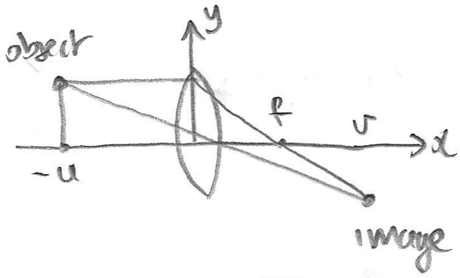
Proof of the lensmaker's formula

Thin lens equation

$$\boxed{\frac{1}{u} + \frac{1}{v} = \frac{1}{f}}$$



Lensmaker's formula : $\boxed{\frac{1}{f} \simeq (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$

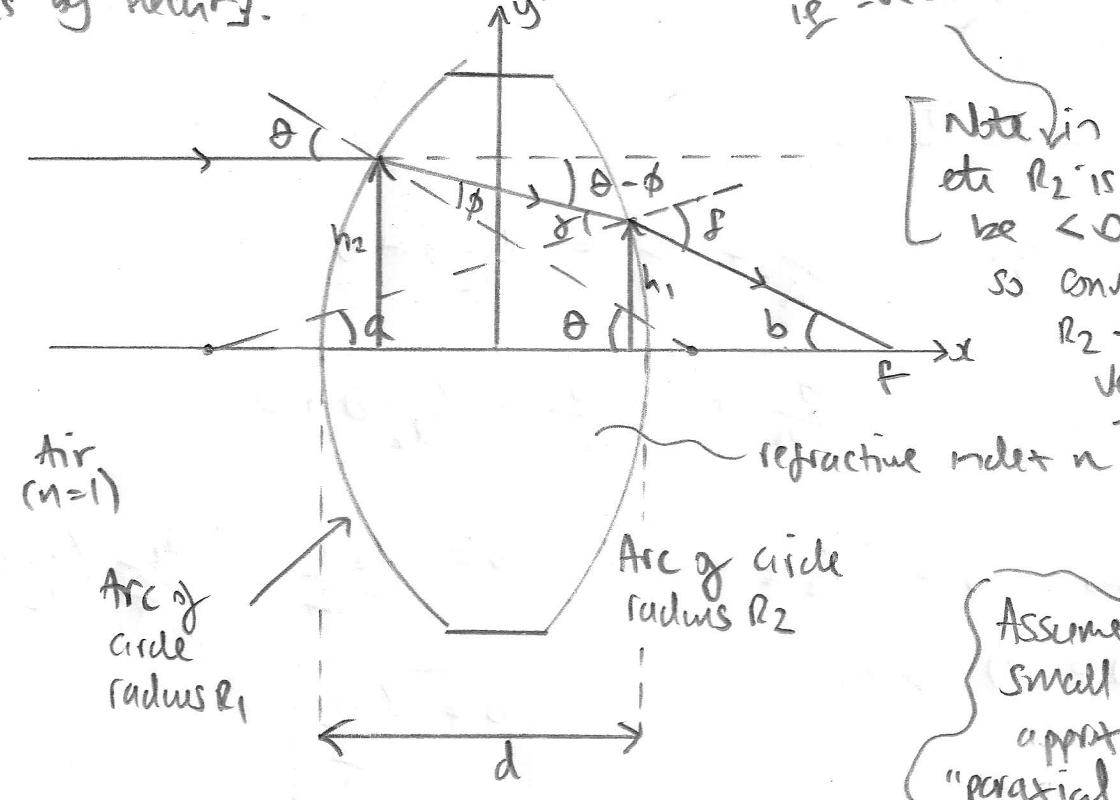
R_1, R_2 are curvature radii of lens surfaces

[Better formula for thicker lenses is : $\frac{1}{f} \simeq (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{(n-1)d}{nR_1R_2}\right)$
But this is hard to prove! It is not even in Optics by Hecht].

Correction term.

ie "-ve curvature"

[Note in Hecht etc R_2 is set to be < 0 (!!) so conventionally $R_2 \to -R_2$ <u>very</u> confusing!]



Assume small angle approximation "paraxial rays"

Snell's law: $\sin\theta = n \sin\phi$

so $\boxed{\theta \simeq n\phi}$ (1)

$\sin\delta = n \sin\gamma$

so $\boxed{\delta \simeq n\gamma}$ (2)

$h_1 \cong \left(f - \frac{d}{2}\right) \tan b \simeq \boxed{fb}$ (3)

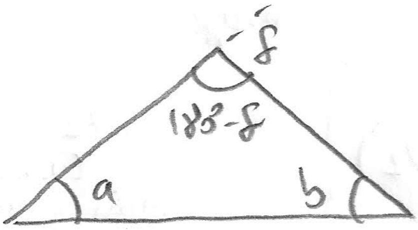
(1) $h_1 = R_2 \sin a \simeq \boxed{R_2 a}$ (4)

