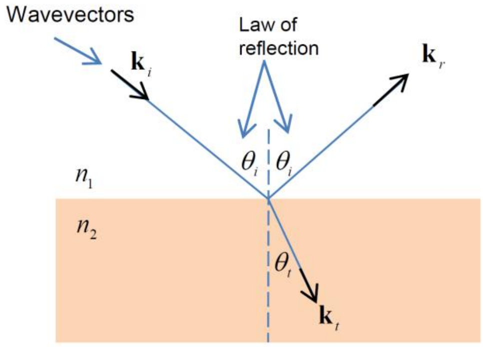
Understanding the nature of light has been a motivator for the development of much of Physics. Indeed, *visible* light (and its larger and smaller wavelength variants) is in most cases the *only* viable means of gaining information about the Cosmos beyond our closest planetary neighbours. Light is an ***Electromagnetic (EM) wave***, a disturbance in the electric and magnetic field that exists in space from, respectively, charges and moving charges. A *charge* in an electric field will feel a *force*. An additional force will be experienced if the charge is moving in a magnetic field. Unlike a sound wave or water wave, electromagnetic waves can travel through empty space. No 'medium' is required – it itself moves. In a vacuum, light travels at speed $c = 2.998 \times 10^8 \,\mathrm{ms}^{-1}$. *This is the speed limit for the conveyance of information in the Universe.* Like other forms of waves, **light will travel in straight lines unless the wave speed changes**. If light passes through water or glass, the presence of charges in the atomic structure of these substances will impede its progress, making the path more tortuous, and longer. The effective speed of light c' will therefore reduce by factor n i.e. $c' = c/n$, where n is the **refractive index**. For a vacuum, $n = 1$, for air $n \approx 1.00$, for water $n \approx 1.34$ and for glass $n \approx 1.50$.

Light can travel through any path between two points, but the *most probable*[1] is the path which takes the *least time*. This is ***Fermat's Principle.*** The result is that light **refracts** (bends) at a boundary of wave speeds (e.g. glass to air), or bends continuously if the wave speed also varies. This also explains the law of **refection** i.e. the angle of incidence from the normal to a surface = the angle of reflection from this normal.



Snell's law of refraction:

$$\frac{\sin\theta_i}{c/n_1} = \frac{\sin\theta_t}{c/n_2} \Rightarrow \boxed{n_1 \sin\theta_i = n_2 \sin\theta_t} \quad \therefore \theta_t = \sin^{-1}\left(\frac{n_1 \sin\theta_i}{n_2}\right)$$

Limiting case when refraction angle is 90°. ***Total internal reflection*** for greater incident angles than the *critical angle:*

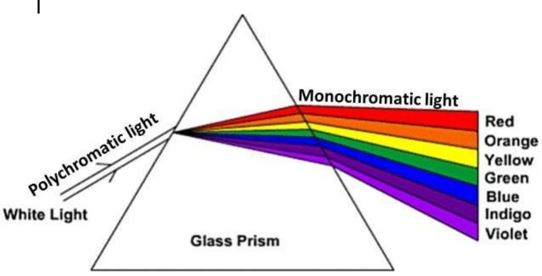
$$n_1 \sin\theta_c = n_2 \sin 90^0 \quad \therefore \theta_c = \sin^{-1}\left(n_2/n_1\right).$$

The number of waves passing per second (i.e. the frequency f) must be a *conserved quantity* on both sides of the boundary, otherwise there must be some form of energy, and also information, input. Since the speed of waves is the frequency multiplied by the wavelength:

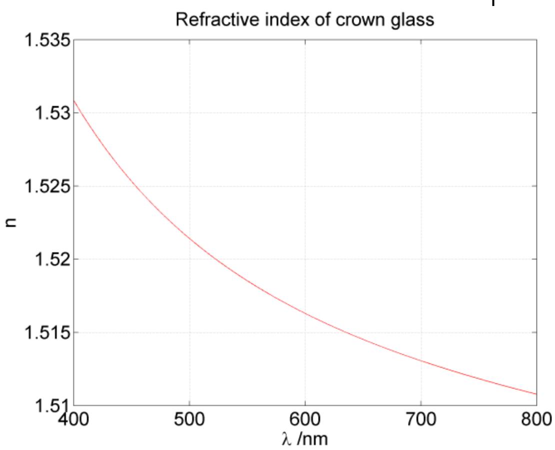
$$\frac{c}{n_1} = f\lambda_1, \quad \frac{c}{n_2} = f\lambda_2 \quad \therefore \boxed{\frac{\lambda_2}{\lambda_1} = \frac{n_1}{n_2}}$$

. The total ***power*** of reflected and transmitted rays must equate to the incident power. The exact balance depends upon the angle of incidence, the ratio of refractive indices and the polarization of the light.[2] Note power is proportional to the $A^2 f^2$ where A is the amplitude of the EM wave.

