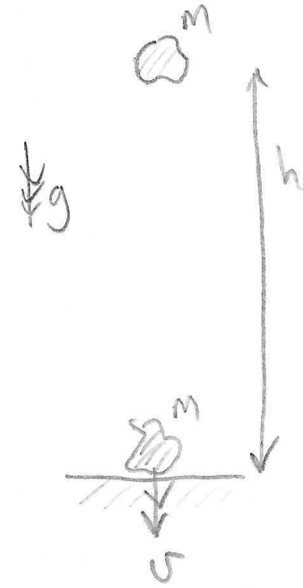
BPhO 1 2020 Section 1 AF SOLUTIONS

a) Estimate from what height, under free-fall conditions, a heavy stone would need to be dropped if it were to reach the surface of the Earth at the speed of sound of 330 $\mathrm{m\,s^{-1}}$. **[2]**



Assume $Mg \gg \frac{1}{2}C_D \rho A v^2$

ie weight >> air resistance

* A is the cross sectional area of the stone

* ρ is air density

∴ Acceleration (downwards) $= g = 9.81\ \mathrm{m/s^2}$

So $\boxed{v^2 = 2gh}$ "SUVAT"

∴ $h = \frac{v^2}{2g}$

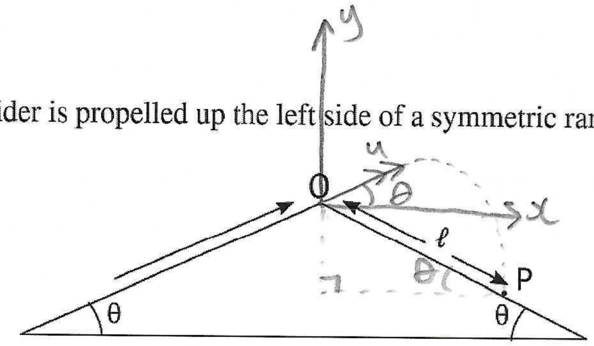
So $h = \frac{330^2}{2 \times 9.81}$ (m)

$= \boxed{5550\ \mathrm{m}}$

[Alternatively: $\underbrace{\tfrac{1}{2}Mv^2}_{KE} = \underbrace{Mgh}_{GPE}$ since no air resistance

$\Rightarrow \boxed{v^2 = 2gh}$]

b) A motorcycle rider is propelled up the left side of a symmetric ramp shown in **Figure 1**.



$\cancel{+}g = 9.81\,m/s^{-2}$

Figure 1

The rider reaches the apex of the ramp at speed of u, and falls to a point P on the descending ramp. In terms of u, θ and g, obtain expressions for,

(i) The time t_a for which the rider is airborne.

(ii) The distance OP ($= \ell$) along the descending ramp.

[4]

(i) Using Cartesian (x, y) coordinates centred on O (when $t=0$)

$$\boxed{\begin{aligned} x &= ut\cos\theta \\ y &= ut\sin\theta - \tfrac{1}{2}gt^2 \end{aligned}}$$

At point P: $\boxed{\begin{aligned} x &= l\cos\theta \\ t &= t_a \end{aligned}}$ so $l\cancel{\cos\theta} = ut_a \cancel{\cos\theta}$

$\therefore \boxed{t_a = \dfrac{l}{u}}$

(ii) At point P: $\boxed{y = -l\sin\theta}$

$$\therefore -l\sin\theta = ut_a\sin\theta - \tfrac{1}{2}gt_a^2$$

$$-\cancel{l}\sin\theta = \cancel{u}\frac{\cancel{l}}{\cancel{u}}\sin\theta - \tfrac{1}{2}g\frac{l^2}{u^2}$$

$$\tfrac{1}{2}g\frac{l}{u^2} = 2\sin\theta$$

$$\therefore \boxed{l = \frac{4u^2\sin\theta}{g}}$$

c) Two buckets hang from a rope over a frictionless pulley as in **Figure 2**. The bucket on the right has a mass m_2, which is greater than the mass of the bucket on the left m_1 ($m_2 > m_1$). Bucket 2 starts at height h above the ground. If the buckets are released from rest, determine:

(i) the speed with which **bucket 2** hits the ground in terms of m_1, m_2, h, and the acceleration due to gravity g, and

(ii) the further increase in height of **bucket 1** after bucket 2 hits the ground and stops.

Ignore resistive effects and assume the rope is long compared to the height above the ground.

Assume "light inextensible rope"

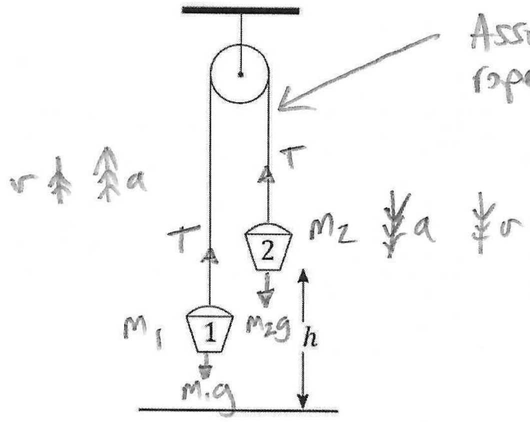


Figure 2

So M_2 goes down as much as M_1 goes up.

[4]

(i) NII for bucket 2: $\quad M_2 a = M_2 g - T \quad (1)$

" " " 1: $\quad M_1 a = T - M_1 g \quad (2)$

$(1)+(2)$: $\quad (M_2+M_1)a = (M_2-M_1)g \quad \therefore \boxed{a = \frac{M_2-M_1}{M_2+M_1} g}$

So constant acceleration motion:

$\Rightarrow$ Bucket 2 hit the ground at speed v where

$$v^2 = 2ah$$

$$\Rightarrow \boxed{v = \sqrt{2gh}\left(\frac{M_2-M_1}{M_2+M_1}\right)^{\frac{1}{2}}}$$

Bucket 1 is now moving with acceleration g downwards once Bucket 2 hits the floor and rope tension $\to 0$

So further height gain is Δh where $v^2 = 2g\Delta h$

$$\Rightarrow \Delta h = \frac{v^2}{2g} = \boxed{h\left(\frac{M_2-M_1}{M_2+M_1}\right)}$$

d) A rugby pitch lies in a north-south direction. In this question $\hat{\mathbf{i}}$ represents a unit vector due east, and $\hat{\mathbf{j}}$ represents a unit vector due north. Rugby player **Y** collects the ball and runs with a velocity $(3\hat{\mathbf{i}}+4\hat{\mathbf{j}})\ \mathrm{m\,s^{-1}}$. Player **X**, starting 20 m due east of player **Y**, immediately gives chase at a speed of $8\ \mathrm{m\,s^{-1}}$. She is an expert player and runs in a straight line to intercept player **Y**. Calculate

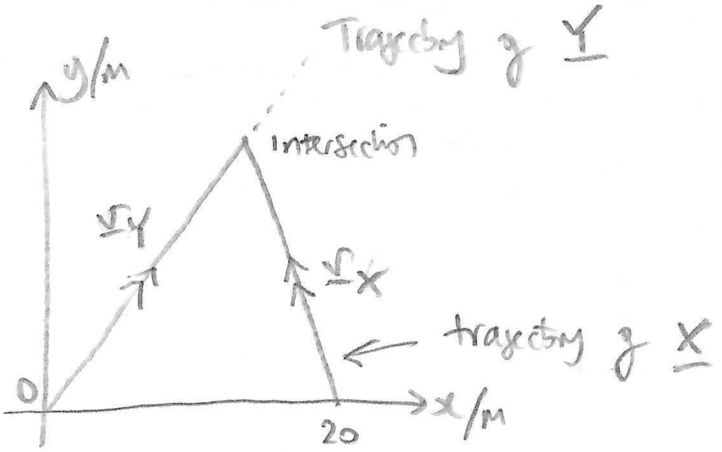
MS works out speed! i.e. 5 m/s $\quad \underline{v}_Y = \begin{pmatrix}3\\4\end{pmatrix}$ m/s

(i) the velocity of player **Y**,

(ii) the time taken for the players to meet, and

(iii) the displacement of **Y** from her original position at their point of contact.