The idea is to distinguish: ( ) from)(in terms of +ve/-ve curvature

$$h_2 = R_1 \sin\theta \simeq \boxed{R_1\theta} \quad (5)$$

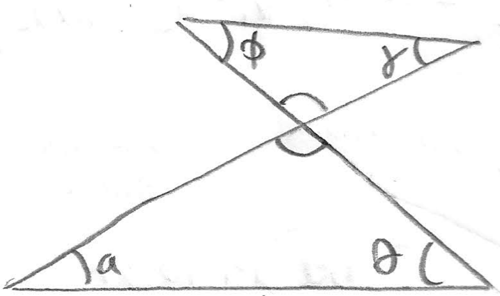
Assume $h_1 \approx h_2 \Rightarrow R_1\theta \approx R_2 a$

$$\therefore \boxed{a \simeq \frac{R_1}{R_2}\theta} \quad (6)$$



$$180^\circ = a + b + 180^\circ - \delta$$

$$\Rightarrow \boxed{a + b = \delta} \quad (7)$$



$$\text{so } \boxed{\phi + \gamma = a + \theta} \quad (8)$$

From (8):

$$\frac{\theta}{n} + \frac{\delta}{n} = \frac{R_1}{R_2}\theta + \theta$$

$$\frac{\theta}{n} + \frac{a+b}{n} = \frac{R_1}{R_2}\theta + \theta$$

Now $fb = R_2 a$

(3) = (4) is h_1

$$\therefore b = \frac{R_2 a}{f} = \frac{R_2}{f}\frac{R_1}{R_2}\theta$$

$$\boxed{b = \frac{R_1\theta}{f}}$$

$$\therefore \frac{\theta}{n} + \frac{R_1}{R_2}\frac{\theta}{n} + \frac{R_1\theta}{nf} = \frac{R_1}{R_2}\theta + \theta$$

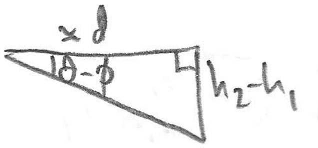
$$\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{f} = \frac{n}{R_2} + n/R_1$$

$$\frac{1}{R_1}(n-1) + \frac{1}{R_2}(n-1) = \frac{1}{f}$$

$$\Rightarrow \boxed{\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

Possible extension to simple lensmaker formula which looks (close) to the "official" formula!

Rather than $h_2 \approx h_1$, note $h_2 - h_1 \approx d\tan(\theta - \phi)$



$$\Rightarrow \boxed{h_2 - h_1 \approx d\left(\theta - \theta/n\right)}$$

$h_2 = R_1\theta$ and $h_1 = \left(f - \frac{d}{2}\right)b$

so $h_2 - h_1 = R_1\theta - fb + \frac{db}{2} = d\theta\left(1 - \frac{1}{n}\right) = d\theta\frac{(n-1)}{n}$

$$\therefore \left(R_1 - \frac{d(n-1)}{n}\right)\theta = b\left(f - \frac{d}{2}\right)$$

$$\therefore \boxed{b = \frac{R_1 - \frac{d(n-1)}{n}}{f - d/2}\,\theta} \qquad (9)$$

Now from (8): $\phi + \gamma = a + \theta$

$$\frac{\theta}{n} + \frac{\delta}{n} = a + \theta$$

$$\therefore \boxed{\frac{\theta}{n} + \frac{a+b}{n} = a + \theta} \qquad (10)$$

Now $h_1 = R_2 a$, so $R_1\theta - R_2 a = d(\theta - \theta/n)$

$$\therefore \boxed{\frac{\theta\left(R_1 - \frac{d(n-1)}{n}\right)}{R_2} = a} \qquad (11)$$

So $a + b = \left(R_1 - \frac{d(n-1)}{n}\right)\left(\frac{1}{R_2} + \frac{1}{f - d/2}\right)\theta$

$\therefore$ in (10): $\frac{\theta}{n} + \frac{\theta}{n}\left(R_1 - \frac{d(n-1)}{n}\right)\left(\frac{1}{R_2} + \frac{1}{f - d/2}\right) = \theta\left(\frac{R_1 - \frac{d(n-1)}{n}}{R_2} + 1\right)$

$$\therefore \left(R_1 - \frac{d(n-1)}{n}\right)\left(\frac{1}{R_2} + \frac{1}{f-d/2}\right) = \frac{\left(R_1 - \frac{d(n-1)}{n}\right)n}{R_2} + n - 1$$

$$\frac{1}{R_2} + \frac{1}{f-d/2} = \frac{n}{R_2} + \frac{n-1}{R_1 - \frac{d(n-1)}{n}}$$

$$\frac{1}{f-d/2} = (n-1)\left(\frac{1}{R_2} + \frac{1}{R_1}\frac{1}{\left(1-\frac{d(n-1)}{R_1 n}\right)}\right)$$

$$\frac{1}{f-d/2} = (n-1)\left(\frac{1}{R_2} + \frac{1}{R_1}\left(1+\frac{d(n-1)}{R_1 n}\right)\right)$$

$$\left(1 - \frac{d(n-1)}{R_1 n}\right)^{-1} \simeq 1 + \frac{d(n-1)}{R_1 n}$$

if $R_1 \gg d$

$$\therefore \frac{1}{f-\frac{d}{2}} = (n-1)\left(\frac{1}{R_2} + \frac{1}{R_1} + \frac{d(n-1)}{R_1^2 n}\right)$$

So if $f \gg d$

$$\Rightarrow \boxed{\frac{1}{f} \simeq (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{d(n-1)}{R_1^2 n}\right)}$$

which is similar to, but not quite the correct formula (eg from Hyperphysics)

$$\boxed{\frac{1}{f} \simeq (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{(n-1)d}{nR_1R_2}\right)}$$

(again using $R_2 > 0$, not < 0 as is conventional)

↳ with this sign convention: $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right)$