Variation of refractive index n results in ***dispersion*** (i.e. a different angle of refraction) of light of different wavelengths. This explains why a glass prism can separate white light into a spectrum of colours, and how a rainbow is formed.



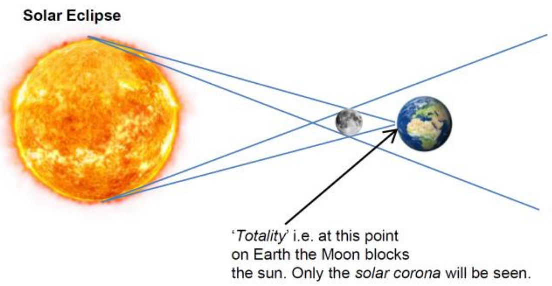
Colour	Wavelength λ /nm
Violet	380-450
Blue	450-495
Green	495-570
Yellow	570-590
Orange	580-620
Red	620-750

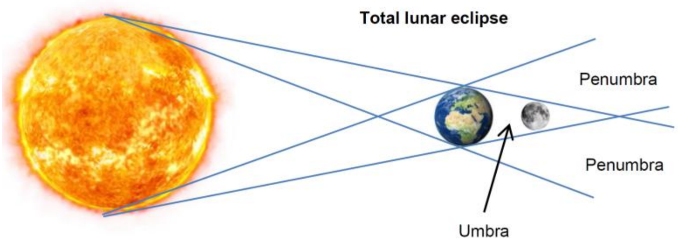


[1] This idea is explored in the theory of *Quantum Electrodynamics* (QED).
[2] The balance of transmitted and reflected power is determined by the *Fresnel equations*

Light might be a thin laser beam, but it might also be a broad illumination from an extended source such as the Sun. In both extremes we can think of the *direction* that light propagates. We call this the **wavevector**, or more simply, a **ray**.

Consideration of rays from extended sources (such as the Sun) explain the *penumbra* feature of *eclipses*.





Geometric Optics is in essence the drawing of *diagrams* that represent the *directions light rays take from source to observer*. When multiple rays can be drawn between source and observer, the *intersection of these different paths will predict where an image of a light source will form.*

In most geometric optics problems, consider the *intersection* of an undeviated ray, and one which starts horizontally and is refracted or reflected by a known amount.

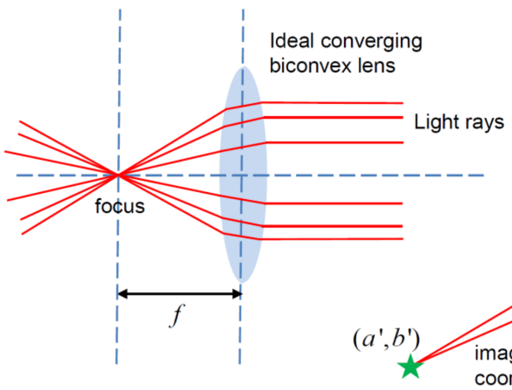
Question 1

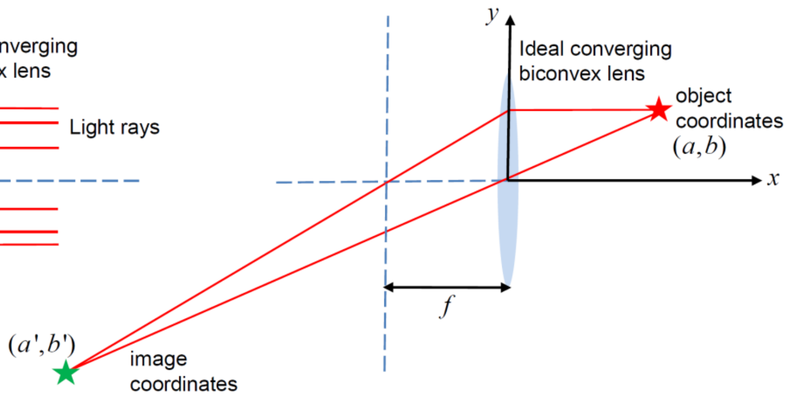
(i) Calculate the critical angles for: (a) a glass to air interface and (b) a glass to water interface. Assume $n_{water} = 1.34$ and $n_{glass} = 1.50$.

