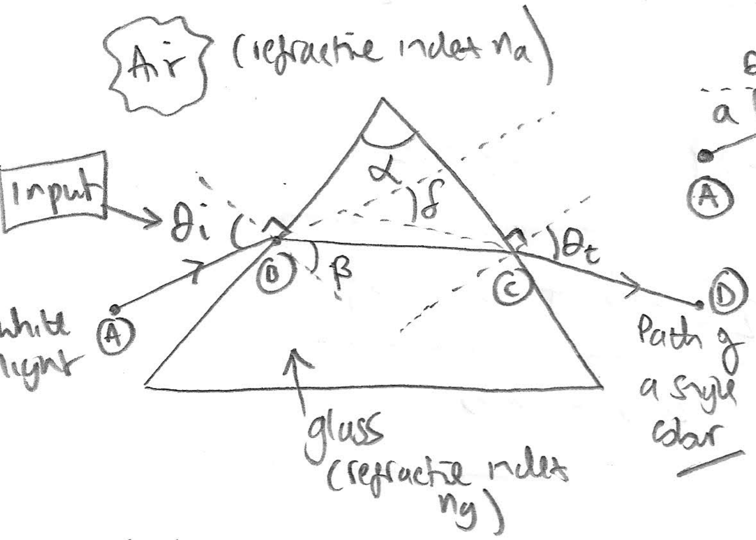
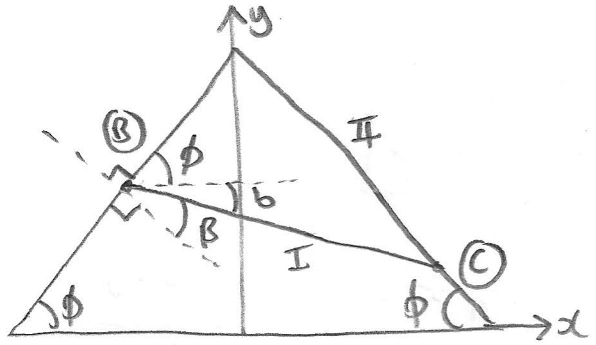
# Dispersion of light via a prism




Snell's law:

$$n_a \sin\theta_i = n_g \sin\beta$$

$$\therefore \boxed{\beta = \sin^{-1}\left(\frac{n_a \sin\theta_i}{n_g}\right)}$$




$$90^\circ = \phi + b + \beta$$

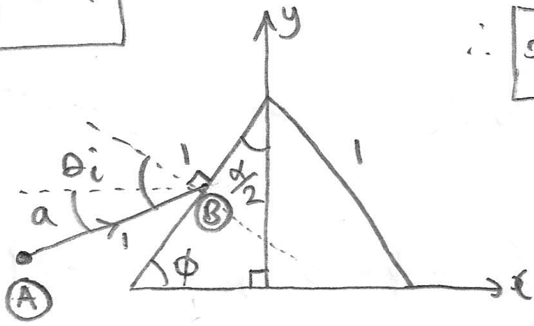
$$\therefore b = 90^\circ - \phi - \beta$$

$$b = 90^\circ - 90^\circ + \alpha/2 - \beta$$

$$\boxed{b = \alpha/2 - \beta}$$

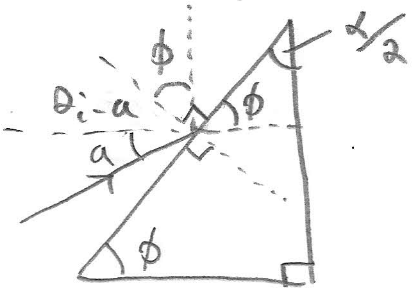
$$\phi + \frac{\alpha}{2} + 90^\circ = 180^\circ$$