[4]



$\boxed{\text{Velocities}}$

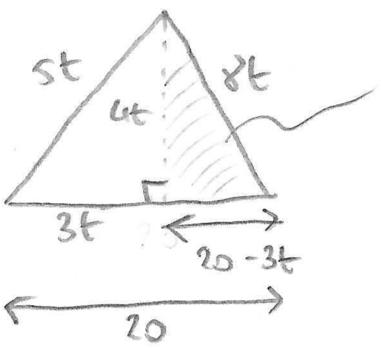
$\underline{v}_Y = \begin{pmatrix}3\\4\end{pmatrix}$ m/s and $|\underline{v}_Y| = 5$ m/s

$\underline{v}_X = \begin{pmatrix}-u\\v\end{pmatrix}$ m/s

where $\sqrt{u^2+v^2} = 8$

Positions relative to O after time t

$\underline{r}_X = \begin{pmatrix}3t\\4t\end{pmatrix} \qquad \underline{r}_Y = \begin{pmatrix}20-ut\\vt\end{pmatrix}$



Pythagoras to solve for t

$(8t)^2 = (4t)^2 + (20-3t)^2$

$64t^2 = 16t^2 + 400 - 120t + 9t^2$

$39t^2 + 120t - 400 = 0$

$t = \dfrac{-120 \pm \sqrt{120^2 - 4(39)(-400)}}{78}$

(we lost!)

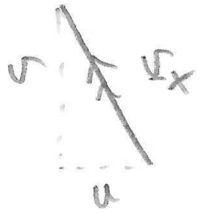
$= \dfrac{-120 + 277.13}{78}$

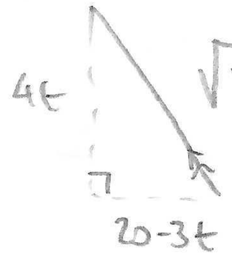
$= \boxed{2.014\ \text{s}} \rightarrow$ calc memory

Now intersection is $\underline{r}_x = \underline{r}_y = \begin{pmatrix}3t\\4t\end{pmatrix}$

$$= \begin{pmatrix}3\times 2.0145\ldots\\4\times 2.0145\ldots\end{pmatrix} \approx \begin{pmatrix}6\\8\end{pmatrix}$$

[Extra:





$$\sqrt{16t^2 + (20-3t)^2} = 16.12$$

So $\underline{v}_x = \begin{pmatrix}-20+3t\\4t\end{pmatrix} \times \frac{8}{16.12}$

$$= \begin{pmatrix}-6.93\\4\end{pmatrix} \text{ m/s} \quad]$$

So $u = 6.93$ m/s $\quad v = 4$ m/s

☺

d R

e) A long wire of uniform diameter 1.40 mm has a resistance of 0.478 Ω. It is wrapped into a ball in order to find its weight, which is 4.60 N. When weighed in water it is 4.08 N. Calculate the resistivity of the wire. ρ

Density of water = 1000 kg m^{-3}

$\left[\rho = 2.14 \times 10^{-8}\ \Omega m\right]$

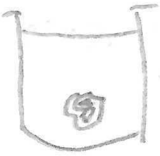
[3]

$R = \dfrac{\rho l}{\pi \left(\frac{d}{2}\right)^2}$

so $\boxed{\rho = \dfrac{\pi d^2 R}{4l}}$

So we need the length of the wire.

wrapped into ball.



when placed in water, the weight of water displaced is $(4.60 - 4.08)$ N

$= \boxed{0.52\,N}$

{ 4.08N is weight of 4.60N − upthrust; Upthrust is the weight of water displaced (Archimedes' principle) }

So $\boxed{\text{Volume of water displaced} = \text{Volume of wire}}$ ← key idea.

$(W = \rho V g)$

↳ so $V = \dfrac{W}{\rho g}$

$= \dfrac{0.52}{9.81 \times 1000}$ m^3

$= \boxed{5.3 \times 10^{-5}\ m^3}$

$\Rightarrow l = \dfrac{5.3 \times 10^{-5}}{\pi \times (0.7 \times 10^{-3})^2}$ (m) $= \boxed{34.4\ m}$

$\therefore \rho = \dfrac{\pi \times (1.4 \times 10^{-3})^2 \times 0.478}{4 \times 34.4} = \boxed{2.14 \times 10^{-8}\ \Omega m}$

* unpack problem in (t, v) graphs first! *

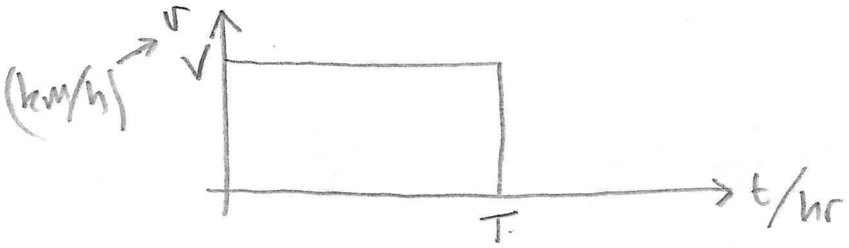
f) A train travels at a constant speed for one hour but is then delayed on the line for half an hour. When it restarts, its speed is reduced to 75% of its previous speed. It arrives at its destination $1\frac{1}{2}$ hours later than if it had travelled at its initial speed throughout. If the delay had occurred 45 km further on, then the train would only have been 1 hour late. Determine,

(i) the distance travelled, and x

(ii) the initial speed of the train. V

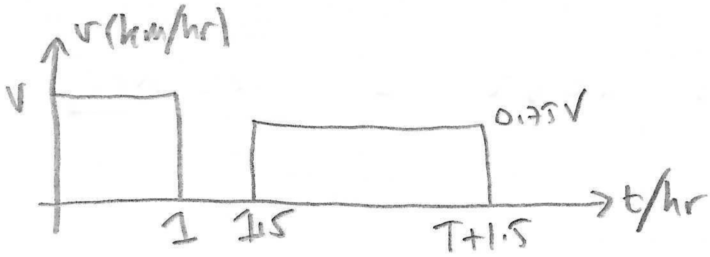
[4]

① $\boxed{\text{Constant speed throughout}}$



$\boxed{x = VT}$ (in km)