(ii) 520nm green light (in air) from the *Aurora Borealis* passes through ice of refractive index 1.31. The light penetrates about 5cm into the ice before being undetectable. Calculate how many wavelengths this is *in the ice.*

(iii) An ideal converging biconvex lens will cause all horizontal rays to converge to the *focus* $(-f, 0)$ of the lens, as shown below. The centre of the lens is the origin of the coordinate system. Rays straight through the centre of the lens are undeviated. By considering the intersection of horizontal and 'straight through' rays, show that light rays from an object at displacement (a,b) from the centre of the lens, will converge at (a',b') where

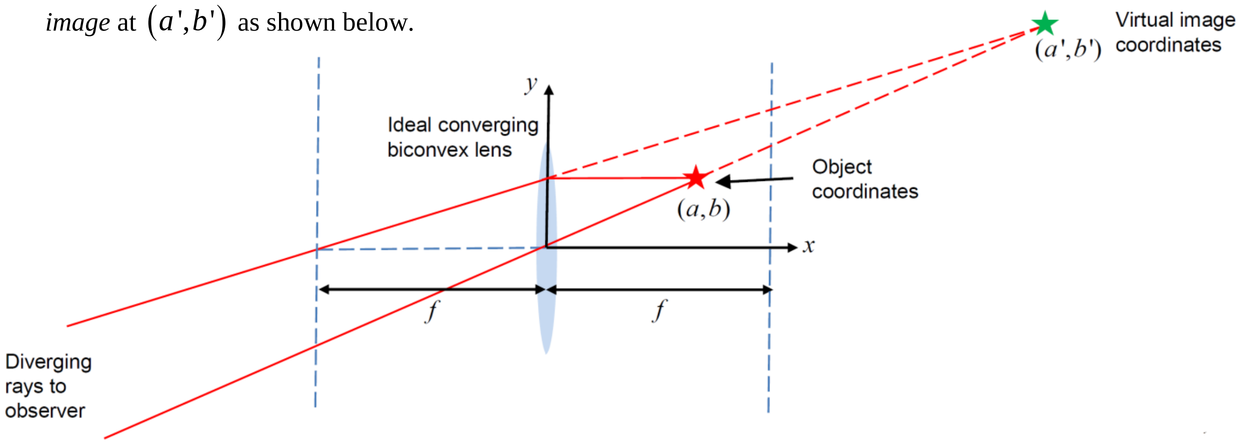
$a' = -\left(\frac{1}{f} - \frac{1}{a}\right)^{-1}$, $b' = -\frac{b}{a}\left(\frac{1}{f} - \frac{1}{a}\right)^{-1}$. Assume $a > f$.





A projector has a focal length f of 20.0mm. An image of Christiaan Huygens of height 1.6m is projected on a screen 5.0m away from the lens. Determine the height of the (probably LED) light source corresponding to Mr Huygens, and also calculate how far away from the lens it is.

(iv) A converging biconvex lens can be used as a magnifying glass if an object is placed within the focal length f of the lens. The human brain will interpret the diverging rays emerging from the lens as emanating from a *virtual image* at (a',b') as shown below.



(a) Show that the vertical magnification factor is: $M = \frac{f}{f-a}$.

(b) Sherlock Holmes investigates a curious mould on the floor of 221B Baker Street. His hand-held lens offers 5.0x magnification if the stain is 8.0cm from the lens. What is the focal length?

(v) The Moon is very slowly moving away from the Earth (about 3.8cm per year). Calculate the Earth-Moon distance when the *umbra* (or '*totality*') in a *solar eclipse* exists only at a *single point* on the surface of the Earth. The Earth-Sun distance (1Astronomical Unit, AU) is $1.496\times10^{8}\,\text{km}$, solar radius is 696,340km and lunar radius is 1,737.1km. The Earth radius is 6,371km. If the current Earth-Moon distance (*between centers*) has a minimum (*perigee*) of 356,500km, approximately how many more years before total solar eclipses cannot be seen on Earth?