$$\therefore \boxed{\phi = 90^\circ - \frac{\alpha}{2}}$$




A : $\left(-\cos a - \frac{1}{2}\cos\phi,\ \frac{1}{2}\sin\phi - \sin a\right)$

B : $\left(-\frac{1}{2}\cos\phi,\ \frac{1}{2}\sin\phi\right)$




$$90^\circ = \theta_i - a + \phi$$

$$\therefore a = \theta_i + \phi - 90^\circ$$

$$a = \theta_i + 90^\circ - \frac{\alpha}{2} - 90^\circ$$

$$\boxed{a = \theta_i - \frac{\alpha}{2}}$$

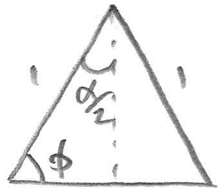
Cartesian eq. of line I is

$$y_I = -\tan b\, x + c$$

using B coordinates: $\frac{1}{2}\sin\phi = -\tan b\left(-\frac{1}{2}\cos\phi\right) + c$

$$\therefore c = \frac{1}{2}\sin\phi - \frac{1}{2}\cos\phi \tan b$$

line II is : $y_{II} = -\tan\phi\, x + \sin\phi$



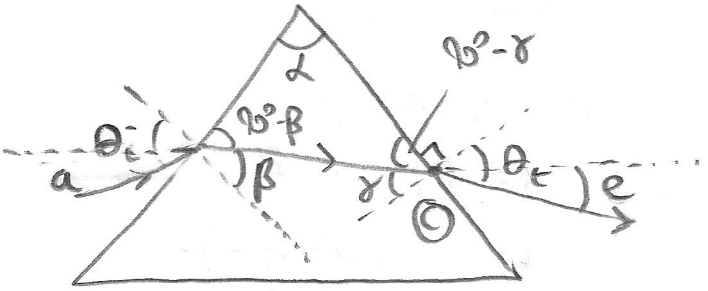
So to find Ⓒ: $y_I = y_{II}$ Ⓒ is $(x_c, y_c)$

$$\therefore -\tan b\, x_c + \tfrac{1}{2}\sin\phi - \tfrac{1}{2}\cos\phi \tan b = -\tan\phi\, x_c + \sin\phi$$

$$x_c(\tan\phi - \tan b) = \tfrac{1}{2}\sin\phi + \tfrac{1}{2}\cos\phi \tan b$$

$$\therefore \boxed{x_c = \frac{\tfrac{1}{2}(\sin\phi + \cos\phi\tan b)}{\tan\phi - \tan b}}$$

$$\therefore \boxed{y_c = -\tan\phi\, x_c + \sin\phi}$$ (using $y_{II}$)



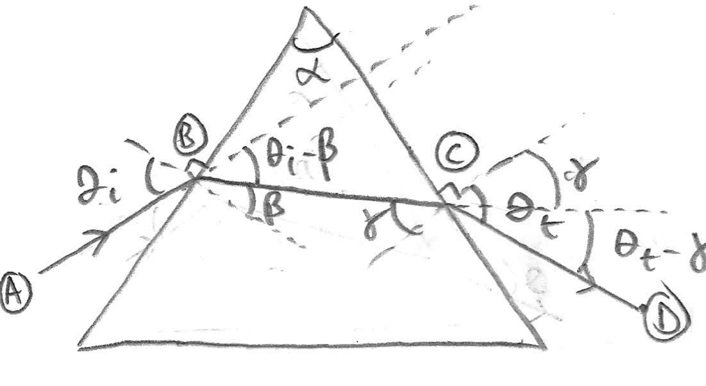

Now $\alpha + 90° - \beta + 90° - \gamma = 180°$

$\therefore \alpha = \beta + \gamma$

$\therefore \boxed{\gamma = \alpha - \beta}$

Snell: $n_a \sin\theta_t = n_g \sin\gamma$

$$\therefore \boxed{\theta_t = \sin^{-1}\left(\frac{n_g \sin\gamma}{n_a}\right)}$$



