BPhO Round 1 Section 1. 2022

AF SOLUTIONS

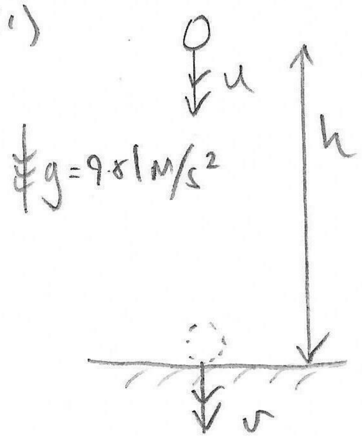
a) A steel ball is thrown down with a speed of 3.0 m s^{-1} (u) on to a hard surface from a height of 2.0 m (h). It retains 70% (η) of its energy on each bounce. Calculate

(i) the speed (v) at which it hits the ground for the first time, and

(ii) the maximum height it reaches after the 4$^{\text{th}}$ bounce.

[2]

(i)



If ball has mass m kg and no energy is lost during the fall....

... By conservation of energy: $\boxed{\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mgh}$

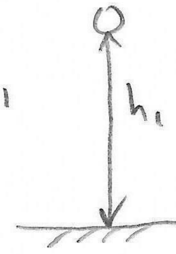
$$\therefore v = \sqrt{u^2 + 2gh}$$

$$= \sqrt{3.0^2 + 2 \times 9.81 \times 2.0} \quad (\text{m/s})$$

$$= \boxed{6.9 \text{ m/s}}$$

(ii) After first bounce, ball has energy (KE) of $\eta\left(\frac{1}{2}mu^2 + mgh\right)$.

By conservation of energy it will rise to height h_1 where $mgh_1 = \eta\left(\frac{1}{2}mu^2 + mgh\right)$.



After the next bounce, $mgh_2 = \eta mgh_1$

$$\Rightarrow \boxed{h_n = \eta^n\left(\frac{u^2}{2g} + h\right)}$$

So $h_4 = 0.7^4\left(\frac{3.0^2}{2 \times 9.81} + 2.0\right)$ (m)

$= 0.7^4 \times 2.46$ (m) $= \boxed{0.59 \text{ m}}$

[$h_1 = 0.7 \times 2.46$ m $= 1.72$ m]

b) A long-distance cyclist uses a cycle computer that is dual-powered: it has an internal battery and a solar panel. While the cyclist is riding in direct sunlight, the solar panel on the computer provides energy at a rate equal to $\frac{1}{3}$ of the power consumption. If a fully-charged cycle computer lasts 10 hours when riding at night, calculate how long a fully charged computer will run for in direct sunlight.

[2]

* Assume power consumption P is constant, and a fully charged computer constitutes energy E

So $\boxed{E = P t_N}$ where t_N = 10 hours (ie night time)

* During t_D hours in direct sunlight it will expend $P t_D$ Joules and gain $\frac{1}{3} P t_D$ Joules.

So $\underbrace{P t_D}_{\text{total energy spend}} = \underbrace{P t_N}_{\text{initial charge energy}} + \underbrace{\tfrac{1}{3} P t_D}_{\text{gain from solar panel}}$

$\therefore \quad \frac{2}{3} t_D = t_N \Rightarrow \boxed{t_D = \frac{3}{2} t_N}$

$\Rightarrow \boxed{t_D = 15 \text{ hours}}$

[if solar panel energy input is ηP watts $\leftarrow 0 < \eta < 1$

$\Rightarrow \quad P t_D = P t_N + \eta P t_D$ is the energy balance

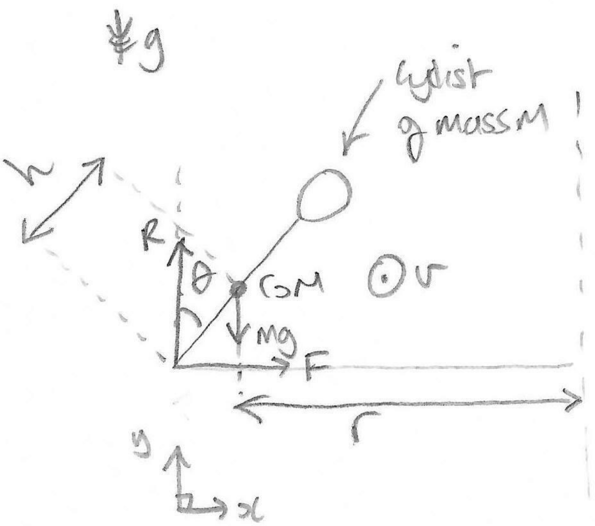
$\therefore \quad t_D (1 - \eta) = t_N$

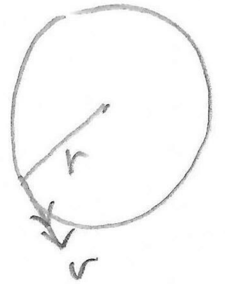
$\boxed{t_D = \frac{1}{1-\eta} t_N}$].

v θ

c) Riding round a corner at 10 km h^{-1} a cyclist leans over at an angle of 12° to the vertical. At what angle would they lean over if they went round the corner at 15 km h^{-1}? **[3]**

Same if r = constant



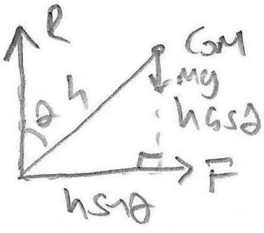


Note this is the circle (section) made by the cyclist CoM.

Circular motion $\Rightarrow$ NII // x direction

$$\frac{Mv^2}{r} = F \quad (1)$$

If it is accelerating!



And NII // y direction

$$0 = R - mg \quad (2) \quad \Rightarrow \quad \boxed{R = mg}$$

You can't choose ANY location since mr in eq.