② $\boxed{\text{with delay}}$



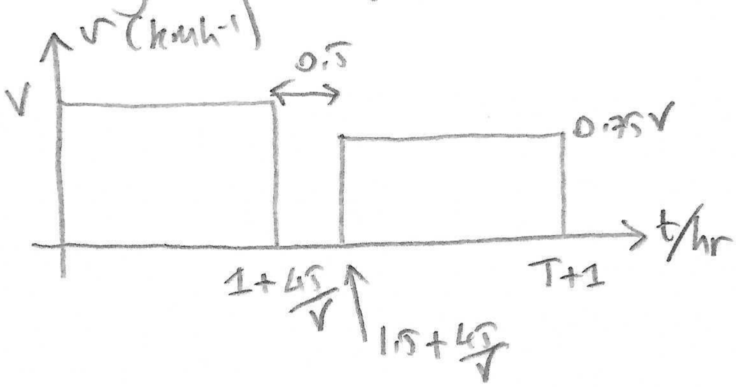
Area under (t, v) graph is distance travelled – which must be the same. So comparing areas ① and ②

$$VT = V(1) + \frac{3}{4}V(T+1.5-1.5)$$

$$T = 1 + \frac{3}{4}T$$

$$\frac{T}{4} = 1 \quad \therefore \quad \boxed{T = 4}$$

③ Delay 45 km further on is at $1 + 45/V$ hours



Comparing distance ③ to ①:

$$v\left(1+\frac{45}{v}\right) + \frac{3}{4}v\left(T+1-1.5-\frac{45}{v}\right) = vT$$

using $T = 4$:

$$1+\frac{45}{v} + \frac{3}{4}\left(5-1.5-\frac{45}{v}\right) = 4$$

$$\therefore \frac{45}{4v} = 4-1+\frac{9}{8}-\frac{15}{4} = \frac{3}{8}$$

$$\therefore v = \frac{45\times 8}{3\times 4} = 15\times 2 = \boxed{30} \text{ (km/h)}$$

So distance travelled is $x = vT = \boxed{120 \text{ km}}$

g) Emmy walks into a lift with a set of (bathroom) scales. She stands on the scales, presses the button for the 30^{th} floor, and starts a timer as the lift begins to move. She notices that the reading on the scales varies with time, according to the equation below: (!)

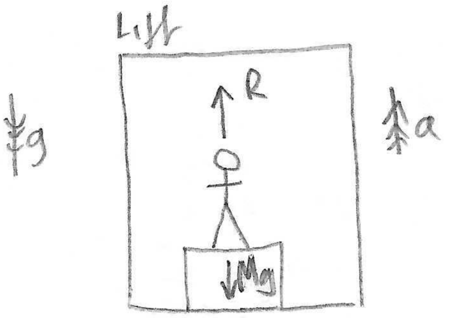
$$m(t) = 60\left(1 + \frac{t}{10} - \frac{t^2}{100}\right)$$

(i) Write down an expression for the acceleration of the lift as a function of time.

(ii) How fast is the lift moving after 10 seconds?

(iii) After the initial 10 seconds, the lift decelerates at a constant rate until it arrives at the 30^{th} floor. Given that the 30^{th} floor is 100 m above the ground, calculate the minimum value of the mass reading (in kg) shown on the scales during this deceleration.

[4]



NII: $\boxed{Ma = R - Mg}$ ①

where M is Emmy's actual mass $= \boxed{60\,kg}$ (i.e. $m(0)$).

Scale mass is $\boxed{m = R/g}$

since force on scales is equal and opposite to reaction force R on Emmy.

So since $R = M(a+g)$ from ①

$\therefore$ $m = \dfrac{M(a+g)}{g}$ $\therefore$ $\dfrac{mg - Mg}{M} = a$

$\therefore$ lift acceleration $\boxed{a = g\left(\frac{m}{M} - 1\right)}$

$\therefore$ Since $M = m(0) = 60\,kg$

(i) $\boxed{a(t) = g\left(\frac{t}{10} - \frac{t^2}{100}\right)}$

(ii) $v = \int_0^t a(t')\,dt' = \boxed{g\left(\frac{t^2}{20} - \frac{t^3}{300}\right)}$ ($v=0$ at $t=0$)

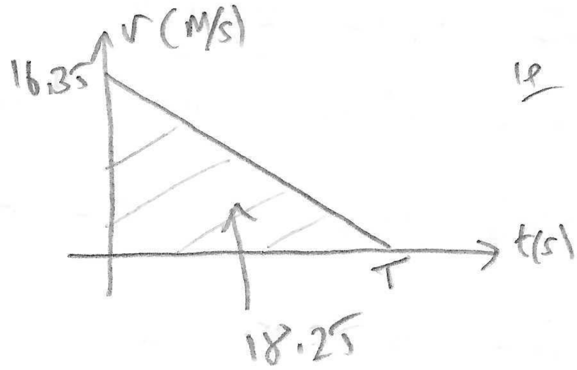
So $v(10) = 9.81 \times \left(\frac{10^2}{20} - \frac{10^3}{300}\right) = \boxed{16.4\,m/s}$ (16.35)

(ii) During first 10s the lift has travelled $x_{10} = \int_0^{10} v(t')\,dt'$

$$= \left[g\left(\frac{t^3}{60} - \frac{t^4}{1200} \right) \right]_0^{10} = 9.81 \left(\frac{1000}{60} - \frac{10{,}000}{1200} \right)$$

$$= \boxed{81.75 \text{ m}}$$

So if 30th floor is at 100m, this means 18.25m to go. (v,t) graph for this final phase is:



ie constant deceleration motion

[Could also use $v^2 = 2ax$

$\therefore a = \frac{16.35^2}{2 \times 18.25}$

$= 7.32 \text{ m/s}^2$]

$\frac{1}{2}(T)(16.35) = 18.25 \quad \therefore T = \frac{18.25 \times 2}{16.35}$ (s)

$$= \boxed{2.23 \text{ s}}$$

$\therefore$ Constant (deceleration) $= \frac{16.35}{2.23}$ m/s²

$$= \boxed{7.32 \text{ m/s}^2}$$

So since $\boxed{m = 60\text{ kg} \times \left(\frac{a}{g} + 1 \right)}$ (mass on scales)

and (in this case) $a = -7.32 \text{ m/s}^2$, minimum m in this phase is:

$$m_{min} = 60\text{ kg} \times \left(\frac{-7.32}{9.81} + 1 \right) = \boxed{15.2 \text{ kg}}$$

h) Two transparent miscible liquids of refractive indices $n_a = 1.15$ and $n_b = 1.52$ can be mixed together to produce a liquid of refractive index n by mixing volumes V_a and V_b of the liquids. The refractive index of the mixture varies linearly with the volumes of the two liquids. The refractive index of powdered glass, n_g, poured into the mixture can be found by adjusting the liquid mixture until the powdered glass cannot be observed in the liquid.

(i) Obtain an expression for n_g in terms of n_a, V_a, n_b and V_b.

(ii) If the powdered glass is poured into 100 ml of liquid A and is seen to disappear when 64 ml of liquid B is added, what is the refractive index of the glass?

[3]

Composite liquid from a and b has refractive index

(i)

$$\boxed{n = \frac{n_a V_a}{V_a + V_b} + \frac{n_b V_b}{V_a + V_b}} \quad *$$

ie $\frac{V_a}{V_a + V_b}$ is the volume proportion of a to the total volume $V_a + V_b$.