(vi) Calculate the maximum Earth-Moon distance (between their centers) such that the moon is entirely within the umbra region during a lunar eclipse. Use the Sun and Earth data in (v).

(vii) The spacecraft *Cassini* acquired an extraordinary image of Saturn *occulting* (i.e. blocking out the Sun) on Sept 15th 2006. If the radius of Saturn is 58,232km and Saturn orbits the Sun at a radius of 9.957AU, calculate the maximum distance from the Surface of Saturn that *Cassini* could be in order to take the photograph.

(viii) An optic fiber consists of a *core* of pure silica glass, of radius 4.0 μm, surrounded by a *cladding* of doped silica glass.[3] At a particular frequency, the refractive indices of the core and cladding are, respectively, 1.4475 and 1.4440.

(a) Calculate the critical angle θ_c for light to propagate with minimal transmission loss in the fiber.

(b) Show that if the light in the core travels $2r$ between reflections, compared to $2x$ of core length, $r = x/\sin\theta$, where θ is the angle of internal reflection in the fiber.

(c) Hence determine the time it takes light to travel along a fiber that equals the circumference of the Earth. (Don't forget the speed is c/n_{core}). The Earth radius is 6,371km.

(d) London to Sydney, Australia is about 16,983km. Bruce sends an email to his English brother-in-law. Calculate the minimum time delay ('latency') in the fiber part of their communications link.

(ix) If light is S-polarized (i.e. the electric field oscillates in a perpendicular direction to a plane containing the incident, reflected and transmitted rays), the fraction of light power reflected $|r_\perp|^2$ is given by (for non-magnetic material):

$$|r_\perp|^2 = \left(\frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}\right)^2 .$$

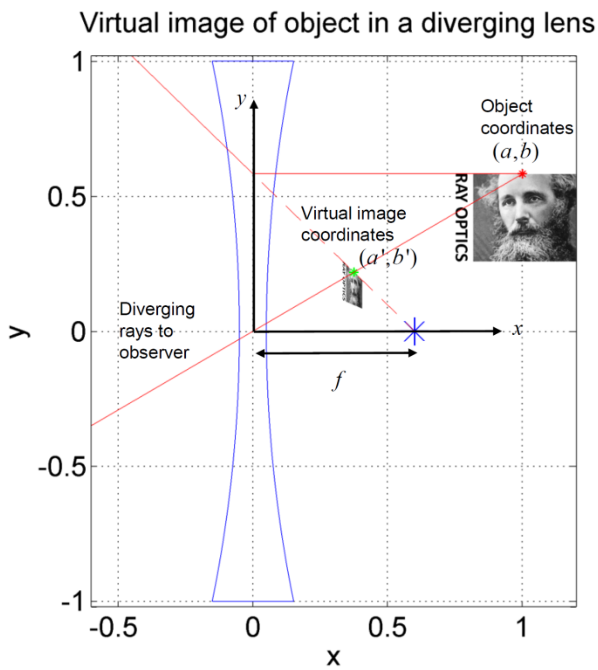
Hence calculate the fraction of incident light power that is *transmitted* through an air, glass interface with $\theta_i = 42^\circ$, $n_1 = 1.00, n_2 = 1.50$.

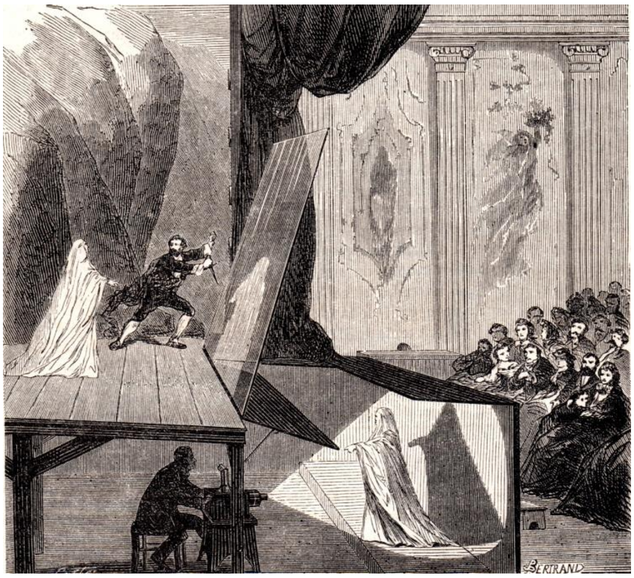
[3] Beyond the core and cladding you have additional layers (the 'buffer' and 'jacket') which provide mechanical protection to the glass core and cladding of an optical fibre.

