(a) The circuit in Figure 1.(a) contains a cell of emf E, a known variable resistance R_0, an unknown resistance R and an ammeter. When X and Y are short circuited $E = I_0R_0$

When R is inserted the current is αI_0, where α is a constant.

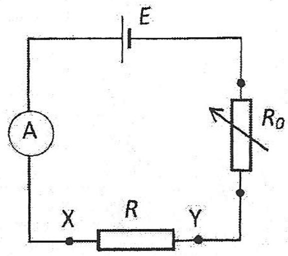


Figure 1.(a)

(i) Express R in terms of R_0 and α, giving the range of validity of R and α.

(ii) In order to extend the range of α, modify the circuit by putting R in parallel with R_0. Determine the ranges of R and α for the modified circuit.

[4]

(i) Kirchoff II (or simply ohm's law for entire circuit)

$\boxed{E = I(R+R_0)}$ Now: $E = I_0R_0$, $I = \alpha I_0$ } This is given in the text.

So $I_0R_0 = \alpha I_0(R+R_0)$ $\therefore R+R_0 = \dfrac{R_0}{\alpha}$

$\therefore \boxed{R = R_0\left(\frac{1}{\alpha}-1\right)}$ [MS gives this as $R = R_0\left(\frac{1-\alpha}{\alpha}\right)$ which is the same expression]

Now $R \geq 0$ so $\boxed{\alpha \leq 1}$ (if $\alpha > 1$ then $R < 0$)

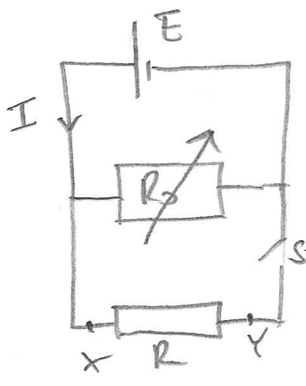
$I \geq 0$ so $\boxed{\alpha \geq 0}$

$\therefore \boxed{0 \leq \alpha \leq 1}$

So using $R = R_0(1/\alpha - 1) \Rightarrow \boxed{0 \leq R \leq \infty}$

$\alpha = 1$ $\alpha = 0$

(ii)



let $I = \alpha I_0$ as before, and with X→Y cut (eg via a switch) $E = I_0R_0$.

Total resistance is $R_{TOT} = \left(\frac{1}{R_0}+\frac{1}{R}\right)^{-1}$

and $E = IR_{TOT}$

$$\therefore \quad \underbrace{I_0 R_0}_{E} = \underbrace{\alpha I_0}_{I} \times \left(\frac{1}{R_0} + \frac{1}{R}\right)^{-1}$$

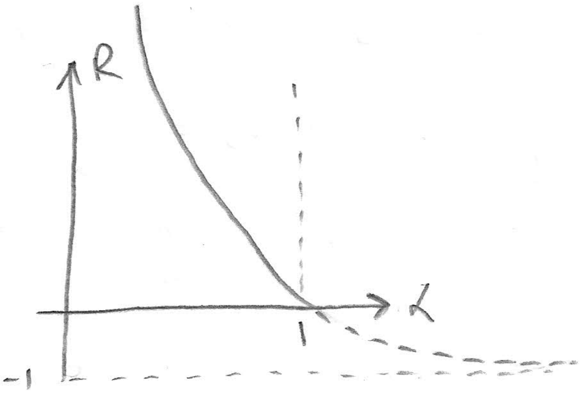
$$\therefore \quad \frac{1}{R_0} + \frac{1}{R} = \alpha / R_0$$

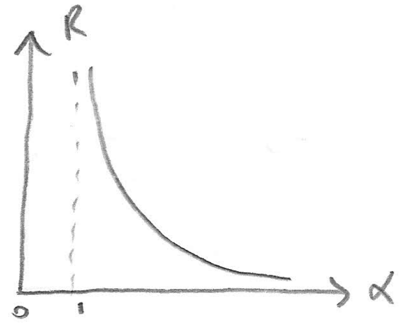
$$\therefore \quad \frac{1}{R} = \frac{1}{R_0}(\alpha - 1)$$

$$\therefore \quad \boxed{R = \frac{R_0}{\alpha - 1}}$$

Now $\boxed{\underset{\uparrow \atop \alpha = 1}{0} \leq R \leq \underset{\uparrow \atop \alpha = \infty}{\infty}}$ and $\boxed{1 \leq \alpha \leq \infty}$

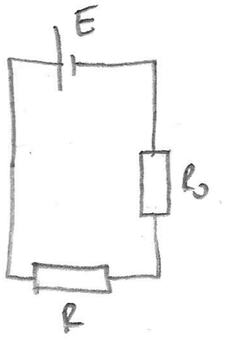
(i)

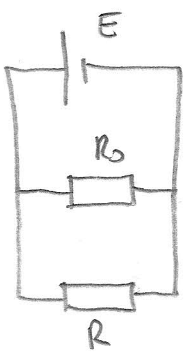




$$\boxed{R = R_0\left(\frac{1}{\alpha} - 1\right)}$$

$$\boxed{R = \frac{R_0}{\alpha - 1}}$$





(b) A man, on an open wagon of a train travelling along a straight horizontal track at a constant speed of 10 ms^{-1}, throws a ball into the air in line with the track, that he judges to be at 60° to the horizontal. A woman standing on the ground observes the ball rise vertically.