(ii) Powdered glass vanishes when $n = n_g$ (ie there is no reflection at a boundary of refractive index)

So $$n_g = \frac{1.15 \times 100 + 1.52 \times 64}{100 + 64}$$

$$= \boxed{1.29}$$

* which simplifies to $$\boxed{n = \frac{n_a V_a + n_b V_b}{V_a + V_b}}$$

(A bit like centre of mass calculations $R = \frac{\sum r_i M_i}{\sum M_i}$).

i) (i) Five resistors, $R_1 \ldots R_5$ are connected in a circuit between points **A** and **B**, as in **Figure 3**. Resistors R_2 and R_4 can be changed in value to be connecting wires with $R = 0$, finite values, or open circuit with $R = \infty$. Write down the values of R_2 and R_4 so that the network between **A** and **B** is equivalent to

i. three resistors in series,
ii. three resistors in parallel, and
iii. two identical resistors in parallel.

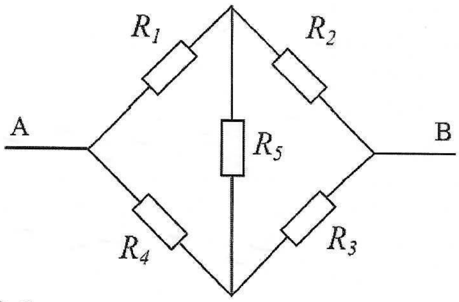
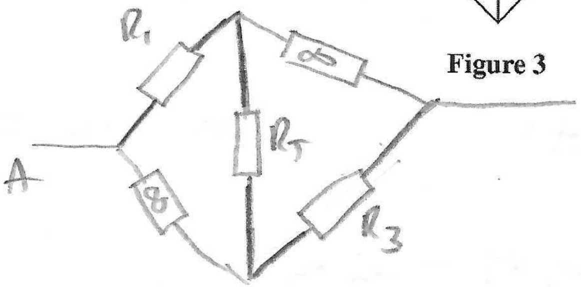


Figure 3

(i)



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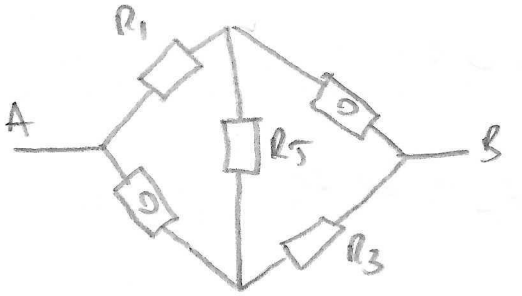
ie $\boxed{R_2 = R_4 = \infty}$

(ii)



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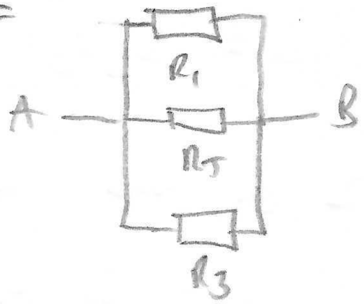


ie $\boxed{R_2 = R_4 = 0}$

(iii) Balance wheatstone Bridge if $\boxed{R_1 R_3 = R_2 R_4}$

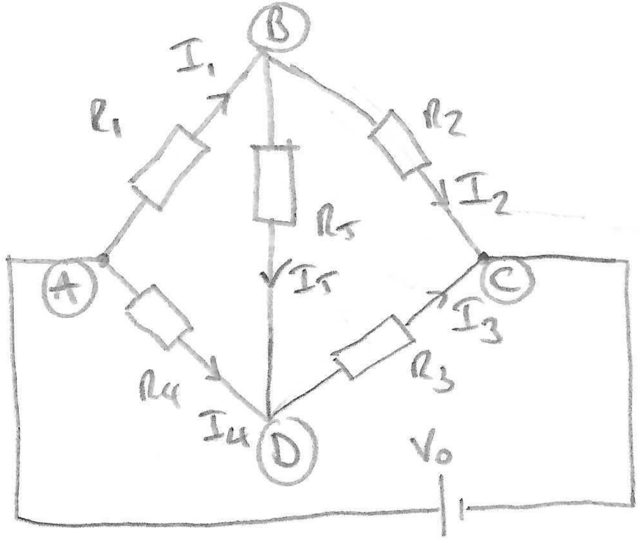
ie PD across $R_5 = 0$

So equivalent circuit is

$R_1 + R_2 = \boxed{2R_1}$

$R_4 + R_3 = \boxed{2R_1}$

If $R_1 + R_2 = R_4 + R_3$ AND
$R_1 R_3 = R_2 R_4 \Rightarrow$
all resistors the same ie $R_2 = R_3$ and $R_4 = R_1$ to make the balance work



Extra: $\boxed{\text{Wheatstone bridge circuit}}$ derivation.

{ Note from potential divider idea:

$$R_5 I_5 = V_0\left(\frac{R_2}{R_1+R_2} - \frac{R_3}{R_3+R_4}\right)$$ }

Kirchoff I at B : $I_1 = I_2 + I_5$ (1)

D : $I_3 = I_4 + I_5$ (2)

Kirchoff II for loop ABDA: { Note no EMF ! }

$$0 = I_1 R_1 + I_5 R_5 - I_4 R_4 \quad (3)$$

Kirchoff II for loop DCBD :

$$0 = I_3 R_3 - I_2 R_2 + I_5 R_5 \quad (4)$$

Kirchoff II for loop cell → ADC → cell

$$V_0 = I_4 R_4 + I_3 R_3 \quad (5)$$

If bridge is balanced $\boxed{I_5 = 0}$ ∴ $I_1 R_1 = I_4 R_4$ from (3)

$I_2 R_2 = I_3 R_3$ from (4)

From (1) and (2): $I_1 = I_2$ and $I_3 = I_4$ if $I_5 = 0$

So $I_1 R_1 = I_3 R_4$ (6) ∴ $\frac{(6)}{(7)}$ $\frac{R_1}{R_2} = \frac{R_4}{R_3}$

$I_1 R_2 = I_3 R_3$ (7)

or $\boxed{R_1 R_3 = R_2 R_4}$ to balance the bridge

(ii) **Figure 4** shows a simple circuit with two cells and three resistors. If the current in the ammeter shown in the figure is 2.0 A determine the unknown e.m.f. ε, of the battery, assuming the batteries and ammeter have no internal resistance.

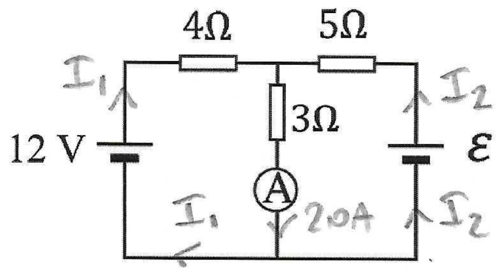


Figure 4

[4]

KI: $I_1 + I_2 = 2.0$ (1)

KII: (left loop) : $12 = 4I_1 + 3 \times 2.0$ (2)

$\therefore$ $I_1 = \frac{12-6}{4} = \boxed{1.5\text{ A}}$

$\therefore$ in (1): $I_2 = 2.0 - 1.5 = \boxed{0.5}$ A

KII (Right loop) : $\varepsilon = 5I_2 + 2.0 \times 3$

$\varepsilon = 5 \times 0.5 + 2.0 \times 3$

$= \boxed{8.5}$ (V)

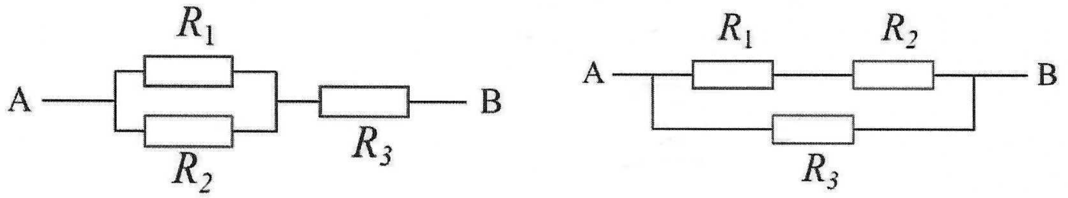
Check: PD across 3Ω resistor is $3\Omega \times 2.0\text{A} = \boxed{6V}$

$\therefore$ PD across 4Ω resistor is $12 - 6 = \boxed{6V}$

So $I_1 = 6/4 = \boxed{1.5A}$

j) (i) In **Figure 5a**, resistors R_1, R_2 and R_3 are connected between **A** and **B**. Derive an expression for R_3 in terms of R_1 and R_2, if the equivalent resistance, R_{AB} is equal to R_1.