Question 2

(i) Show that rays from (a,b) beyond the focus of an ideal diverging biconcave lens will appear to originate from (a',b') where: $a' = \left(\frac{1}{a}+\frac{1}{f}\right)^{-1}$; $b' = \frac{b}{a}a'$.

(ii) Hence show the vertical magnification factor (well, 'demagnification factor!) is: $M = \frac{f}{f+a}$.





Question 3

The diagram on the left is from the Wikipedia page for the stage illusion *"Pepper's Ghost",* which is a method for the creation of a realistic spectral figure on stage, as viewed by the audience. Draw a ray diagram to explain the effect, and also why it may not be immediately apparent that there is a glass sheet between the audience and the stage.

Question 4

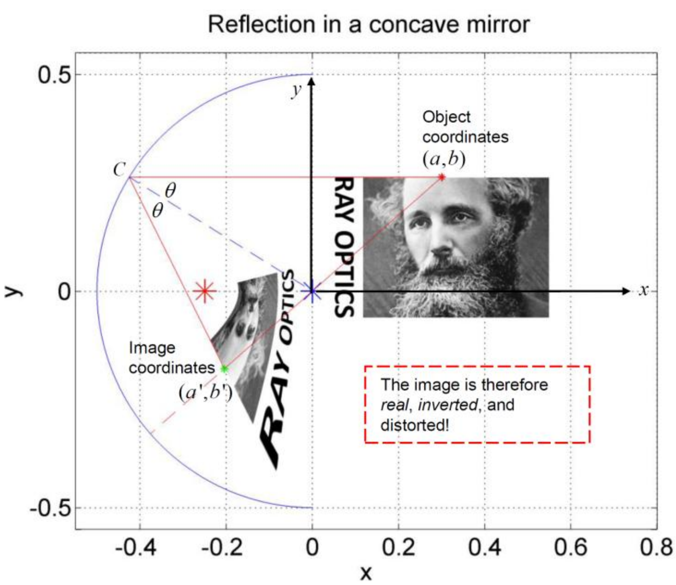
A curious optical illusion occurs when you dangle an object close to the geometric centre of a *concave mirror*. As shown below, the convergence of direct and reflective rays from (a,b) result in a real, inverted image at (a',b') . The object appears to magically float in space. It is traditional in Physics laboratories to use a plastic animal, hence the demo is known as the 'flying cow' experiment.

(i) If the circle which defines the mirror has radius R , show that the coordinates of the reflection point is: $\left(-\sqrt{R^2-b^2}, b\right)$

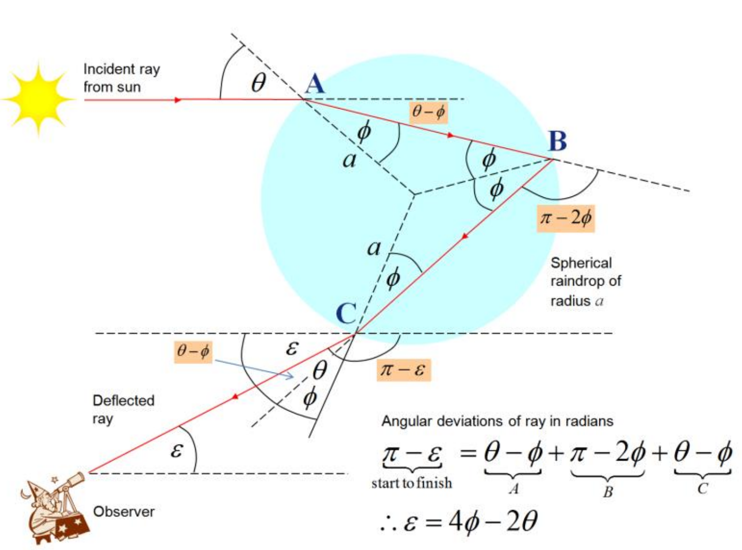
and hence $\theta = \tan^{-1}\left(\dfrac{b}{\sqrt{R^2-b^2}}\right)$.