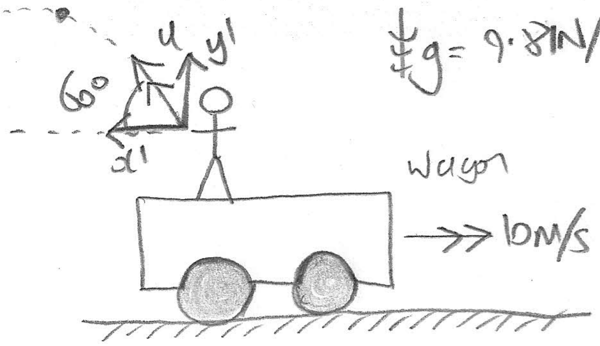
How high does the ball rise relative to

(i) the man and;

(ii) the woman?

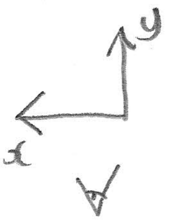
[5]

But unlike special relativity $t = t'$ and velocities add in a Galilean sense!



$g = 9.81 N/kg$

like special relativity, the ' frame is that of the man in the wagon.



Woman's frame of reference.

$x = x' = 0$ at $t = 0$

VELOCITIES

Wagon: $\boxed{v_{y'} = u\sin 60^\circ - gt \quad v_{x'} = u\cos 60^\circ}$

Woman: $\boxed{v_y = v_{y'} = u\sin 60^\circ - gt \quad v_x = u\cos 60^\circ - 10}$

Now woman observes $v_x = 0$ (m/s)

$\therefore \boxed{u = 10/\cos 60^\circ}$

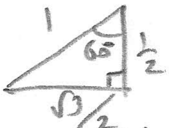
$\therefore v_y = v_{y'} = \boxed{10\tan 60^\circ - gt}$

Now $\boxed{y = u\sin 60^\circ t - \frac{1}{2}gt^2}$ { same for wagon and woman frames since $v_y = v_{y'}$ $\Rightarrow y = y'$ }

Apogee when $v_y = 0 \Rightarrow \boxed{t = 10\tan 60^\circ / g}$

$\therefore y_{max} = 10\tan 60^\circ \times \frac{10\tan 60^\circ}{g} - \frac{1}{2}g \times 100\tan^2 60^\circ / g^2$

$y_{max} = \frac{50\tan^2 60^\circ}{g}$



$\boxed{\tan 60^\circ = \sqrt{3}}$

$y_{max} = \frac{50 \times 3}{9.81} = \boxed{15.3m}$

{ Note could use "$v^2 = 2gh$" $\Rightarrow y_{max} = \frac{(10\tan 60^\circ)^2}{2g}$ }

3

(c) A glass block of refractive index $\mu = 1.5$ has an 'L' cross-section, Figure 1.(c), and is of constant width and thickness.

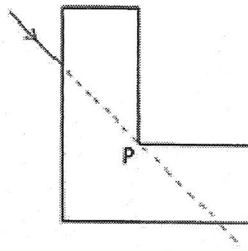


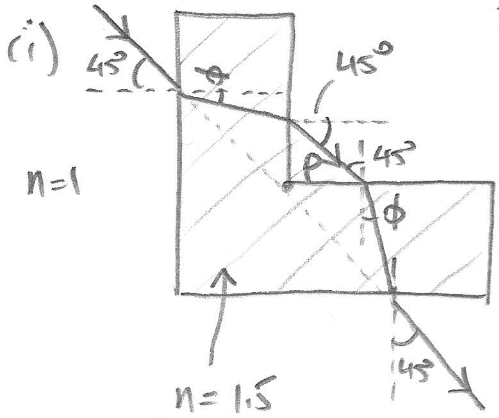
Figure 1.(c)

(i) A laser beam enters the block from the left, as indicated in Figure 1.(c), at an incident angle of $\theta = 45°$. If the block was absent the beam would pass through the point P. Determine the angle at which the beam will emerge from the bottom face after refraction through the block. ← it is 45° !

(ii) If this beam enters the block below the horizontal through P, determine its possible subsequent path(s).