(ii) A different arrangement is shown in **Figure 5b** for resistors R_1, R_2 and R_3. Again the equivalent resistance, $R_{AB} = R_1$, and the ratio $\frac{R_3}{R_2} = 6$. Determine the value of the ratio of $\frac{R_1}{R_2}$.



(a) **(b)**

Figure 5

[4]

(i) $R_{AB} = \dfrac{1}{\frac{1}{R_1}+\frac{1}{R_2}} + R_3$

if $R_{AB} = R_1$ $\therefore$ $R_3 = R_1 - \dfrac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$

$\therefore R_3 = R_1 - \dfrac{R_1R_2}{R_1+R_2} = \dfrac{R_1^2 + \cancel{R_1R_2} - \cancel{R_1R_2}}{R_1+R_2}$

$\therefore$ $\boxed{R_3 = \dfrac{R_1^2}{R_1+R_2}}$

{A "Show that" would have been useful here}

(ii) $R_1 = R_{AB} = \dfrac{1}{\frac{1}{R_1+R_2}+\frac{1}{R_3}}$

$\Rightarrow \dfrac{1}{R_1+R_2} + \dfrac{1}{R_3} = \dfrac{1}{R_1}$

$\Rightarrow \dfrac{1}{\frac{R_1}{R_2}+1} + \dfrac{1}{R_3/R_2} = \dfrac{1}{R_1/R_2}$

let $x = R_1/R_2$ and $R_3/R_2 = a = 6.$

So $\dfrac{1}{x+1} = \dfrac{1}{x} - \dfrac{1}{a}$

$\dfrac{1}{x+1} = \dfrac{a-x}{ax}$

$x+1 = \dfrac{ax}{a-x}$

$\cancel{ax} - x^2 + a - x = \cancel{ax}$

$x^2 + x - a = 0$

$\left(x+\frac{1}{2}\right)^2 - a - \frac{1}{4} = 0$

$\boxed{x = -\frac{1}{2} + \sqrt{a+\frac{1}{4}}}$ (+ve root)

So if $a = 6$:

$$x = \frac{R_1}{R_2} = -\frac{1}{2} + \sqrt{\frac{25}{4}}$$

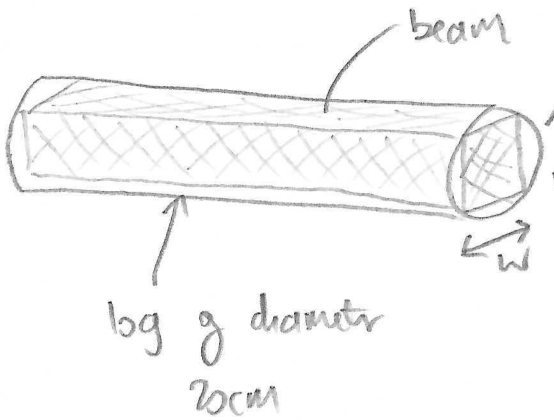
$$= -\frac{1}{2} + \frac{5}{2}$$

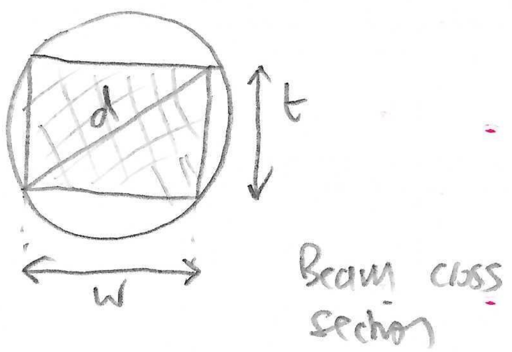
$$= \boxed{2}$$

k) The stiffness, S, of a beam of rectangular cross-section, with width w, and thickness t, is directly proportional to its width and the cube of its thickness, (t^3); that is, stiffness, $S \propto wt^3$. Determine the cross-sectional dimensions of the stiffest rectangular cross-section wooden beam that can be cut from a log of diameter 20 cm.

d

[3]

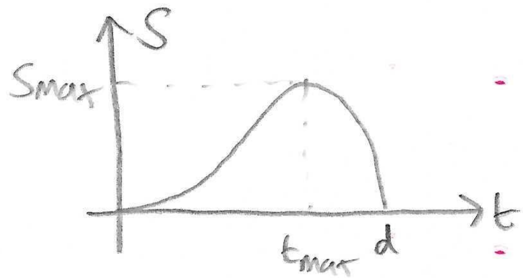




Pythagoras: $d^2 = w^2 + t^2$ so $\boxed{w^2 = d^2 - t^2}$ (1)

Stiffness $\boxed{S = kwt^3}$ k constant of proportionality

$\therefore$ $\boxed{S = k(d^2 - t^2)^{\frac{1}{2}} t^3}$



$t = t_{max}$ when $S = S_{max}$

ie when $\frac{dS}{dt} = 0$

$$\frac{dS}{dt} = \frac{1}{2}k(d^2 - t^2)^{-\frac{1}{2}}(-2t)t^3 + 3k(d^2 - t^2)^{\frac{1}{2}}t^2$$

$\Rightarrow \frac{dS}{dt} = 0$ when $\frac{t^4}{\sqrt{d^2 - t^2}} = 3t^2\sqrt{d^2 - t^2}$

$\Rightarrow t^2 = 3(d^2 - t^2)$

$4t^2 = 3d^2$

$\boxed{t_{max} = d\frac{\sqrt{3}}{2}}$

$\therefore W_{max} = \sqrt{d^2 - \frac{3}{4}d^2} = \boxed{\frac{d}{2}}$

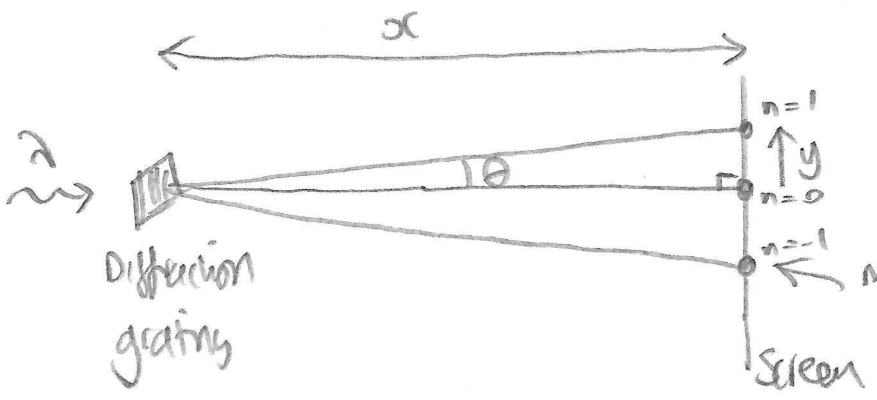
So if $d = 20cm$

$\boxed{\begin{array}{l} t_{max} = 17.32cm \\ W_{max} = 10cm \end{array}}$

only three spots seen.