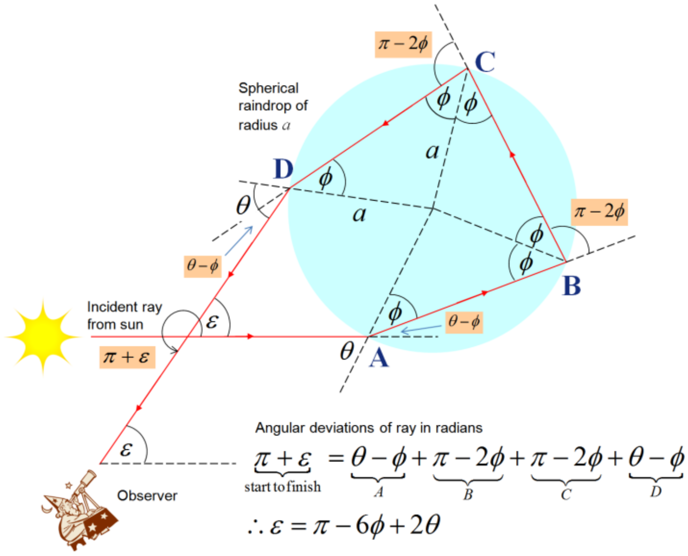
(ii) Hence show that, if $m = \tan 2\theta$

$$a' = -\frac{m\sqrt{R^2-b^2}-b}{m+\frac{b}{a}};\; b' = -\frac{b}{a}\,\frac{m\sqrt{R^2-b^2}-b}{m+\frac{b}{a}} \; .$$



Question 5 The ray diagrams for internal reflection of a light beam inside a spherical raindrop are given below (for a single reflection, and for two reflections). The can be used to explain the formation of *rainbows*.





(i) Show using *Snell's Law*, that the angle of elevation of rays relative to the *anti-solar direction*[4] are:

$\varepsilon = 4\sin^{-1}\left(\frac{\sin\theta}{n}\right) - 2\theta$ for a single internal reflection, and $\varepsilon = \pi - 6\sin^{-1}\left(\frac{\sin\theta}{n}\right) + 2\theta$ for a double internal reflection. where θ is the angle of incidence and n is the refractive index of water in the raindrop.

(ii) Explain why $d\varepsilon/d\theta = 0$ implies a maxima in light intensity, and why the elevation angle ε when this true corresponds to the elevation angle of the rainbow above the anti-solar direction.

(iii) Prove $\left(\frac{d\phi}{d\theta}\right)^2 = \frac{1-\sin^2\theta}{n^2-\sin^2\theta}$ from Snell's law to help show that for a primary rainbow: $\theta = \sin^{-1}\sqrt{\frac{4-n^2}{3}}$, and for a secondary rainbow: $\theta = \sin^{-1}\sqrt{\frac{9-n^2}{8}}$.

(iv) Use a spreadsheet such as Microsoft Excel (or better, a programming language like MATLAB or Python) to plot ε vs θ for single and double internal reflections, for a range of optical frequencies (colour these appropriately), and also the rainbow elevation angles vs light frequency. Use a frequency range: $405\text{THz} < f < 790\text{THz}$.

The semi-empirical relationship for the refractive index of water over this range is:

$$n = \sqrt{1 + \left(1.731 - 0.261\left(\frac{f}{10^{15}\,\text{Hz}}\right)^2\right)^{-\frac{1}{2}}}$$

and this should be plotted against frequency too.

Colour for plots:

Colour	**R,G,B**	**Frequency range /THz**
Red	1,0,0	405 to 480
Orange	1, 127/255, 0	480 to 510
Yellow	1,1,0	510 to 530
Green	0,1,0	530 to 600
Cyan	0,1,1	600 to 620
Blue	0,0,1	620 to 680
Violet	137/255, 0, 1	680 to 790

[4] When the sun is high, the rainbow elevation may not make it above the horizon. So the best rainbows are viewed when the Sun is low i.e. at dawn or dusk. If you view a rainbow from an aircraft cockpit , you will see the complete circle!

Question 6

The *Fresnel Equations* define the proportions of transmitted and reflected power of *polarized* light at a boundary of different refractive indices n_1, n_2. In Q1 (ix) the equation for the proportion of reflected S-polarized light is given as:

$$|r_\perp|^2 = \left(\frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}\right)^2$$ which means the transmitted power fraction is: $|t_\perp|^2 = 1 - |r_\perp|^2$.