[6]

(i)



Although we can work out ϕ from Snell's law:

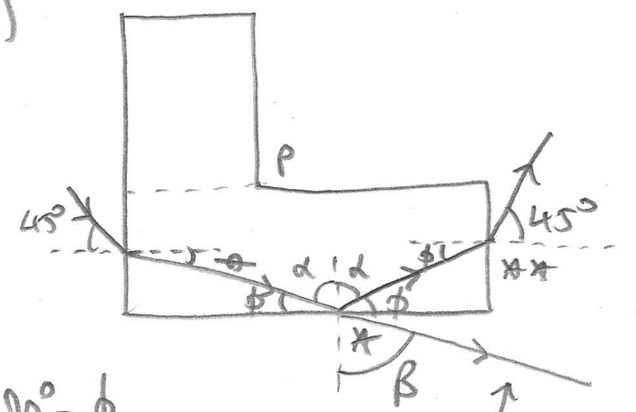
$$1.5 \sin\phi = 1.0 \sin 45°$$

$$\Rightarrow \quad \phi = \sin^{-1}\left(\frac{1.0}{1.5}\sin 45°\right)$$

$$\boxed{\phi = 28.1°}$$

By the ray tracing on the left, the ray must exit the block ∥ to the incident beam.

(ii)



This does not exist!

So ray will exit as shown at **, but will it exit at * also?

Snell:

$$\boxed{1.5 \sin\alpha = 1.0 \sin\beta}$$

$\alpha = 90° - \phi$

$= \boxed{61.9°}$

Now $1.5 \sin 61.9° = 1.32$, so there is no valid β. α is beyond the critical angle $\left[1.5 \sin\theta_C = 1.0 \sin 90° \Rightarrow \theta_C = \sin^{-1}\frac{1}{1.5} = 41.8°\right]$ so only $\boxed{\text{total internal reflection}}$ at *.

(4)

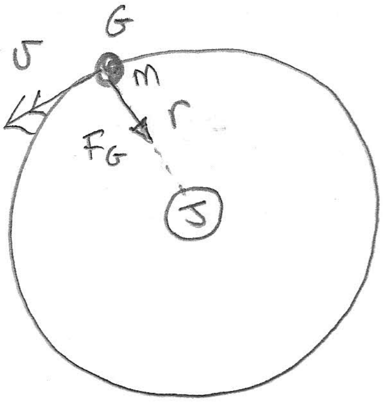
J G

(d) The largest moon of Jupiter, Ganymede, revolves around the planet in a circular orbit of radius 1.07×10^6 km and period 7.16 days. Determine the mass of Jupiter, M_J, in terms of the mass of the Earth, M_E.

r T

The radius of the Earth $R_E = 6.38 \times 10^6$ m

[5]



Newton II: (for Ganymede's orbit about Jupiter)

$$\boxed{\frac{Mv^2}{r} = \frac{GM_J M}{r^2}}$$

$$v = \frac{2\pi r}{T}$$

So $$\frac{4\pi^2 r^2}{rT^2} = \frac{GM_J}{r^2}$$

$$\Rightarrow \boxed{\frac{4\pi^2 r^3}{T^2 G} = M_J}$$

{Kepler III}

$$T^2 = \frac{4\pi}{GM} r^3$$

Now $$\boxed{g = \frac{GM_E}{R_E^2}}$$

(gravitational field strength at the surface of Earth)

So $$G = g R_E^2 / M_E$$

$$\therefore \boxed{M_J = \frac{4\pi^2 r^3}{T^2 g R_E^2} M_E}$$

$$\therefore M_J = \frac{4\pi \times (1.07 \times 10^6 \times 10^3)^3}{(7.16 \times 24 \times 3600)^2 \times 9.81 \times (6.38 \times 10^6)^2} \times M_E$$

$$\cong \boxed{316 M_E}$$

i.e. yields G

[So this explains why if you know g, M_E, R_E you can "weigh" the other planets if you know T, r from astronomy]

e.g. Schiehallion experiment

5

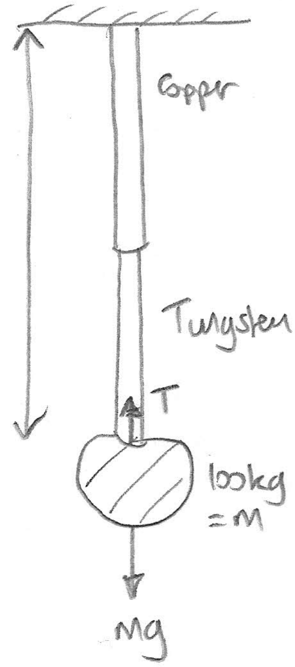
ℓ

(e) Two 1.00 m lengths of wire, one copper and one tungsten, are joined vertically end to end. The copper wire has a diameter of 0.500 mm. When a 100 kg block is suspended from one end, the combined length of wire stretches by 6.00 cm. What is the diameter, d_T of the tungsten wire if the Young's modulus for copper is 12.4 x 10^{10} Pa, and that for tungsten is 35.5 x 10^{10} Pa?

d_c Y_c Y_T

[6]

$l = 1.00$ m



$g = 9.81$ N/kg

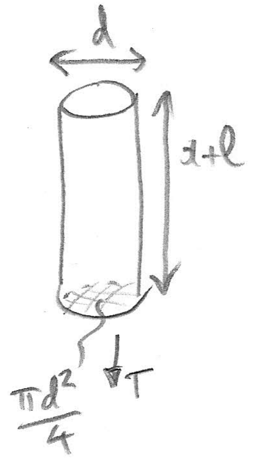
let x be extension of copper wire
y " " " tungsten wire

$$\boxed{x + y = 0.06} \quad \text{(metres)} \quad (1)$$

Now assume same tension $\boxed{T = Mg}$ in both copper and tungsten wires.