1) A narrow beam of ~~monochromatic~~ light is passed normally through a diffraction grating of 6×10^5 lines per metre. Three spots of light are observed on a screen placed 80 cm away from the grating, and the outer spots are 30 cm away from the central spot. Determine the wavelength of the light used.

[2]

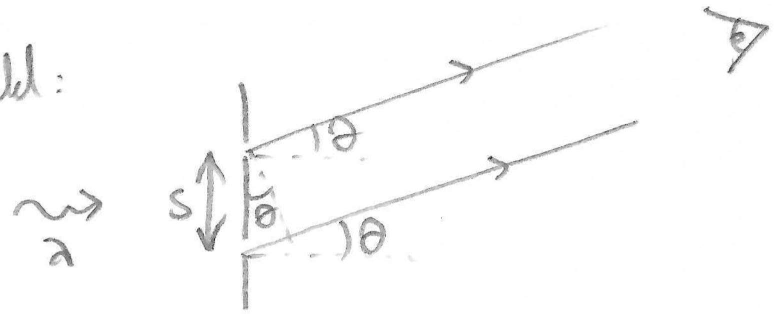


$y_1 = 30\,\text{cm}$

$x = 80\,\text{cm}$

Geometry: $\boxed{y = x\tan\theta}$

Grating far field:



Constructive interference if $\boxed{s\sin\theta = n\lambda}$

So maxima at $\boxed{\theta_n = \sin^{-1}\left(\frac{n\lambda}{s}\right)}$

Slit spacing $\boxed{s = \frac{1\,\text{m}}{\#\text{ lines per m}}}$

So for $n=1$ spot: $\theta_1 = \tan^{-1}\left(\frac{30}{80}\right) = \boxed{20.56^\circ}$

$\therefore\ \lambda = \frac{1.0\,\text{m}}{6\times10^5}\times\sin\theta_1 = 5.85\times10^{-7}\,\text{m}$

$= \boxed{585\,\text{nm}}$

m) Large craters can be produced on the Earth by meteorites. The size of a crater with diameter d is dependent on the kinetic energy of the meteorite E, the density of the rock removed from the crater ρ, and the field strength g, since the rock must be lifted out of the crater. We can express this as

$$d = kE^{\alpha}\rho^{\beta}g^{\gamma}$$

where k is a numeric constant and $k \approx 1$. k dimensionless

(i) By considering the dimensions (or units) of the quantities above, obtain an expression relating the diameter of the crater to E, ρ and g.

(ii) The Barringer Crater in Arizona was made by a meteorite that landed there 30 000 years ago. It has a diameter of 1200 m and is in rock of typical density 3000 kg m^{-3}. If the impact speed was 15 km s^{-1}, estimate the mass of the meteorite.

(iii) If the spherical meteorite was made of iron of density 8000 kg m^{-3} what was its diameter?

[6]

(i)

$[d] = m$ $\qquad$ $[E] = [\frac{1}{2}mv^2] = kg\, m^2 s^{-2}$

$[E^{\alpha}\rho^{\beta}g^{\gamma}] = kg^{\alpha} m^{2\alpha} s^{-2\alpha}\, kg^{\beta} m^{-3\beta}\, m^{\gamma} s^{-2\gamma}$

Comparing powers:

$m: \quad 1 = 2\alpha - 3\beta + \gamma \quad (1)$

$kg: \quad 0 = \alpha + \beta \quad (2)$

$s: \quad 0 = -2\alpha - 2\gamma \quad (3)$

$(2): -\alpha = \beta \qquad (3)\ \gamma = -\alpha$

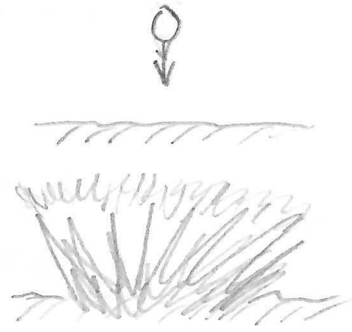
$\therefore$ in (1): $1 = 2\alpha + 3\alpha - \alpha \Rightarrow \boxed{\alpha = \frac{1}{4}}$

$\therefore \boxed{\beta = -\frac{1}{4}}, \boxed{\gamma = -\frac{1}{4}}$

So $\boxed{d = k\left(\frac{E}{\rho g}\right)^{\frac{1}{4}}}$

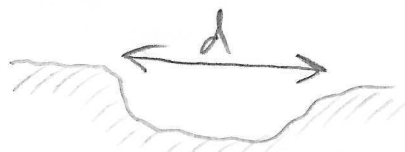
(ii) Asteroid that made $\boxed{\text{Barringer crater}}$

$$\left.\begin{aligned} d &= 1200\,m \\ \rho &= 3000\ kg/m^3 \\ v &= 15{,}000\,m/s \end{aligned}\right\} \quad \text{So } E = \tfrac{1}{2} m v^2$$



using $\boxed{d \simeq \left(\dfrac{E}{\rho g}\right)^{\frac{1}{4}}}$ if assume $k \approx 1$

$\therefore \quad d^4 \rho g = \frac{1}{2} m v^2$



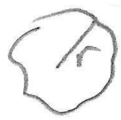
Impact crater

$\therefore \quad \boxed{m = \dfrac{2 d^4 \rho g}{v^2}}$

$$\therefore \quad m = \frac{2 \times (1200)^4 \times 3000 \times 9.81}{15{,}000^2} \quad \text{(kg)}$$

$$= \boxed{5.42 \times 10^8\ kg}$$

(iii) If the meteorite was diameter $2r$



Assume ≈ spherical

$\boxed{m = \frac{4}{3}\pi r^3 \rho_m}$. $\quad \rho_m = 8000\ kg/m^3$.

$$\therefore \quad r = \left(\frac{3m}{4\pi\rho_m}\right)^{\frac{1}{3}}$$

$$\therefore \text{ Diameter is } \boxed{2\left(\frac{3m}{4\pi\rho_m}\right)^{\frac{1}{3}}}$$

$$= 2\left(\frac{3 \times 5.42 \times 10^8}{4\pi \times 8000}\right)^{\frac{1}{3}} = 50.6\,m \simeq \boxed{51\,m}$$

v L x

n) The petrol engine of a car consumes 5.3 litres of petrol for every 100 km travelled at a speed of $100\,\mathrm{km\,h^{-1}}$ with the outside temperature being 16 °C. The heat of combustion of the petrol is 30 MJ per litre, and 23 % of this energy finds its way to the water cooling system. Calculate the mass rate of flow of cooling water such that the temperature rise of the cooling water is limited to 40 °C.
Specific thermal capacity of water $= 4180\,\mathrm{J\,kg^{-1}\,°C^{-1}}$.

[4]

* Per litre of fuel used, $30\times10^6\times0.23$ J results in heating water.