Recall, S-polarized light is when the electric field oscillates *perpendicular* to the plane containing all the incident, reflected and transmitted rays. P-Polarized light is when the electric field *only* oscillates in the plane containing the rays.

P-polarized light has a slightly different reflection coefficient: $|r_\parallel|^2 = \left(\frac{n_1\cos\theta_t - n_2\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t}\right)^2$, but as before, $|t_\parallel|^2 = 1 - |r_\parallel|^2$

(i) Use an Excel sheet and plot $|r_\parallel|^2$, $|t_\parallel|^2$, $|r_\perp|^2$, $|t_\perp|^2$ vs θ_i over range $0 \le \theta_i \le 90^\circ$ for $n_2 = 2,\ n_1 = 1$.

(ii) Observe that $|r_\parallel|^2$ has a zero about 63.4°. What practical application might this effect have?

Sketch how the curves might alter if $n_2 = 1,\ n_1 = 2$.

(iii) Show that the angle (which is called the *Brewster angle*) when $|r_\parallel|^2 = 0$ is: $\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right)$.

Question 7

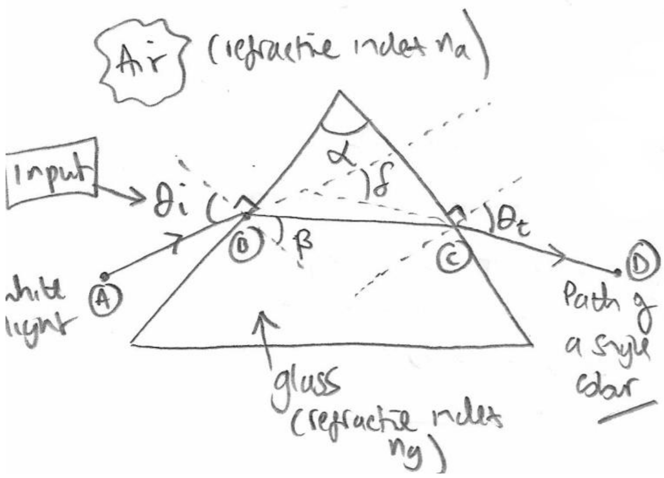
A ray of light travelling in air strikes the surface of a triangular prism at angle of incidence θ_i. n is the refractive index of the prism and the prism has apex angle α.

(i) Re-draw the diagram and show that the deviation angle of the incident ray is: $\delta = \theta_i + \theta_t - \alpha$.

(ii) Prove that $\sin\theta_t = \sqrt{n^2 - \sin^2\theta_i}\,\sin\alpha - \sin\theta_i\cos\alpha$.

(iii) The refractive index of crown glass is given by the *Sellmeier* empirical equations, where λ is the wavelength of light (in a vacuum) in μm.

$$n^2 - 1 = \sum_{i=1}^{3} \frac{a_i\lambda^2}{\lambda^2 - b_i}$$

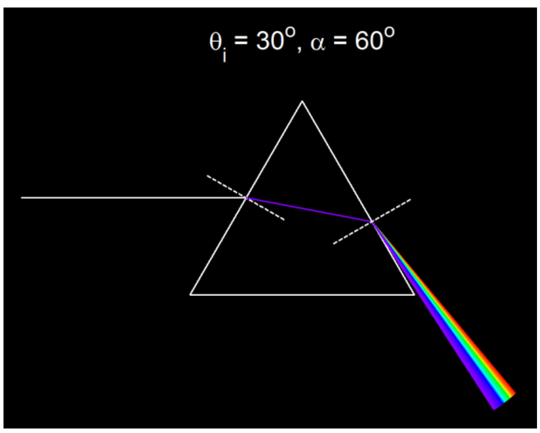


And example set of coefficients (for 'BK7 Crown glass') is:

$a_1 = 1.03961212,\ a_2 = 0.231792344,\ a_3 = 1.01146945$
$b_1 = 0.00600069867,\ b_2 = 0.0200179144,\ b_3 = 103.560653$

Use this information (and the colour values for visible light in Question 5) to construct an Excel, Python or MATLAB program to plot δ vs θ_i for various values of α. Use an average visible light frequency of 542.5THz.