Young's modulus = $\frac{\text{Stress}}{\text{Strain}}$

$$Y_c = \frac{T/\pi\left(\frac{d_c}{2}\right)^2}{x/\ell} \qquad Y_T = \frac{T/\pi\left(\frac{d_T}{2}\right)^2}{y/\ell}$$



So $$Y_c = \frac{4T\ell}{\pi d_c^2 x} \qquad Y_T = \frac{4T\ell}{\pi d_T^2 y}$$

$$\therefore \quad d_T = \sqrt{\frac{4T\ell}{\pi y Y_T}}$$

Now $$x = \frac{4T\ell}{\pi d_c^2 Y_c} \qquad \therefore y = \underset{\Delta}{0.06} - \frac{\overset{T = Mg}{4T\ell}}{\pi d_c^2 Y_c}$$

$$\therefore \quad \boxed{d_T = \sqrt{\frac{4Mg\ell}{\pi Y_T\left(\Delta - \frac{4Mg\ell}{\pi d_c^2 Y_c}\right)}}} = \sqrt{\frac{4 \times 100 \times 9.81 \times 1.00}{\pi \times 35.5 \times 10^{10}(0.06 - *)}}$$

$$= \boxed{0.423 \text{ mm}}$$

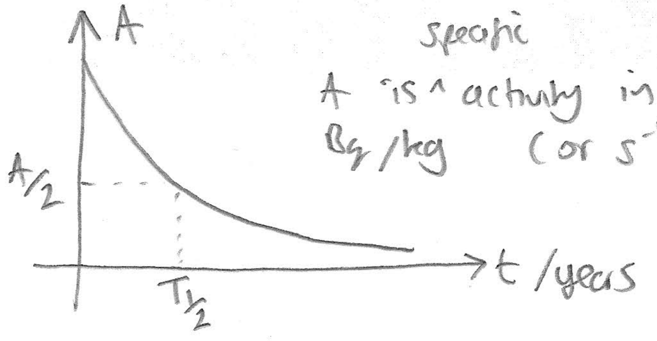
ie 4.23×10^{-4} m

$$\left[* = \frac{4 \times 100 \times 9.81 \times 1.00}{\pi \times (0.500 \times 10^{-3})^2 \times 12.4 \times 10^{10}} = 0.04 \text{m} \right]$$

(f) Wood from the coffin of an Egyptian mummy showed a specific activity of 1.2×10^2 $s^{-1}kg^{-1}$. Comparable living wood has a value of $2.0 \times 10^2\ s^{-1}kg^{-1}$. The half life of carbon-14 is 5.70×10^3 years. What is the time interval, T_B, in years, since the burial?

[5]

${}^{14}_{6}C$ decay curve is



A is specific activity in Bq/kg (or $s^{-1}\ kg^{-1}$)

$$\boxed{A = \frac{A_0}{2^{t/T_{1/2}}}}$$

Assume $A_0 = 2.0 \times 10^2\ s^{-1}\ kg^{-1}$

Now $2^{t/T_{1/2}} = A_0/A$

$$\frac{t}{T_{1/2}} \ln 2 = \ln(A_0/A)$$

$$\therefore \quad \boxed{t = T_{\frac{1}{2}} \ln(A_0/A) / \ln 2}$$

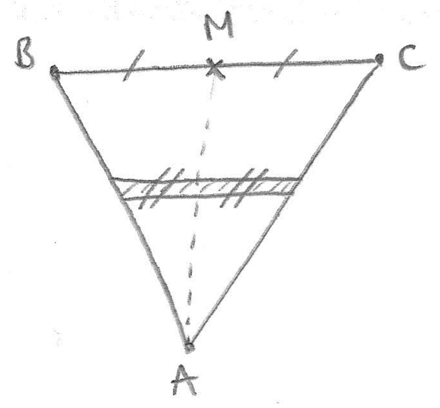
So $T_B = 5.70 \times 10^3 \text{ years} \times \ln\left(\frac{2.0 \times 10^2}{1.2 \times 10^2}\right) / \ln 2$

$= \boxed{4.2 \times 10^3 \text{ years}}$

(g) Explain why the centre of gravity of a triangular plate lies along a median; the line joining a vertex to the midpoint of the opposite side. An equilateral triangular plate, sides of length b, has a triangle, formed by two corners and the centre of gravity of the original plate, removed. Determine the centre of gravity of the remaining plate. The centre of gravity of a triangular plate is at a point two thirds along the length of a median measured from the vertex.