* Mass of water is M such that:

$$\underbrace{30\times10^6\times0.23}_{\text{Energy supplied to water}} = \underbrace{4180\times M\times(40-16)}_{cm\Delta T}$$

Assume water entering the cooling system is at 16 °C:

$$\therefore\ M = \frac{30\times10^6\times0.23}{4180\times24}\ (\mathrm{kg})$$

$$= \boxed{68.9\ \mathrm{kg}}\ \text{of water}\ \boxed{\text{per kg of fuel used.}}$$

Now it takes $t = x/v$ seconds to travel x metres

So 1 hour $= \boxed{3600\,s}$ to travel 100 km and uses 5.3 litres of fuel.

$\therefore$ $\boxed{\text{time per litre}}$ is: $\dfrac{3600\,s}{5.3} = \boxed{679.2\,s}$

$\therefore$ Mass rate of cooling water flow is $\dfrac{68.9\ \mathrm{kg}}{3600\,s/5.3} = \boxed{0.101\ \mathrm{kg/s}}$

or $\boxed{365\ \mathrm{kg/hour}}$

* Easy! *

o) Stefan's Law states that for a given perfectly radiating surface at an absolute temperature T, the radiated power Φ is directly proportional to T^4. The radiated energy is distributed over a range of wavelengths of electromagnetic radiation, with the peak in emission occurring at λ_{max}, as shown in **Figure 6**. The value of λ_{max} is determined by Wien's Law which states λ_{max} is inversely proportional to absolute temperature. Determine:

(i) The ratio of the radiative powers of the surface at 500 °C and 1000 °C, i.e. evaluate Φ_{1000}/Φ_{500},

(ii) the wavelength of maximum emission at 1000 °C if the wavelength of maximum emission at 500 °C is 3750 nm.

[4]

$\boxed{\Phi = \sigma A \varepsilon T^4}$ ← kelvin

(A area, ε emissivity)

(i) $\dfrac{\Phi_{1000}}{\Phi_{500}} = \left(\dfrac{1000+273}{500+273}\right)^4$

$= \boxed{7.36}$

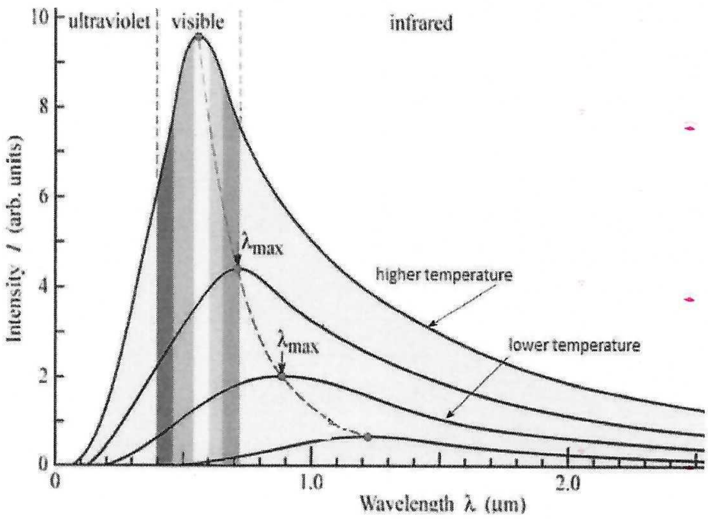


Figure 6

(ii) Wien's law

$\boxed{\lambda_{max} = \dfrac{b}{T}}$

So $\boxed{\dfrac{\lambda_{max}}{3750\,\text{nm}} = \left(\dfrac{T}{773\,\text{K}}\right)^{-1}}$

∴ if $T = 1273\,\text{K}$

$\Rightarrow \lambda_{max} = 3750\,\text{nm} \times \left(\dfrac{1273}{773}\right)^{-1}$

$= \boxed{2277\,\text{nm}}$

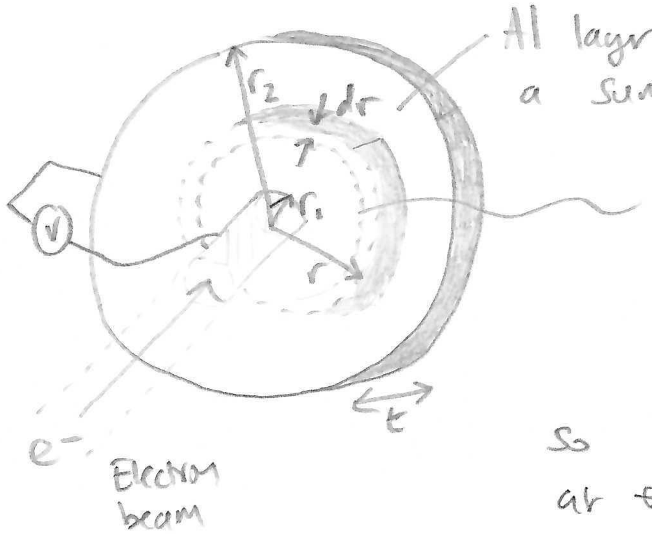
i.e. $\boxed{2280\,\text{nm}}$ to 3 s.f.

p) Electron beam lithography is used for etching microscopic patterns on surfaces. Typically, a 5 nm layer of aluminium deposited on an insulating substrate will remove the incident charge. If the beam current is 1.0 nA, and the electron beam of width 15 nm is incident centrally on a circular aluminium covered surface of diameter 5.0 cm, calculate the electrical resistance and potential difference between the edge of the spot and the edge of the surface.

Resistivity of aluminium = $2.8 \times 10^{-8}\,\Omega\,\mathrm{m}$.

[5]

t   I $2r_1$ $2r_2$ ρ



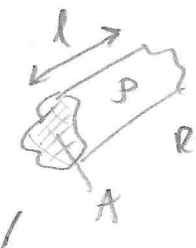
Al layer is a disc, composed of a sum of conducting rings of width dr and cross sectional area $2\pi r t$

* Each ring is effectively a resistor in series *

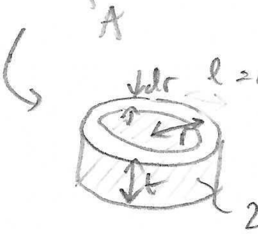
So total resistance of disc, from r_1 at the beam edge, to r_2 at the disc edge is:

$$R = \int_{r_1}^{r_2} \frac{\rho\, dr}{2\pi r t}$$

ie



$R = \rho \frac{l}{A}$



$l = dr$

$2\pi r t = A$

$$\therefore \boxed{R = \frac{\rho}{2\pi t} \ln\left(\frac{r_2}{r_1}\right)}$$

$$\therefore R = \frac{2.8 \times 10^{-8}}{2\pi \times 5 \times 10^{-9}} \ln\left(\frac{5.0 \times 10^{-2}}{15 \times 10^{-9}}\right) = \boxed{13.4\,\Omega}$$

$\therefore$ PD is $V = IR = 1.0 \times 10^{-9} \times 13.4\,\Omega$

$= \boxed{13.4\,\mathrm{nV}}$

$(1.34 \times 10^{-8}\,\mathrm{V})$

So "internal resistance" r is in // with capacitor

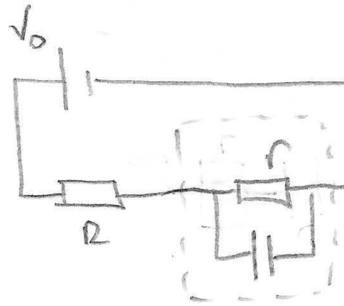
q) A leaky capacitor is one which does not have a perfectly insulating dielectric layer between the plates upon which the charge is stored. Such a capacitor contains a dielectric material filling the space between the plates, and with an effective resistivity ρ of $1.5 \times 10^{12}\,\Omega\,\mathrm{m}$. The area of the plates is $A = 0.603\,\mathrm{m}^2$, and the dielectric film thickness $d = 0.82\,\mu\mathrm{m}$. The capacitor is charged from a 24 V supply ($\uparrow V_0$) connected in series with a $4.7\,\mathrm{M\Omega}$ (R) resistor.