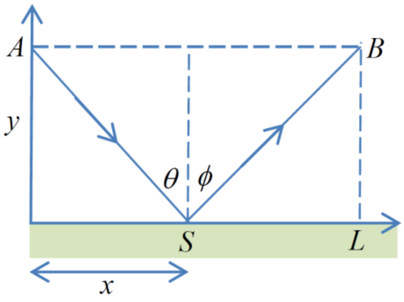
Better still, use the model to actually plot correctly coloured rays emerging from a computer generated prism, and implement a 'keypress' functionality which allows the rays (for say 500 colours within the range 405 to 790THz) to be automatically updated when θ_i and α are modified by pressing specific keys.

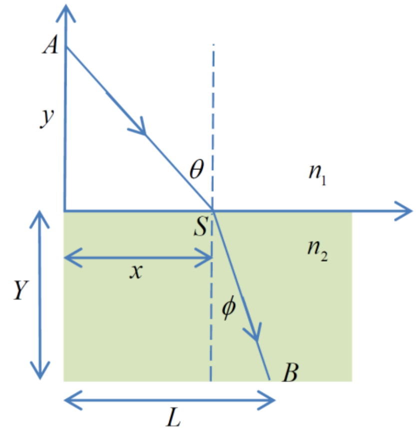


Question 8

Use the diagrams on the right, and *Fermat's principle*, to prove the *law of reflection*, and *Snell's law of refraction.*

Hint: find travel time $\Delta t_{ASB}(x)$ and determine a relationship between θ and ϕ that results in $\dfrac{\partial \Delta t_{ASB}}{\partial x} = 0$.





Question 9

A real biconvex lens, made from glass of refractive index n , is of thickness d and has curved surfaces which are arcs of circles with radii R_1 and R_2 .

If incident rays are close to the 'optical axis' (i.e. the horizontal *x* axis in the diagram) we can use *a small angle approximation.*

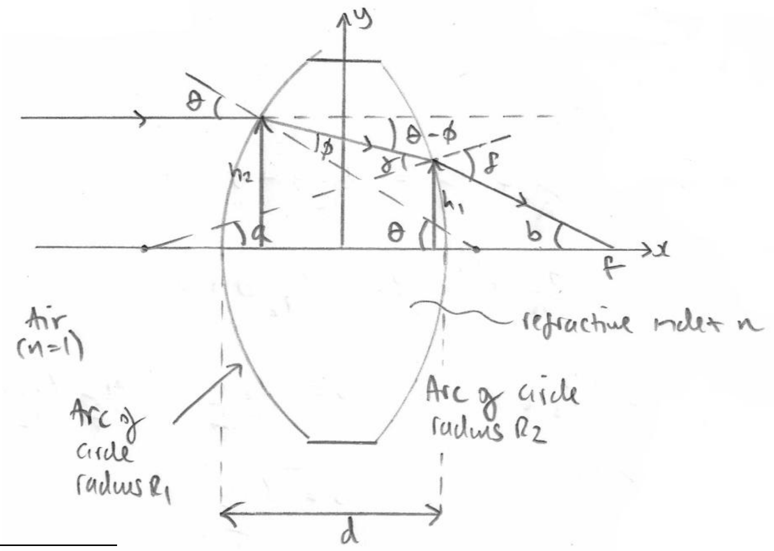
i.e. $\sin\theta \approx \tan\theta \approx \theta$ (and this is true for *all* angles $a, b, \theta, \phi, \gamma, \delta$). Note all angles are in radians! Rays like these are called *paraxial*.

Also, given the lens is deemed '*thin*': $f, R_1, R_2 \gg d$ and $h_1 \approx h_2$

By applying Snell's law, and using the thin lens and small angle approximations above, show that the focal length of the lens can be determined via the following *"Lensmaker's formula"* :[5]

$$\frac{1}{f} \approx (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Hint: Explain why $\phi + \gamma = a + \theta$ and $a + b = \delta$. Then express all angles in terms of θ .



[5] A slightly more accurate formula, incorporating lens thickness d is: $\dfrac{1}{f} \approx (n-1)\left(\dfrac{1}{R_1} + \dfrac{1}{R_2} - \dfrac{(n-1)d}{nR_1R_2}\right)$. However, this is pretty hard to derive. It is not even in *Optics* by Hecht! Another major health warning is that conventionally R_2 is defined to be *negative*. So to match the formula in many references, change $R_2 \to -R_2$. But since this is in reality merely a *double negative*, it seems needlessly confusing in my view. The reason for this strangeness is probably to use the *same formula* for both convex and concave lenses. To swap between these, change the sign of the radii.