Now one assumes no rotation about the CoM, so no net torque ↺+

$$\therefore \quad 0 = R \times h\sin\theta - F \times h\cos\theta \quad \Rightarrow \quad \boxed{F = R\tan\theta}$$

$$\therefore \text{ in (1)} \quad \frac{mv^2}{r} = mg\tan\theta$$

$$\Rightarrow \quad \boxed{\tan\theta = \frac{v^2}{rg}}$$

$$\tan 12° = \frac{(10\text{ km/h})^2}{rg}$$

so if rg = constant

$$\frac{\tan\theta_{15}}{\tan 12} = \left(\frac{15}{10}\right)^2$$

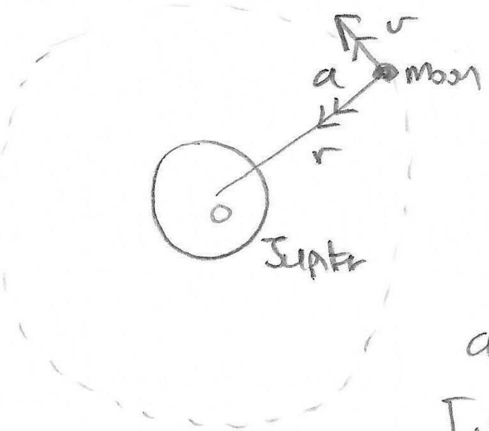
$$\Rightarrow \theta_{15} = \tan^{-1}(\tan 12° \times 1.5^2)$$
$$= \boxed{26°}$$

d) Io and Europa are both moons of Jupiter. Europa takes twice as long as Io to complete an orbit. What is the ratio of the centripetal acceleration of Io and Europa, $\frac{a_{\text{Io}}}{a_{\text{Europa}}}$?

You may use the result that for this gravitational system, $\omega^2 r^3$ = constant, where ω is the angular velocity and r is the radius of the orbit.

[This is a statement of Kepler's third law since $\omega = 2\pi / T$ ← period and $\therefore$ $\boxed{T^2 \propto r^3}$]

[2]



$$a = r\omega^2$$

$$[v = r\omega]$$

So $$\boxed{\frac{a_{Io}}{a_{Europa}} = \frac{r_{Io}\,\omega_{Io}^2}{r_{Europa}\,\omega_{Europa}^2}}$$

$$\omega = 2\pi / T$$

Now $$\boxed{\omega_{Io}^2\, r_{Io}^3 = \omega_{Europa}^2\, r_{Europa}^3}$$

So $$\frac{r_{Io}}{r_{Europa}} = \left(\frac{\omega_{Europa}}{\omega_{Io}}\right)^{2/3} = \left(\frac{T_{Io}}{T_{Europa}}\right)^{2/3}$$

$$\therefore \quad \frac{a_{Io}}{a_{Europa}} = \left(\frac{T_{Io}}{T_{Europa}}\right)^{2/3}\left(\frac{T_{Europa}}{T_{Io}}\right)^{2}$$

$$= \left(\frac{T_{Europa}}{T_{Io}}\right)^{4/3} = 2^{4/3} = \boxed{2.52}$$

(Since $T_{Europa} = T_{Io} \times 2$)

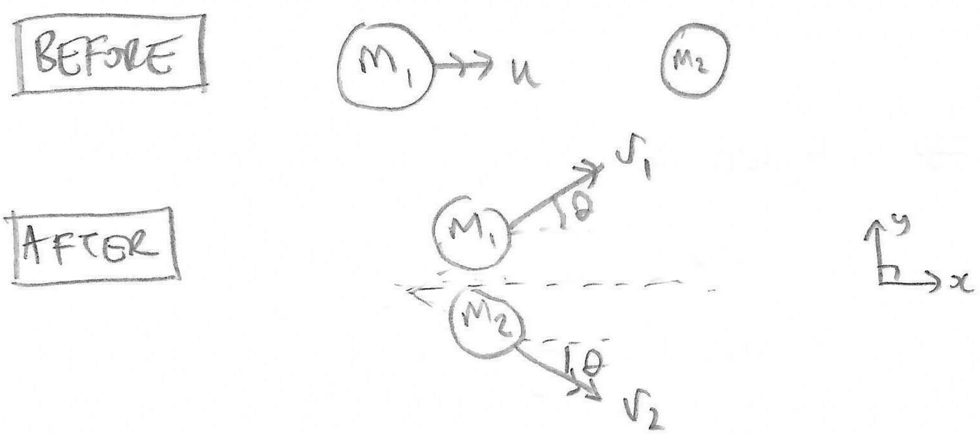
e) A particle of mass m_1 and initial speed u makes an elastic collision with a stationary particle of mass m_2. The particles move off at speeds v_1 and v_2 respectively, at equal angles θ either side of the initial incident direction of m_1.

(i) What is the largest ratio of $\frac{m_1}{m_2}$ for which this equal angle condition can occur?

(ii) If $m_1 = m_2$, what is the largest angle of deflection, θ, of particle m_1 for this equal angle condition?

[5]

eliminate v_1 and v_2



Conservation of momentum: $/\!/x$ $\quad \boxed{M_1 u = M_1 v_1 \cos\theta + M_2 v_2 \cos\theta}$ ①

$/\!/y$ $\quad \boxed{0 = M_1 v_1 \sin\theta - M_2 v_2 \sin\theta}$ ②

Conservation of energy (since elastic) $\quad \boxed{\frac{1}{2}M_1 u^2 = \frac{1}{2}M_1 v_1^2 + \frac{1}{2}M_2 v_2^2}$ ③

②: $\boxed{v_2 = v_1 \, M_1 / M_2}$

In ①: $\quad M_1 u = M_1 v_1 \cos\theta + M_2 v_1 \frac{M_1}{M_2} \cos\theta = 2 M_1 v_1 \cos\theta$

$\therefore \boxed{v_1 = \frac{u}{2\cos\theta}} \qquad \therefore \; v_2 = \frac{M_1}{M_2} \frac{u}{2\cos\theta}$

$\therefore$ In ③: $\quad \frac{1}{2}M_1 u^2 = \frac{1}{2}M_1 \frac{u^2}{4\cos^2\theta} + \frac{1}{2}M_2 \frac{M_1^2}{M_2^2} \frac{u^2}{4\cos^2\theta}$

$1 = \frac{1}{4\cos^2\theta} + \frac{M_1}{M_2}\frac{1}{4\cos^2\theta} \Rightarrow \boxed{4\cos^2\theta = 1 + \frac{M_1}{M_2}}$

↓ PTO

(i) Now $\cos^2\theta$ has a maximum value of 1 when $\theta = 0$.