(i) Calculate the maximum charge that can be accumulated on one of the capacitor plates. [For a parallel plate capacitor, $C = \dfrac{\epsilon_0 A}{d}$]

(ii) If the capacitor is disconnected from the circuit, calculate the time taken for the capacitor to lose half its maximum charge.

[5]

Model charging circuit as:



$\boxed{r = \rho \dfrac{d}{A}}$

"leaky capacitor effective resistance"

"leaky capacitor"

$$\therefore\ r = \frac{1.5\times10^{12} \times 0.82\times10^{-6}}{0.603} \quad (\Omega)$$

$$= \boxed{2.04\times10^{6}\ \Omega}$$

Potential divider

(i) Maximum voltage across capacitor is $\boxed{V_c = \dfrac{r}{R+r} V_0}$

Since leaky capacitor effective resistance is in series with resistance R of charging circuit.

So using $\boxed{Q = CV_c}$, max charge is:

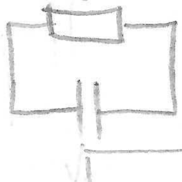
$$Q_{max} = \frac{\varepsilon_0 A}{d} \times \underbrace{\frac{2.04}{4.7+2.04} \times 24}_{7.26\ \text{Volts}} \qquad (C)$$

$\boxed{C = 6.51\times10^{-6}\ \mathrm{F}}$

$$= \frac{8.85\times10^{-12} \times 0.603}{0.82\times10^{-6}} \times 7.26 \qquad (C)$$

$$= \boxed{4.7\times10^{-5}\ \mathrm{C}} \qquad (47\,\mu\mathrm{C})$$

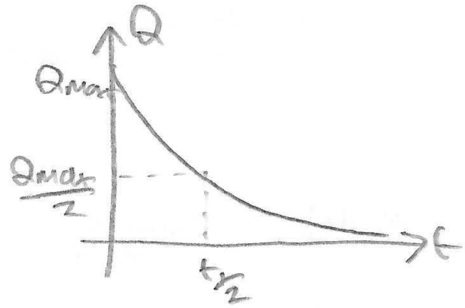
(ii) Discharge situation:



← Another hint why r is // to -||-

Charge on capacitor is: $\boxed{Q = Q_{max} e^{-t/rC}}$

So when $Q = \frac{Q_{max}}{2}$, $t = t_{\frac{1}{2}}$



$2 = e^{t_{\frac{1}{2}}/rC}$

$$\boxed{\ln 2 \times rC = t_{\frac{1}{2}}}$$

$\therefore \quad t_{\frac{1}{2}} = \ln 2 \times 2.04\times10^{6} \times 6.51\times10^{-6}$ (s)

$= \boxed{9.21}$ (s)

r) A fixed mass of gas expands isothermally and the relationship between the pressure p, and the volume of the gas V, is $pV = 380\,\mathrm{Pa\,m^3}$. The volume increases at a rate of $0.005\,\mathrm{m^3\,s^{-1}}$ when the volume is $0.17\,\mathrm{m^3}$. At what rate does the pressure decrease at this point?

k

$\frac{dV}{dt}$

ie $\frac{dp}{dt}$

[3]

$$\boxed{pV = k}$$

$$\therefore \quad p\frac{dV}{dt} + V\frac{dp}{dt} = 0$$

$$\therefore \quad \boxed{\frac{dp}{dt} = -\frac{p}{V}\frac{dV}{dt}} = \boxed{-\frac{k}{V^2}\frac{dV}{dt}}$$

$$\text{So} \quad \left.\frac{dp}{dt}\right|_{V=0.17\,m^3} = -\frac{380}{0.17^2} \times 0.005 \quad (Pa/s)$$

$$= -65.7\ Pa/s$$

So rate of pressure <u>decrease</u> of $\simeq \boxed{66\ Pa/s}$

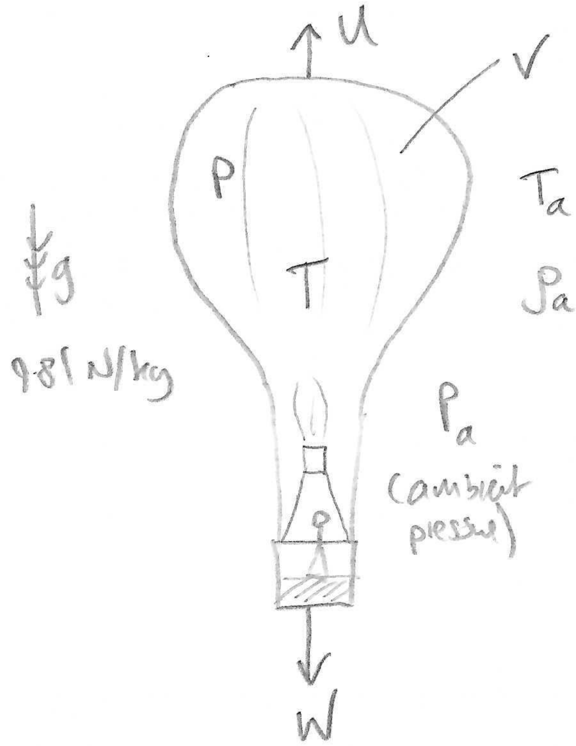
M

s) A hot air balloon uses a gas flame to heat the air it contains to a temperature required to enable it to hover at a small distance above ground level. The mass of the balloon, ropes, basket and riders is 240 kg and the volume of the balloon is 1100 m^3. The temperature of the surrounding air is 15 °C and its density is 1.23 kg m^{-3}. ✓

$T_a - 273$

To what temperature does the air in the balloon need to be heated? **[5]**

ρ



In equilibrium (to enable hovering)

Upthrust U = weight W

$$\boxed{\underbrace{\rho_a V g}_{\text{weight of ambient (cold) air displaced}} = \underbrace{mg}_{\text{weight of balloon, ropes basket, riders}} + \underbrace{Mg}_{\text{weight of hot air in balloon.}}}$$

Assume gas in balloon is ideal. So $\boxed{pV = nRT}$

Desire $\boxed{M = \rho V}$ $\quad \boxed{M = n M_{air}}$ where M_{air} is the molar mass of air molecules. So $pV = \frac{\rho V}{M_{air}} RT$

$\Rightarrow \frac{p M_{air}}{R} = \rho T$ or $\boxed{\rho T = \text{constant}}$ $\boxed{\text{if pressure is constant}}$

"Isobaric"

So $\rho T = \rho_a T_a$ $\therefore$ $\boxed{M = \frac{\rho_a T_a}{T} V}$

Hence: $\rho_a V g = mg + \frac{\rho_a T_a V g}{T} \Rightarrow \frac{\rho_a T_a V g}{T} = -mg + \rho_a V g$

Note independent of g →

$$\boxed{T = \frac{\rho_a T_a V}{\rho_a V - m}} = \frac{1.23 \times (15+273) \times 1100}{1.23 \times 1100 - 240} \text{ (K)} = 350\text{ K} = \boxed{77°C}$$