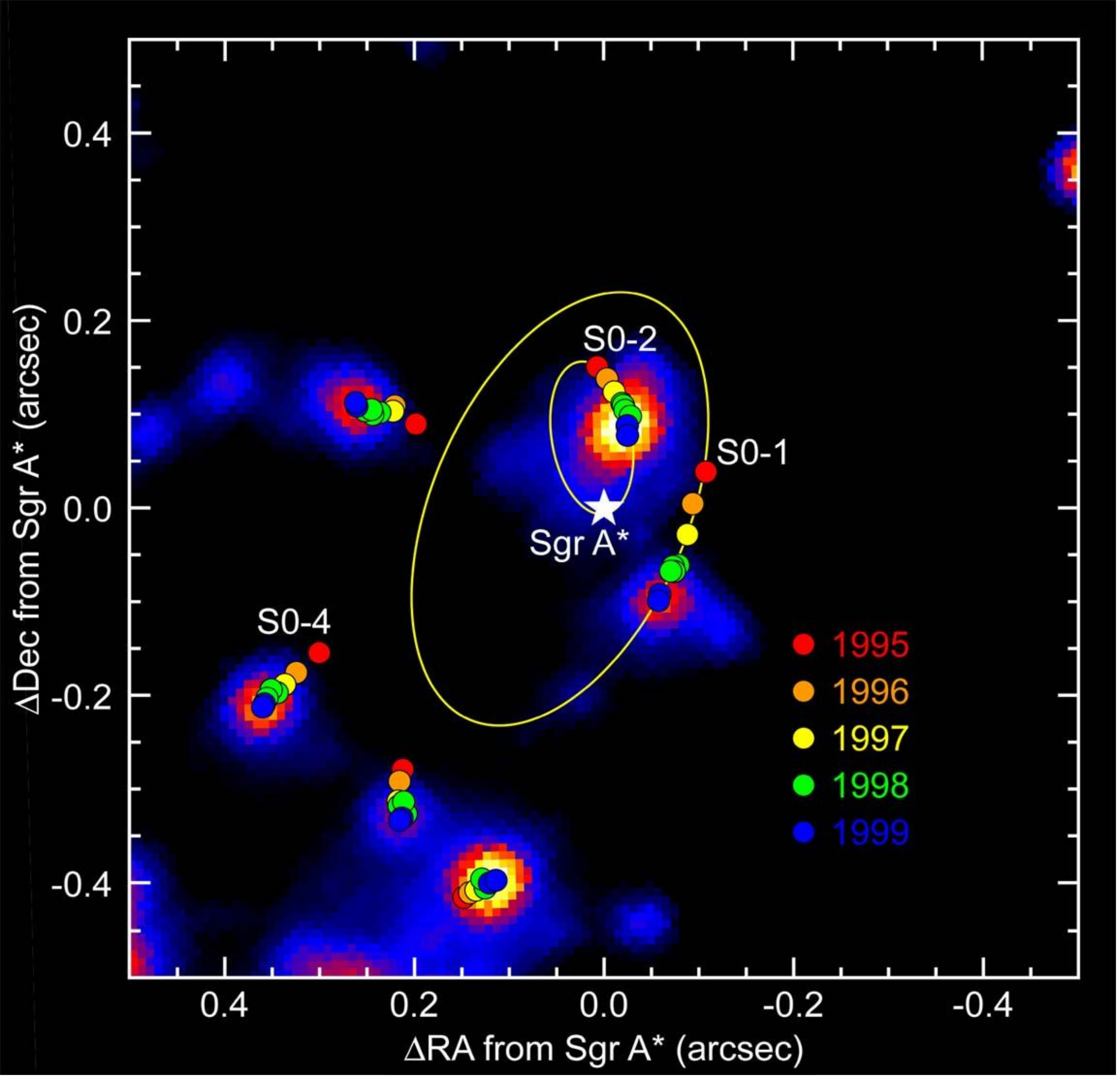

Messier 83 galaxy



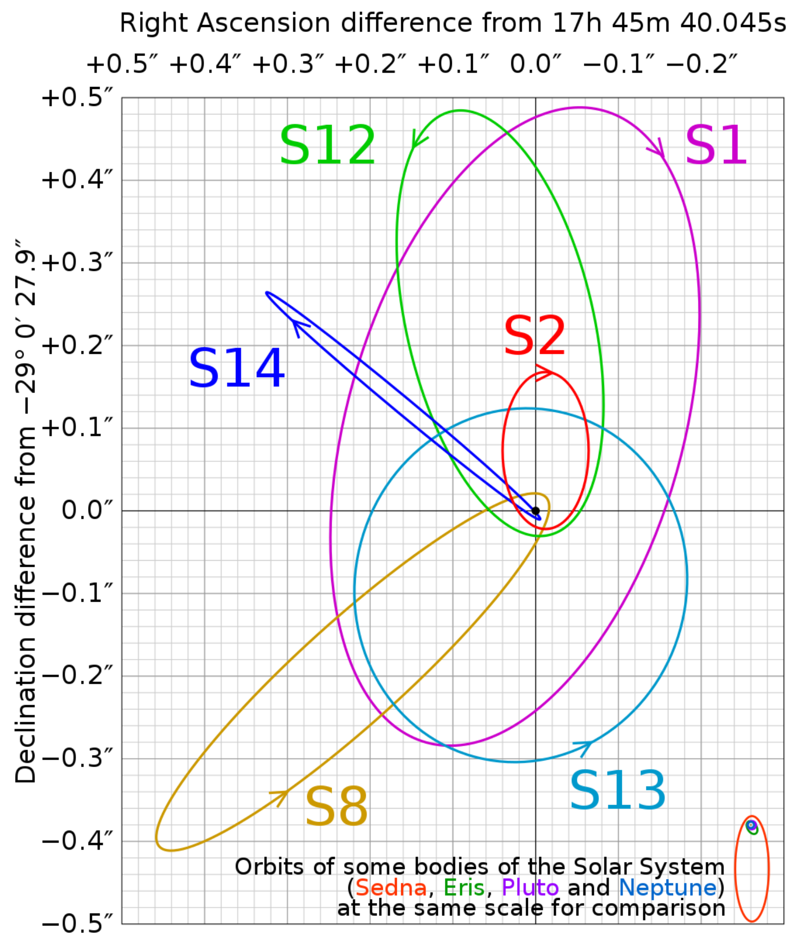
Gravity simulator using Verlet method: 52 masses
20
15
10
5
0
-5
-10
-15
-20
-20
-10
0
10
20
y /AU
x /AU