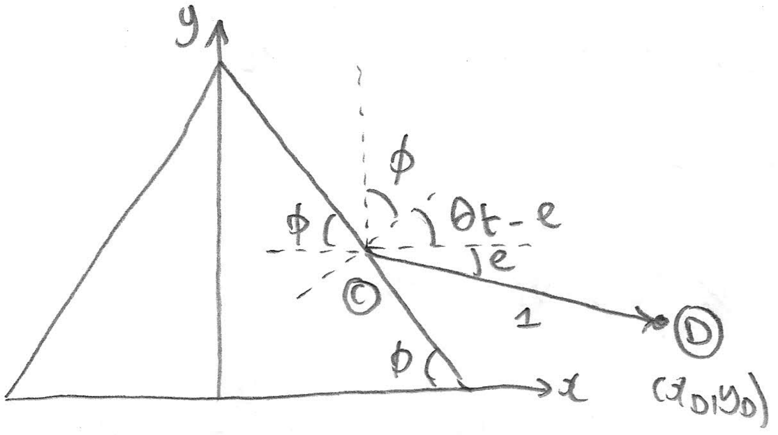

Total deflection is

$$\delta = \theta_i - \beta + \theta_t - \gamma$$

$$\delta = \theta_i + \theta_t - (\beta + \gamma)$$

Now $\gamma = \alpha - \beta$ so $\beta + \gamma = \alpha$

$$\therefore \boxed{\delta = \theta_i + \theta_t - \alpha}$$




D coordinates are:

$$\boxed{\begin{aligned} x_D &= x_C + \cos e \\ y_D &= y_C - \sin e \end{aligned}}$$

$$\phi + \theta_t - e = 90^\circ$$

$$\therefore \quad e = \phi + \theta_t - 90^\circ$$

$$e = 90^\circ - \alpha/2 + \theta_t - 90^\circ$$

$$\boxed{e = \theta_t - \alpha/2}$$

Now $\sin\theta_t = \dfrac{n_g \sin\gamma}{n_a}$

$\gamma = \alpha - \beta$ and $\sin\beta = \dfrac{n_a \sin\theta_i}{n_g}$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$\sin^2\beta + \cos^2\beta = 1 \quad \therefore \quad \cos\beta = \sqrt{1 - \sin^2\beta}$

( ok with just the root, since $\cos\beta > 0$, since $\beta > 0$ and $\beta < 90^\circ$).

$$\therefore \quad \sin\theta_t = \frac{n_g}{n_a}\left[\sin\alpha\sqrt{1 - \frac{n_a^2}{n_g^2}\sin^2\theta_i} - \cos\alpha \frac{n_a}{n_g}\sin\theta_i\right]$$

$$\Rightarrow \quad \boxed{\sin\theta_t = \sin\alpha\sqrt{\frac{n_g^2}{n_a^2} - \sin^2\theta_i} - \cos\alpha\sin\theta_i}$$

Now $\theta_t \leq 90°$, which places a constraint on $\theta_i$

when $\theta_t = 90°$, $\sin\theta_t = 1$

$$\Rightarrow \quad 1 + \cos\alpha \sin\theta_i = \sqrt{\left(n_g/n_a\right)^2 - \sin^2\theta_i}\ \sin\alpha$$

$$1 + 2\cos\alpha\sin\theta_i + \cos^2\alpha\sin^2\theta_i = \sin^2\alpha\left(\left(\frac{n_g}{n_a}\right)^2 - \sin^2\theta_i\right)$$

$$\therefore \quad \left(\cos^2\alpha + \sin^2\alpha\right)\sin^2\theta_i + \left(2\cos\alpha\right)\sin\theta_i + 1 - \left(\frac{n_g}{n_a}\right)^2\sin^2\alpha = 0$$

$$\sin^2\theta_i + 2\cos\alpha\sin\theta_i + 1 - \left(\frac{n_g}{n_a}\right)^2\sin^2\alpha = 0$$

$$\therefore \quad \left(\sin\theta_i + \cos\alpha\right)^2 - \cos^2\alpha + 1 - \left(\frac{n_g}{n_a}\right)^2\sin^2\alpha = 0$$

$$\therefore \quad \boxed{\sin\theta_i = -\cos\alpha \pm \sqrt{\cos^2\alpha + \left(\frac{n_g}{n_a}\right)^2\sin^2\alpha - 1}}$$

Since $\theta_i$ and $\sin\theta_i$ must be +ve

$\Rightarrow$ max $\theta_i$ is:

$$\boxed{\theta_{max} = \sin^{-1}\left(\sqrt{\cos^2\alpha + \left(\frac{n_g}{n_a}\right)^2\sin^2\alpha - 1} - \cos\alpha\right)}$$

So for $\theta > \theta_{max}$, total internal reflection occurs at (c).