So $4\cos^2\theta = 4 \Rightarrow 1 + \frac{M_1}{M_2} = 4$ for the largest ratio $\frac{M_1}{M_2}$

ie $\boxed{\frac{M_1}{M_2} = 3}$ is the largest ratio.

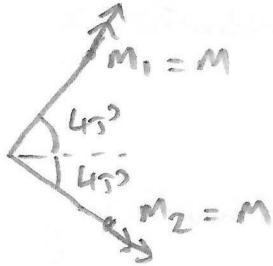
(to satisfy 'equal angle of split' conditions)

(ii) If $\frac{M_1}{M_2} = 1 \Rightarrow 4\cos^2\theta = 2$

$\cos^2\theta = \frac{1}{2}$

$\cos\theta = \frac{1}{\sqrt{2}}$

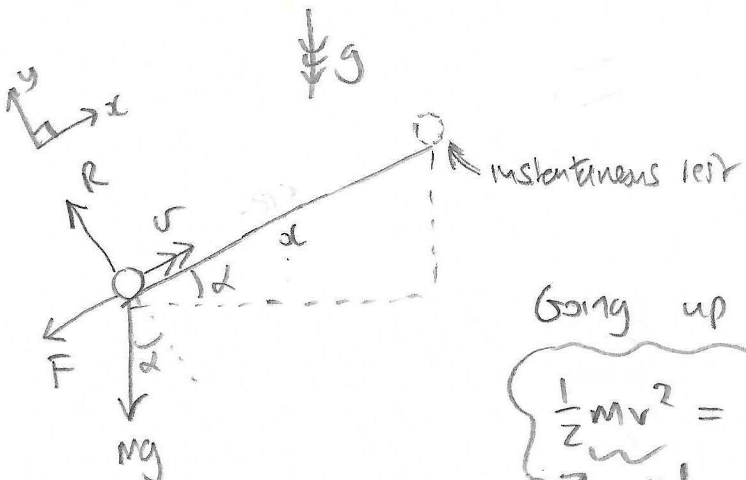
$\Rightarrow \theta = \boxed{45^\circ}$



f) A body is projected with velocity v up a plane inclined at an angle α to the horizontal. When it returns through its starting point it is moving with half the speed with which is was projected.
Determine the coefficient of friction μ, in terms of the angle of the plane.

Hint: The coefficient of friction is given by $F_{\text{friction}} = \mu N$ where N is the normal contact force and F_{friction} is the frictional force on the body.

[4]



$$N\text{II} \,//y: \quad 0 = R - mg\cos\alpha$$

$$\therefore R = mg\cos\alpha$$

$$\therefore \boxed{F = \mu mg\cos\alpha}$$

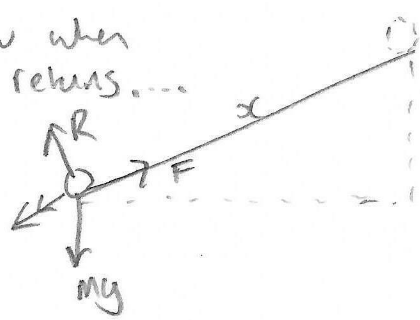
Going up the slope a distance x :

$$\underbrace{\tfrac{1}{2}mv^2}_{\text{initial KE}} = \underbrace{mgx\sin\alpha}_{\text{Work done against gravity}} + \underbrace{Fx}_{\text{work done against friction.}}$$

Conservation of energy

$$\text{So} \quad x = \frac{\tfrac{1}{2}mv^2}{F + mg\sin\alpha} = \frac{\tfrac{1}{2}mv^2}{(\mu\cos\alpha + \sin\alpha)mg} = \boxed{\frac{v^2/2g}{\mu\cos\alpha + \sin\alpha}}$$

Now when it returns....



$$\underbrace{\tfrac{1}{2}mv'^2}_{\text{KE gain}} = \underbrace{mgx\sin\alpha}_{\text{GPE loss}} - \underbrace{Fx}_{\text{Work done against friction}}$$

$$\therefore \tfrac{1}{2}mv'^2 = (mg\sin\alpha - \mu mg\cos\alpha)x$$

$$\text{So} \quad \frac{v'^2}{2g} = \frac{\sin\alpha - \mu\cos\alpha}{\sin\alpha + \mu\cos\alpha}\,\frac{v^2}{2g}$$

$$\Rightarrow \left(\frac{v'}{v}\right)^2 (\sin\alpha + \mu\cos\alpha) = \sin\alpha - \mu\cos\alpha$$

↓ PTO.

$$\mu \left\{ \cos\alpha + \left(\frac{v'}{v}\right)^2 \cos\alpha \right\} = \sin\alpha - \sin\alpha \left(\frac{v'}{v}\right)^2$$

$$\therefore \boxed{\mu = \tan\alpha \left(\frac{1-\left(\frac{v'}{v}\right)^2}{1+\left(\frac{v'}{v}\right)^2} \right)}$$

So if $\frac{v'}{v} = \frac{1}{2}$

$$\Rightarrow \mu = \tan\alpha \times \frac{3/4}{5/4}$$

$$\Rightarrow \boxed{\mu = \frac{3}{5}\tan\alpha}$$

{ Can also solve this by NII // slope and XUVAT, but I think energy conservation gives you more insight }.

g) A cylindrical container is filled with equal volumes of n different liquids which do not mix, so that they form horizontal layers each of height h. The densities of the liquids are ρ, 2ρ, 3ρ, $\cdots$, with the lowest liquid of density $n\rho$. The curved surface area of the cylinder enclosing each liquid is A.