random_stars.m

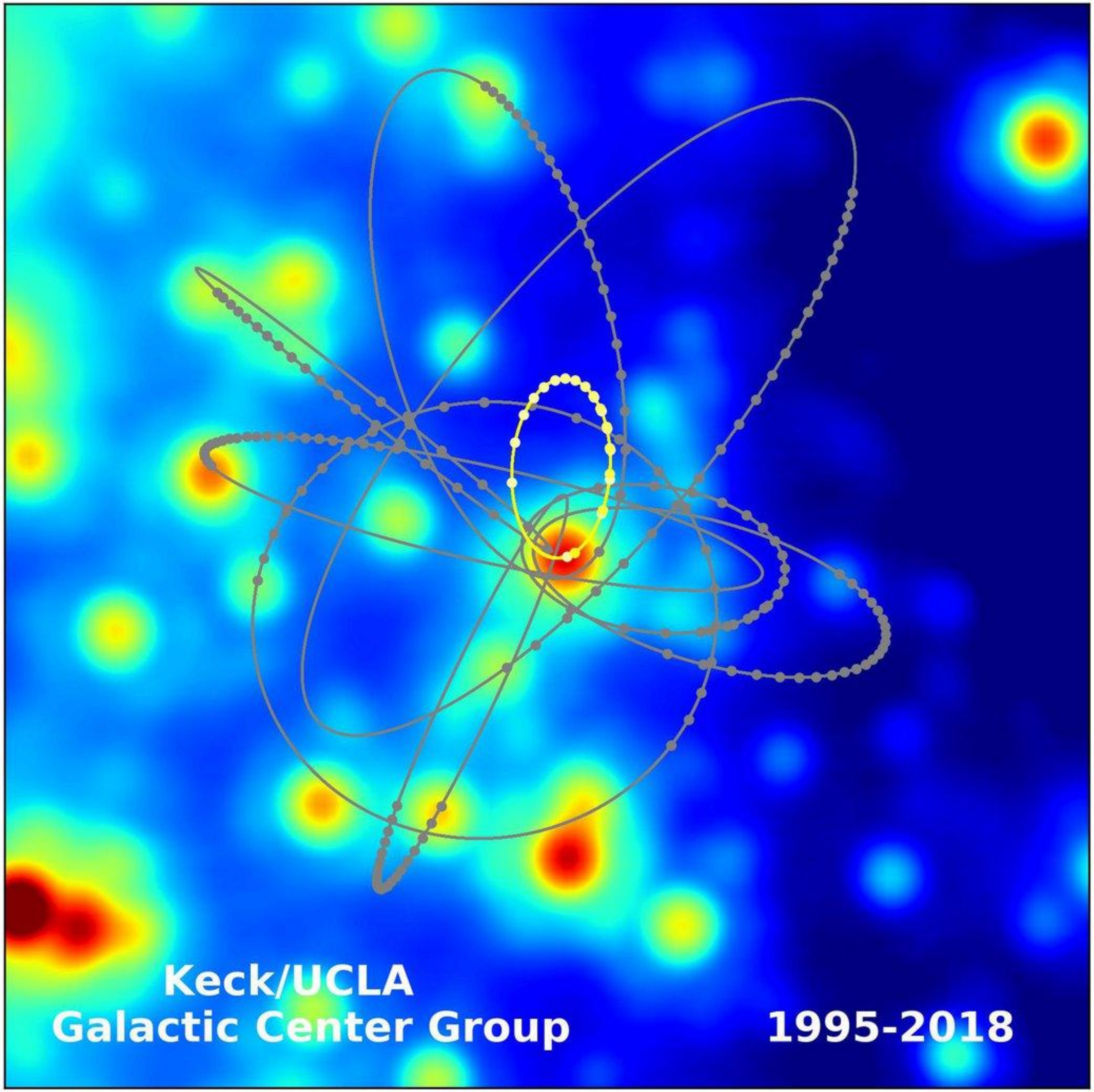




Although nearby star orbits look complex, the distances involved (and the relative mass of the black hole) mean you can model each as an elliptical orbit in a two-body system.



Sagittarius A* is a *supermassive black hole* in the centre of the Milky Way galaxy. It has a mass of about 4.2 **million** solar masses.


Keck/UCLA
Galactic Center Group
1995-2018

BPhO Computational Challenge

- Suggested homework
- Q&A

BPhO British Physics Olympiad