[7]

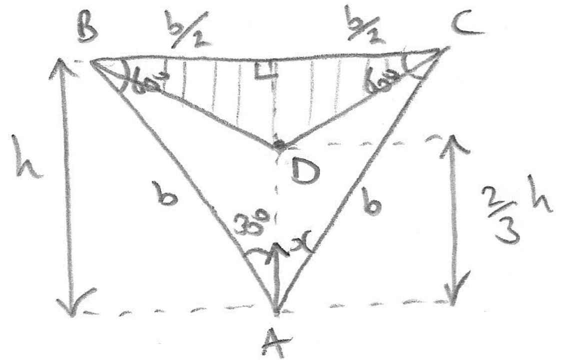
(i)



Consider a general triangle ABC. If you divide it into thin strips // BC then you can see the C.O.M of each strip must be on the median line AM. ∴ the C.O.M of the whole triangle must be on AM.

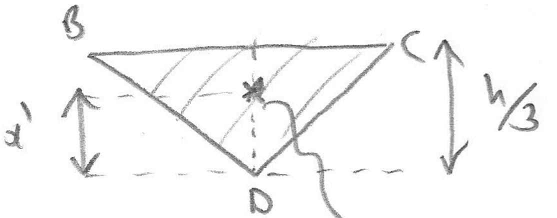
$\bar{x} = \dfrac{\sum m_i x_i}{\sum m_i}$

(ii)



Centre of mass $\bar{x}$ measured upwards from A is:

$$\boxed{\bar{x} = \frac{\frac{2}{3}h\,M_{ABC} - \left(\frac{2}{3}h + \frac{2}{3}\frac{1}{3}h\right)M_{BDC}}{M_{ABC} - M_{BDC}}}$$



Centre of mass

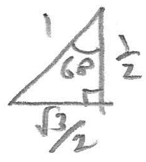
$x' = \frac{2}{3}\frac{h}{3} = \frac{2h}{9}$

If ρ is mass / area of plate

$M_{ABC} = \frac{1}{2}hb\rho$

$M_{BDC} = \frac{1}{2}\frac{h}{3}b\rho$

$h = b\sin 60^\circ = \frac{b\sqrt{3}}{2}$



So $\bar{x} = \dfrac{\frac{2}{3}h\cancel{\frac{1}{2}hb} - \frac{2}{3}\frac{5}{3}h\cancel{\frac{1}{2}\frac{h}{3}b}}{\cancel{\frac{1}{2}hb} - \cancel{\frac{1}{2}\frac{h}{3}b}}$

$\bar{x} = \dfrac{\frac{2}{3}h - \frac{10}{27}h}{1 - \frac{1}{3}}$

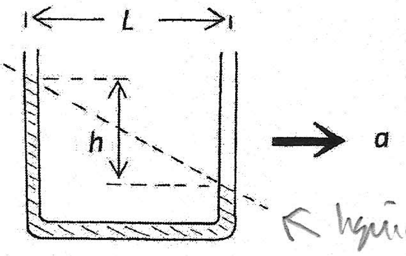
$\bar{x} = \dfrac{\frac{18-10}{27}h}{\frac{2}{3}}$

$\bar{x} = \frac{4}{3^3} \times 3 \times \frac{b\sqrt{3}}{2}$

$\boxed{\bar{x} = \frac{2\sqrt{3}}{9}b}$

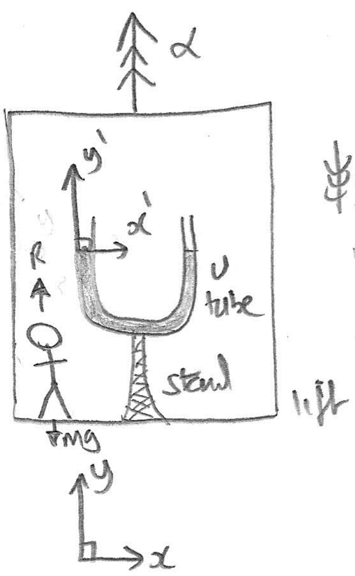
(h) A vertical U-tube, partially filled with liquid, is accelerated vertically upwards in a lift, acceleration α. What is the effective value of 'g', g_v? If the U-tube is mounted in a vehicle accelerating in a horizontal straight line, acceleration a, Figure 1.(h), what is the effective 'g', g_h? Express a in terms of the distance between the arms of the U-tube, L, and the difference in heights, h, of the liquid in the arms.

[7]



liquid surface must always be $\perp$ to "effective" $\underline{g}$ vector.