(i) Give an expression for the force F_1 on the area A surrounding the top liquid in terms of ρ, g, h, A.

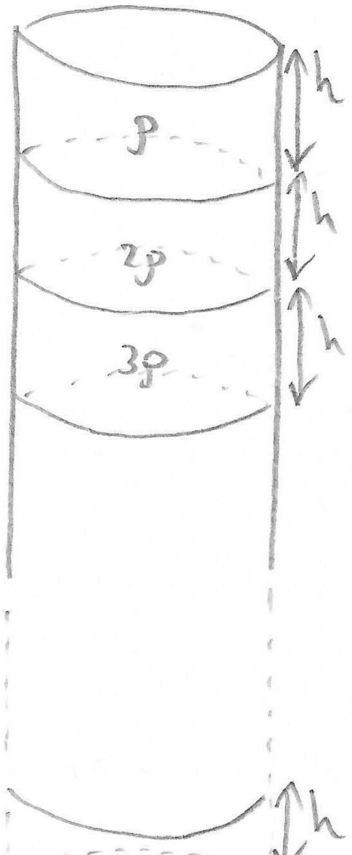
(ii) What is the force F_2 on the surface A surrounding the second liquid down from the top, in terms of F_1?

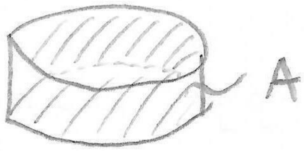
(iii) Give an expression for the force F_n on the area A surrounding the n^{th} liquid at the bottom of the cylinder in term of F_1?

Hint : $1 + 2 + 3 + \cdots + n = \sum_{i=1}^{n} i = \frac{n}{2}(n+1)$

Q g

[4]





(i) Average fluid pressure in top liquid is $\frac{1}{2}\rho g h$

(i.e. zero relative to atmospheric pressure at top and $\rho g h$ at bottom of first layer, and linearly increasing over this range)

$\therefore \boxed{F_1 = \frac{1}{2}\rho g h A}$ (i)

(ii) $F_2 = \Big(\underbrace{\rho g h}_{\text{pressure due to top layer}} + \underbrace{\tfrac{1}{2}(2\rho) g h}_{\text{average pressure of second layer}}\Big) A$

$\therefore \boxed{F_2 = 2\rho g h A}$

PTO

(iii) So force on exterior walls in layer n is

$$F_n = A\left(\rho g h + 2\rho g h + \cdots (n-1)\rho g h + \tfrac{1}{2} n \rho g h\right)$$

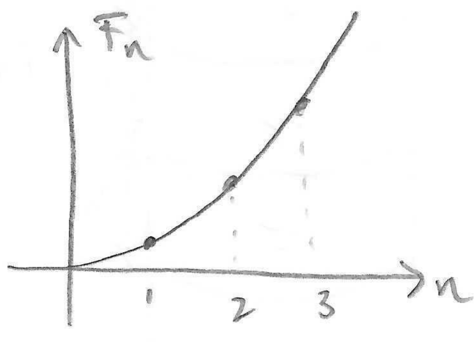
$$F_n = A\rho g h\left(1 + 2 + 3 + \cdots n-1 + \tfrac{n}{2}\right)$$

Now $\boxed{1 + 2 + \cdots\cdots n = \tfrac{1}{2} n(n+1)}$

So $F_n = A\rho g h\left(\tfrac{1}{2} n(n+1) - \tfrac{n}{2}\right)$

$$= \tfrac{1}{2} A\rho g h\left(n^2 + n - n\right)$$

$\therefore$ $\boxed{F_n = \tfrac{1}{2} A\rho g h\, n^2}$



[$S_n = 1 + 2 + 3 + \cdots\cdots n$

$S_n = n + n-1 + n-2 + \cdots\cdots 1$ reverse order

$\therefore$ $2S_n = n(n+1)$ $\Rightarrow$ $\boxed{S_n = \tfrac{1}{2} n(n+1)}$]

h) A rocket of mass $m_r = 5000\,\text{kg}$ contains a further mass $m_0 = 5000\,\text{kg}$ of fuel. Once the fuel is ignited, 50 kg per second of hot gas is expelled downwards at a speed of $2000\,\text{m s}^{-1}$.

μ $\quad$ c

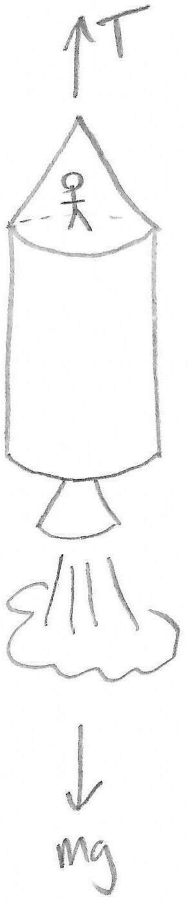
(i) Calculate the thrust, T, applied to the rocket,

(ii) Find an expression for the acceleration of the rocket, a, in terms of its total mass m, T and g,

(iii) Find an expression for the acceleration of the rocket as a function of time, t, in terms of T, g, m_r, m_0 and t_0, the total time for which the thrust acts.

(iv) Calculate the time after launch at which the weight of an astronaut on board will have appeared to double.

[4]



{ Ignore air resistance }

{ Ignore that g changes with altitude
ie ↓ as rocket moves further from Earth }

(i) $\boxed{T = \mu c}$ ie rate of change of momentum

$= 50 \times 2000$ (N)

$= \boxed{10^5\ \text{N}}$

(ii) NII ↑ : $\underbrace{(m_r + m_0 - \mu t)}_{m} a = T - mg$ ← During thrust phase.