Figure 1.(h)



$g = 9.81\ \mathrm{m/s^2}$

For a man in the lift of mass M

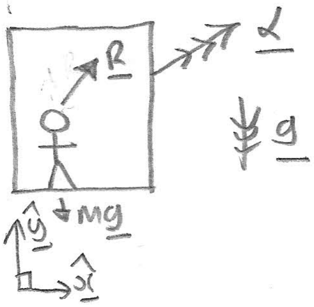
$\boxed{\text{Newton II: } M\alpha = R - Mg}$

$\Rightarrow$ Contact force $\boxed{R = M(\alpha+g)}$

So in the frame of the lift his effective weight $\boxed{Mg_v = R}$

so $\boxed{g_v = \alpha + g}$

Generalizing using vectors:



$\boxed{\text{Newton II}}$

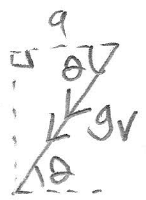
$M\underline{\alpha} = \underline{R} + M\underline{g}$

$\therefore$ if $\boxed{\underline{R} = -M\underline{g}_v}$

$\therefore$ $\boxed{\underline{g}_v = -\underline{\alpha} + \underline{g}}$

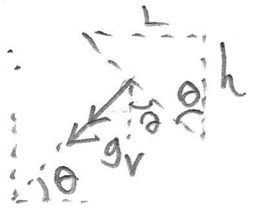
Using lab frame $\hat{\underline{x}}, \hat{\underline{y}}$ coordinates:

$$\underline{g}_v = -\begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -g \end{pmatrix} = \begin{pmatrix} -a \\ -g \end{pmatrix}$$



$g_v = \sqrt{a^2+g^2}$

Now in fig I:



so $\tan\theta = \dfrac{g}{a} = \dfrac{L}{h}$ $\Rightarrow$ $\boxed{a = gh/L}$

(i) In Figure 1.(i) a fixed mirror, a light source and a light receiver are all 0.30 km from a rotating mirror, with angular frequency ω. The distance between the light source and the receiver is 0.60 m. What is the lowest value of ω required for detection of the reflected light?

[4]

[Assume $\Delta\theta \approx \frac{0.6}{0.3\times10^3}$ rad]

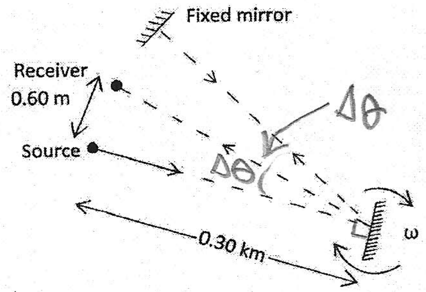


Figure 1.(i)

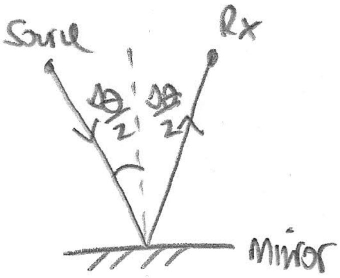
$c \rightsquigarrow c = 2.998\times10^{8}$ m/s

Travel distance from source to rotating mirror to fixed mirror to rotating mirror to receiver is $\boxed{4r}$

∴ total time is $\Delta t = \frac{4r}{c} = \frac{4\times0.30\times10^{3}\,\text{m}}{2.998\times10^{8}\,\text{m/s}}$ {If mirror static}

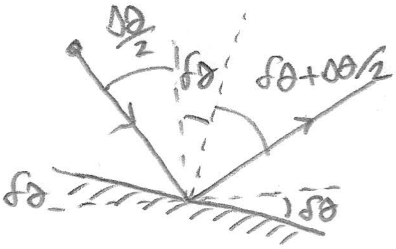
$= \boxed{4.00\times10^{-6}\,\text{s}}$

what happens if it rotates?



"optimum" geometry

→



rotation of mirror by δθ $= \omega\delta t$

Angular deviation of light to Rx is now $\boxed{2\delta\theta}$ from "optimum"

so if mirror rotates δθ, beam deviation is $\boxed{2\delta\theta}$

* To receive the signal via the fixed mirror, $\Delta t = \frac{4r}{c}$ (ie there and back time for fixed mirror) = time for rotating mirror to rotate $\Delta\theta/2$

+ source to Rx distance

So $\boxed{\Delta t\,\omega = \frac{\Delta\theta}{2}}$ $\therefore \omega = \frac{0.6}{0.3\times10^3} \Big/ 4.00\times10^{-6} = \boxed{500\,\text{rad}\,\text{s}^{-1}}$

b

v

(j) A car travelling at 90 km/hr in a straight line sounds its horn continuously, frequency f 400 Hz, as it passes a stationary observer. At the closest point, A, to the observer the car is at a distance D = 100 m from the observer. Determine the frequency heard by the observer when the car is:

(i) at A;

(ii) at a distance x from A, after passing A

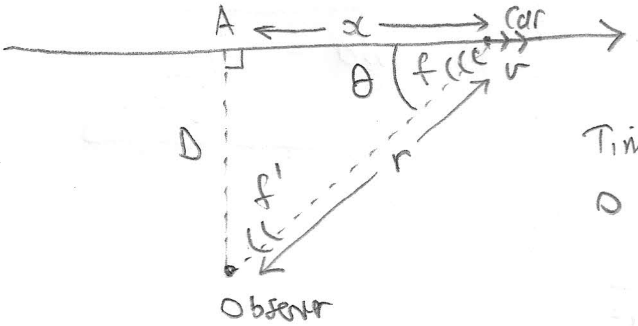
The velocity of sound is v_S = 343 ms^{-1}.

[8]

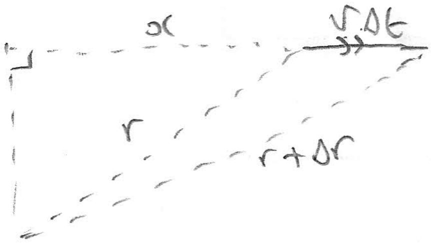
f is horn frequency at source

f' is horn frequency received

$$\boxed{\begin{aligned}\Delta t &= \tfrac{1}{f}\\ \Delta t' &= \tfrac{1}{f'}\end{aligned}}$$



Time between wave crests received at O is $\boxed{\Delta t' = \Delta t + \Delta r / v_S}$



Pythagoras

$$r^2 = x^2 + D^2$$

$$(r+\Delta r)^2 = (x+v\Delta t)^2 + D^2$$

$$\cancel{r^2} + 2r\Delta r + \Delta r^2 = \cancel{x^2} + 2xv\Delta t + v^2\Delta t^2 + \cancel{D^2}$$

$$\Delta r^2\left(1+\frac{2r}{\Delta r}\right) = v^2\Delta t^2\left(1+\frac{2x}{v\Delta t}\right)$$

Δr is extra distance travelled by sound in Δt

Now assume $\Delta r \ll r$ $\therefore$ $\boxed{\Delta r \approx v\Delta t\left(1+\frac{2x}{v\Delta t}\right)^{\frac{1}{2}}}$

$$\therefore \frac{1}{f'} = \frac{1}{f} + \frac{1}{f}\frac{v}{v_S}\left(1+\frac{2x}{v}f\right)^{\frac{1}{2}}$$

[v = 90 km/hr = $\boxed{25 \text{ m/s}}$]

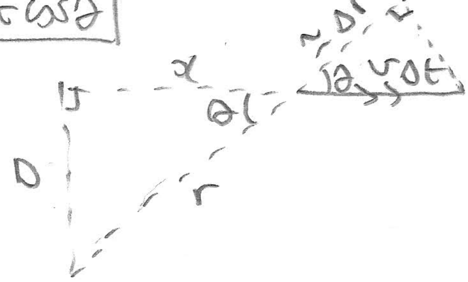
$$\therefore \boxed{f' = f\left(1 + \frac{v}{v_S}\sqrt{1+\frac{2xf}{v}}\right)^{-1}} \quad (*)$$

when $x=0$ (ie at A) : $\boxed{f' = \dfrac{f}{1+v/v_S}} \Rightarrow \dfrac{400}{1+\dfrac{90\times10^3}{3600\times343}}$

$= \boxed{373 \text{ Hz}}$

(ii) If use $\boxed{\text{plane wave approximation}}$ can turn (*) into traditional Doppler formula $\rightarrow$ PTO

Assume : $\boxed{\Delta r \simeq v \Delta t \cos\theta}$



$\therefore \quad \Delta t' = \Delta t + \Delta r / v_s$

$\Rightarrow \quad \frac{1}{f'} = \frac{1}{f} + \frac{v}{v_s f} \cos\theta$

$r\cos\theta = x$

$r^2 = D^2 + x^2$

$\therefore \quad \boxed{f' = f\left(1 + \frac{v}{v_s}\cos\theta\right)^{-1}}$

$\therefore \quad \boxed{\cos\theta = \frac{x}{\sqrt{D^2 + x^2}}}$

$\therefore \quad \boxed{f' = f\left(1 + \frac{v}{v_s}\frac{x}{\sqrt{D^2 + x^2}}\right)^{-1}}$

So $\quad \boxed{f' = \frac{400\,\text{Hz}}{1 + \frac{25.0}{343}\frac{x}{\sqrt{10^4 + x^2}}}}$