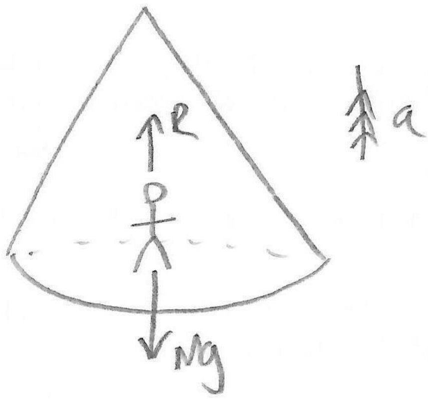
$\therefore \boxed{a = \frac{T}{m} - g}$

(iii) $a = \dfrac{T}{m_r + m_0 - \mu t} - g \qquad \mu = \dfrac{m_0}{t_0}$

So $\boxed{a = \dfrac{T}{m_r + m_0 - \frac{m_0 t}{t_0}} - g}$

PTO

(iii) In astronaut's frame



Astronaut mass

NII $Ma = R - Mg$

So $R = Ma + Mg$

is normal contact force on astronaut.

"weight appears to double" when $R = 2Mg$

$\therefore \quad 2g = a + g \Rightarrow \boxed{a = g}$

when $a = g$ $\qquad g = \dfrac{T}{m_r + M_0 - \frac{tM_0}{t_0}} - g$

$\Rightarrow \quad m_r + M_0 - tM_0/t_0 = T/2g$

$\therefore \quad t = \left(m_r + M_0 - T/2g\right)\dfrac{t_0}{M_0}$

$\therefore \quad t = \left(\dfrac{M_r}{M_0} + 1 - \dfrac{T}{2M_0 g}\right)t_0$

$= \left(\dfrac{5000}{5000} + 1 - \dfrac{10^5}{2 \times 5000 \times 9.81}\right)(100)$ (5)

$= \boxed{98s}$

[if $\dfrac{M_0}{t_0} = 50$ kg/s

$\therefore t_0 = \dfrac{5000\text{ kg}}{50\text{ kg/s}}$

$= 100s$]

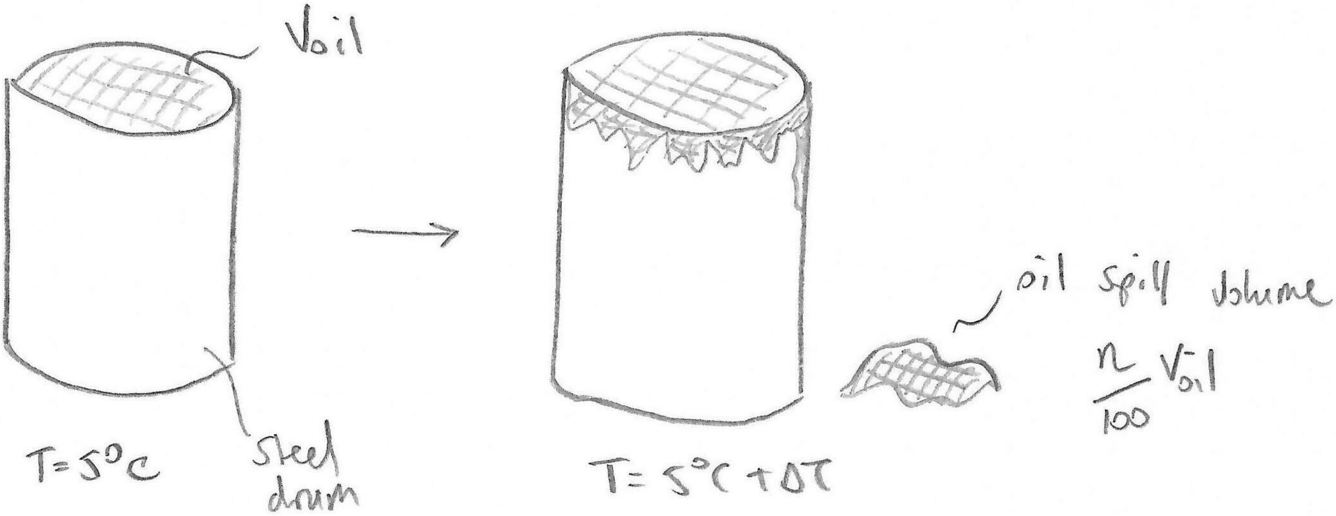
i) An open topped steel drum is completely filled with oil on a day when the temperature is 5.0 °C. On a warm day the temperature rises, and 2.4% of the oil spills out. What is the temperature reached on that day? η

$5°C + \Delta T$

The **volume coefficient** of expansion of oil is $7.0 \times 10^{-4}\,°C^{-1}$ k_{oil}

The **linear coefficient** of expansion of steel is $1.2 \times 10^{-5}\,°C^{-1}$ k_{steel}

[3]



oil leakage volume = Extra oil volume due to thermal expansion − Extra steel drum $\boxed{\text{internal volume}}$ due to thermal expansion.

$$\frac{\eta}{100} V_{oil} = k_{oil} V_{oil} \Delta T - \underbrace{3 k_{steel}}_{} V_{oil} \Delta T$$

initially V_{oil}

The drum expands (isotropically) in 3 dimensions.

$$\therefore \quad \boxed{\Delta T = \frac{\frac{\eta}{100}}{k_{oil} - 3k_{steel}}}$$

$$= \frac{2.4/100}{7.0\times10^{-4} - 3\times1.2\times10^{-5}}\ °C = 36.1°C$$

So temp was $\boxed{41.1°C}$

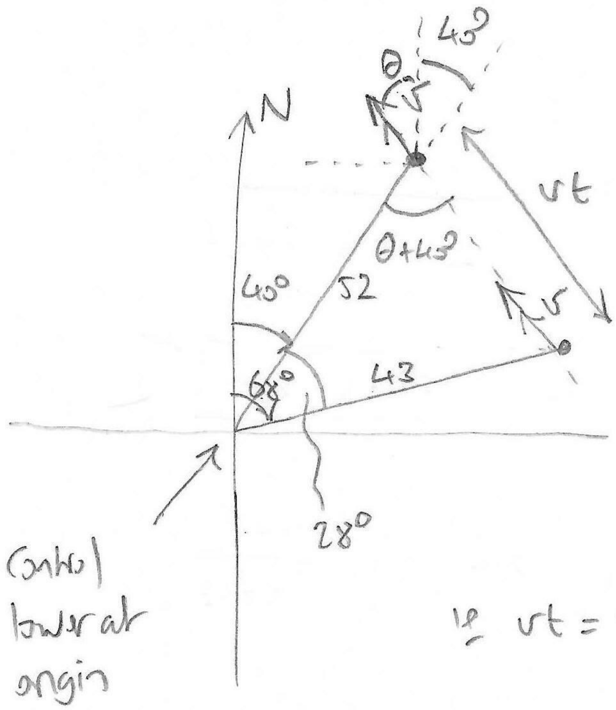
* Assume the aircraft is flying at constant velocity! *

j) Observed from an air traffic control tower, an aeroplane has a bearing of 068° and a range of 43 km. Five minutes later the bearing of the aircraft is 040° with a range of 52 km. Determine $t = 5\times 60\,s$

(i) The speed of the aircraft in $m\,s^{-1}$.

(ii) Its bearing and range 10 minutes after the second sighting.

[4]



(i) Cosine rule:

$$v^2t^2 = 52^2 + 43^2 - 2(52)(43)\cos 28^\circ$$
$$= 604.46..\ (km)^2$$
$$\Rightarrow v = \frac{\sqrt{604.46...}}{5\times 60}\ km/s$$

ie $vt = 24.6\,km$

$$= \boxed{82\ m/s} \rightarrow \text{Calc memory}$$

(ii) Sine rule: $\dfrac{\sin(\theta+40^\circ)}{43} = \dfrac{\sin 28^\circ}{vt}$

$$\therefore \theta = \sin^{-1}\left(\frac{43\sin 28}{vt}\right) - 40^\circ$$
$$= \boxed{15.19^\circ}$$

$$\therefore \underline{v} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \times 82\ m/s = \begin{pmatrix} -21.5 \\ 79 \end{pmatrix} m/s$$

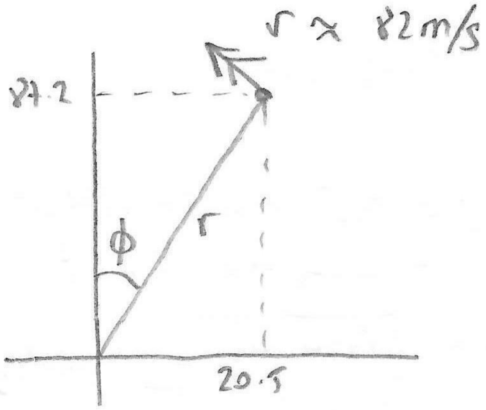
$\therefore$ $600s = t$ after second sighting, position vector is

$$\underline{r} = \begin{pmatrix} 52\sin 40^\circ \\ 52\cos 40^\circ \end{pmatrix} + \begin{pmatrix} -21.5 \\ 79 \end{pmatrix} \frac{t}{1000}\ km.$$

$$\therefore \quad \underline{r} = \begin{pmatrix} 52 \sin 45^\circ \\ 52 \cos 45^\circ \end{pmatrix} + \begin{pmatrix} -21.5 \\ 79 \end{pmatrix} \times \frac{600}{1000} \quad \text{km}$$

$$= \begin{pmatrix} 20.5 \\ 87.2 \end{pmatrix} \text{ km}$$

MS says 89.7 km



$$\therefore \quad \text{range } r = \sqrt{20.5^2 + 87.2^2}$$

$$= \boxed{89.6 \text{ km}}$$

$$\text{bearing } \phi = \tan^{-1}\left(\frac{20.5}{87.2}\right)$$

$$= \boxed{13.2^\circ}$$

{ Ideally we would not round mid-way
↳ this is a good problem to solve algebraically and evaluate using a spreadsheet or programming script }.

[Alternative method is to continue drawing triangles and use sine and cosine rules, but I think it is simpler to use $\underline{r} = \underline{r}_0 + \underline{v}\,t$ i.e. vector geometry. This is how you would plot it on a computer anyway!]

This is actually unhelpful!

k) A gas is found to obey the equation relating p, V, n, R, T

$$p(V-b) = nRT\exp\left(\frac{-a}{nRTV}\right)$$

where p is the gas pressure
V is its volume
R is the molar gas constant
n is the number of moles
a and b are constants.

(i) Determine the SI base units (m, kg, s) in which a and b are expressed.

(ii) If $b \ll V$ and $a \ll nRTV$, show that this expression approximates to the ideal gas equation relating ~~p, V, n, R, T~~ at a particular temperature T_c. Determine T_c in terms of a, b, n and R.

Hint: For $x \ll 1$ $\qquad \exp(x) = e^x \approx 1 + x$ — This is all you need.

[4]

(i) $[b]$ must be m^3 since a volume

$\left[\frac{-a}{nRTV}\right]$ must be dimensionless since an exponent

of $e \simeq 2.7183\ldots$

$\therefore \; [a] = [nRTV] = \cancel{mol} \times J\cancel{mol^{-1}}\cancel{K^{-1}}\cancel{K}\, m^3$

$= kg\,m^2 s^{-2}\, m^3$

$= \boxed{kg\,m^5 s^{-2}}$

$\left[[J] = \underbrace{kg\,m\,s^{-2}}_{\text{SI units of force}} \times \underbrace{m}_{\text{distance}} \right]$

$e^{-\frac{a}{nRTV}} \simeq 1 - \frac{a}{nRTV}$ if $\frac{a}{nRTV} \ll 1$

if $b \ll V \Rightarrow V - b \approx V$ $\qquad \therefore \; pV \approx nRT\left(1 - \frac{a}{nRTV}\right)$

Don't! $\qquad \downarrow$ PTO. $\quad$ Actually don't approx $V-b \simeq V$ yet!

$$\Rightarrow \quad pV - pb \simeq nRT - \frac{a}{V}$$ ← only $\frac{a}{nRTV} \ll 1$ is used

$$\boxed{pV \simeq nRT + pb - \frac{a}{V}}$$

So $pV \simeq nRT$ if $pb - \frac{a}{V} = 0$
(ideal gas eq$\bar{n}$)

$$\Rightarrow \quad pV = \frac{a}{b}.$$ let this occur at $T = T_c$

So $nRT_c = \frac{a}{b}$

$$\boxed{T_c = \frac{a}{bnR}}$$

ie no friction at surface — ie doesn't move!