Example: when $x = 100\,\text{m}$

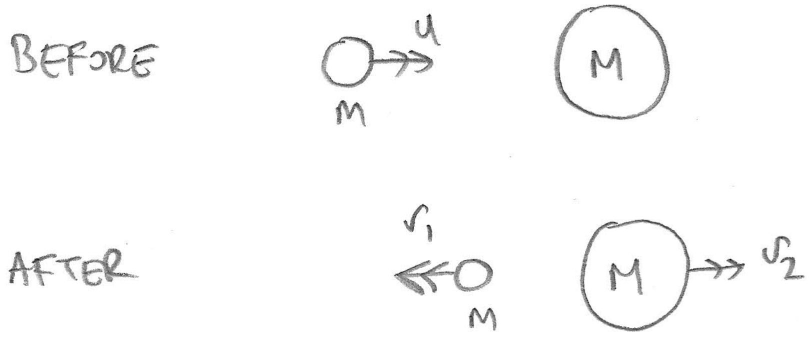
$f' = \frac{400\,\text{Hz}}{1 + \frac{25.0}{343} \times \frac{100}{\sqrt{10^4 + 100^2}}} = \boxed{380\,\text{Hz}}$

(k) A ball of mass m and velocity u collides elastically with a larger ball of mass M, initially at rest. The ball of mass m rebounds along its original line of motion with speed v_1 and the ball of mass M has velocity v_2 in the direction of u.

(i) Write down the conservation equations for the system.

(ii) Deduce the result that $u - v_1 = v_2$.

[5]



$\rightarrow$ +ve conservation of momentum $\boxed{mu = Mv_2 - mv_1}$ (i)

conservation of energy (since elastic)

$$\boxed{\tfrac{1}{2}mu^2 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}Mv_2^2}$$

[Alternatively restitution coefficient = 1 ∴ $\frac{\text{speed of separation}}{\text{speed of approach}} = 1$]

∴ $\frac{v_2 + v_1}{u} = 1$ ∴ $\boxed{u - v_1 = v_2}$

from momentum conservation equation:

$$m(u + v_1) = Mv_2$$

From energy equation: $m(u^2 - v_1^2) = Mv_2^2$

∴ $m(u+v_1)(u-v_1) = Mv_2^2$

∴ $\cancel{m(u+v_1)}(u-v_1) = \cancel{m(u+v_1)}\,v_2$

∴ $\boxed{u - v_1 = v_2}$

(I) A velocity selector, Figure 1.(I), consists of two slotted discs mounted on a common axis a distance d apart. The slots are displaced relative to each other by an angle θ. The axis is driven at an angular velocity ω. Particles in a horizontal beam, with all possible velocities, will get through the first slit, in the first disc, for a short time interval. To subsequently get through the second slit, particles must travel a distance d in the times it takes the second slot to line up with the beam. This will occur, for rotations of the second slit of θ, $2\pi+\theta$, $4\pi+\theta$,.. etc.

If d =1.00 m, ω = 24,000 rpm and θ = 60°, what are the speeds of those particles that pass through the velocity selector?

[5]

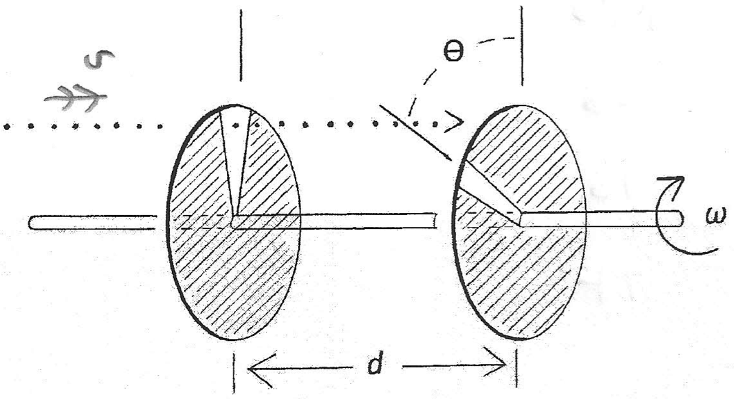


Figure 1.(I)

* If particle speed is v, it will take $\Delta t = d/v$ to travel between the discs.

* In this time, second slit has rotated $\Delta\theta = \omega \Delta t$ radians.

ie complete rotations from "upwards"

* For particles to get through slit #2, $\Delta\theta - \theta = 2\pi n$ where n is an integer.

So $\frac{\omega d}{v} - \theta = 2\pi n$

$\Rightarrow \frac{\omega d}{v} = 2\pi n + \theta$

$\Rightarrow \boxed{\frac{\omega d}{2\pi n + \theta} = v}$

So if $\omega = 24 \times 10^3 \times \frac{2\pi}{60}$ rads^{-1}

$\theta = \pi/3$ rad

$d = 1.00$ m

$\Rightarrow v = \frac{24 \times 10^3 \times \frac{2\pi}{60} \times 1.00}{2\pi n + \frac{\pi}{3} \times \frac{2}{2}}$ (m/s)

$\Rightarrow \boxed{v = \frac{400 \text{ m/s}}{n + \frac{1}{6}}}$ $n = 0, 1, 2 \ldots$

$$\Rightarrow \quad \boxed{v_n = \frac{2400\ \text{m/s}}{6n+1}}$$

n	v_n / ms^{-1}
0	2400
1	343
2	185
3	126
4	96
5	77