1) A thin rod is balanced in a smooth hemispherical bowl fixed to a table, touching both the interior and the rim as shown in **Fig. 1**. The rim of the bowl remains horizontal. Expressed in the simplest form, determine the radius of the bowl r in terms of the length of the rod ℓ, and the angle θ to the horizontal.

Mass M

$\downarrow g = 9.81 \, m/s^2$

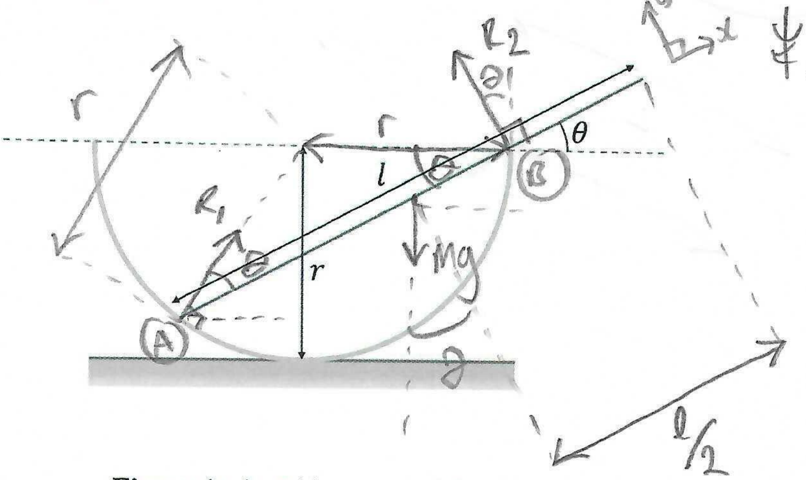
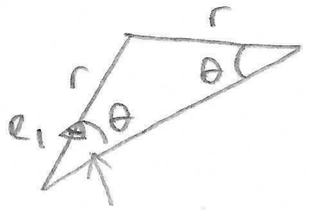


Figure 1: A rod in a smooth bowl.

[6]



This angle must also be θ since isoceles triangle

ie in Eq: Mass × acc = vectorsum of force = 0

NII // rod:

$$R_1 \cos\theta - Mg\sin\theta = 0 \Rightarrow \boxed{R_1 = Mg\tan\theta}$$

NII ⊥ rod:

$$R_1 \sin\theta + R_2 - Mg\cos\theta = 0$$

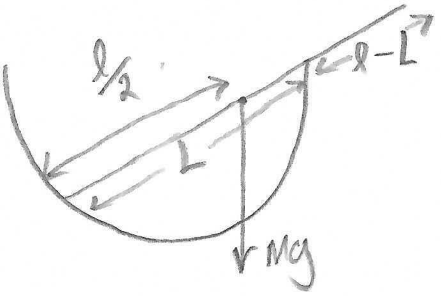
$$\Rightarrow R_2 = Mg\cos\theta - R_1\sin\theta$$

$$\boxed{R_2 = Mg\cos\theta - Mg\frac{\sin^2\theta}{\cos\theta}}$$

More complicated expression, so let's not use it in moments calc.

↻+ moments (note in Eq) about B

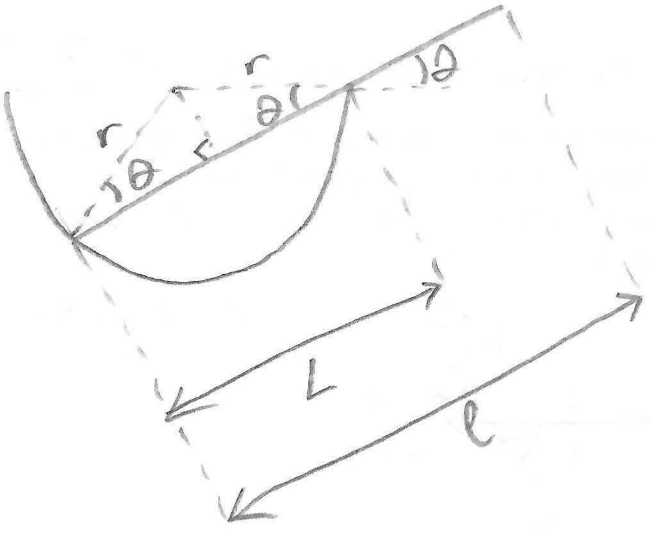
$$-R_1\sin\theta \, L + Mg\cos\theta \left(L - \frac{\ell}{2}\right) = 0$$



$$\text{so } L\left(Mg\cos\theta - Mg\tan\theta\sin\theta\right) = \frac{\ell}{2} Mg\cos\theta$$

$$\Rightarrow L = \frac{\ell}{2} \frac{\cos\theta}{\cos\theta - \tan\theta\sin\theta}$$

↓ PTO



$2r\cos\theta = L$

from geometry

$$\therefore \quad 2r\cos\theta = \frac{l}{2}\,\frac{\cos\theta}{\cos\theta - \tan\theta\sin\theta}$$

$$r = \frac{l}{4}\,\frac{1}{\cos\theta - \frac{\sin^2\theta}{\cos\theta}}$$

$$r = \frac{l}{4}\,\frac{\cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$\boxed{r = \frac{l}{4}\,\frac{\cos\theta}{\cos 2\theta}}$$

$$\Big[\ \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta\ \Big]$$

$\uparrow$

$$\cos(A \pm B) = \cos A\cos B - \sin A\sin(\pm B)$$

$$= \cos A\cos B \mp \sin A\sin B\ \Big]$$

m) In the circuit shown in **Fig. 2**, the three cells each supply an emf of 5.0 V and have an internal resistance of 5.0 Ω. The external resistors also each have a resistance of 5.0 Ω.
What arrangement of switches gives

(i) the maximum current,

(ii) the minimum non-zero current.

(iii) Determine the current in each case.

[3]

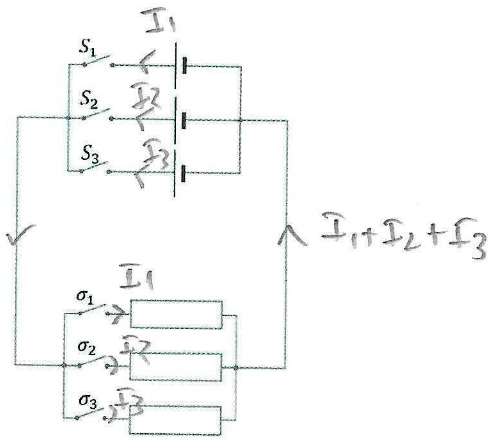


Figure 2: Three cells and switches in parallel.

(i)

* Closing all switches S_1, S_2, S_3 doesn't change the EMF
  (KII: can't have a closed loop including lower resistors which involves more than one 5.0 V EMF)

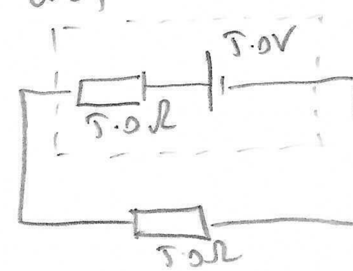
* If all switches $\sigma_1, \sigma_2, \sigma_3$ are closed AND S_1, S_2, S_3

$$I_{1,2,3} = \frac{5.0\,V}{10\,\Omega} = 0.5\,A$$

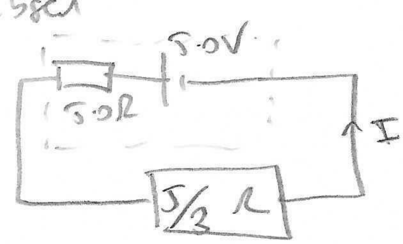
∴ total current is $3 \times 0.5\,A = \boxed{1.5\,A}$ MAX

* one $S_{1,2,3}$ is closed and one $\sigma_{1,2,3}$ then

total current is $I_{1,2,3} = \boxed{0.5\,A}$ MIN

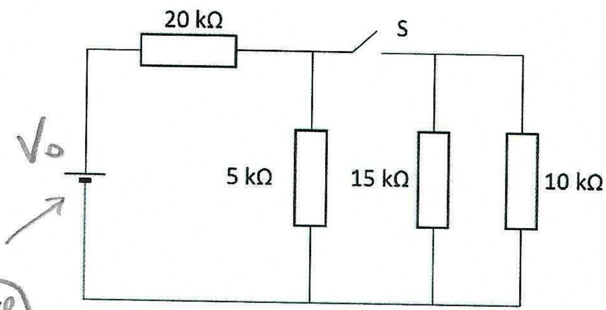


[Note if one $S_{1,2,3}$ is closed and all $\sigma_{1,2,3}$ are closed



$$\therefore I = \frac{5.0}{5 + \frac{5}{3}}\,A = \boxed{0.75\,A}$$]

n) The circuit of **Fig. 3** consists of four resistors and a switch **S**. When the **S** is open, the current flowing through the $5\,\mathrm{k\Omega}$ resistor is I_o. When **S** is closed, the current flowing through the same resistor is I_c. What is the ratio $\frac{I_c}{I_o}$, giving your answer as the ratio of two integers.

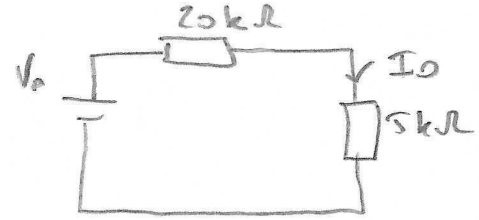


Assume no internal resistance

Figure 3: A switched circuit.

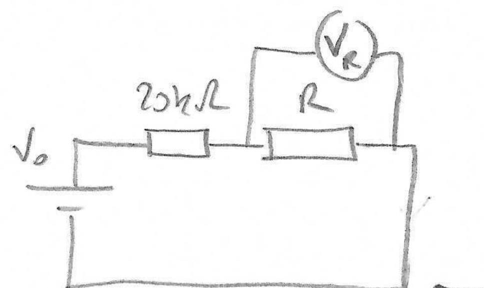
[4]

S open:



$$\boxed{I_o = \frac{V_o}{25\,k\Omega}}$$

S closed: Equivalent circuit is



$$R = \frac{1}{\frac{1}{5}+\frac{1}{15}+\frac{1}{10}}\ k\Omega$$

$$= 2.72\ k\Omega \qquad \left(\boxed{\tfrac{30}{11}\ k\Omega} \text{ as an exact fraction} \right)$$

So $$\frac{V_R}{V_o} = \frac{R}{R+20k\Omega} = \frac{30/11}{30/11+20} = \boxed{3/25}$$

Potential divider

$\therefore$ Since $V_R = I_c \times 5k\Omega \Rightarrow I_c = \dfrac{\frac{3}{25} V_o}{5k\Omega}$

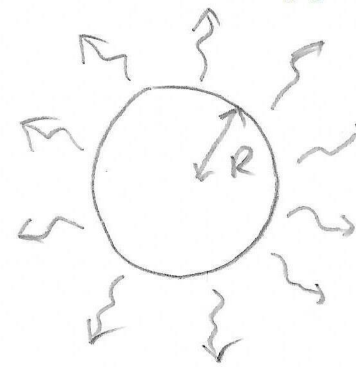
$$\therefore\ I_c/I_o = \frac{3}{125} \times 25 = \boxed{3/5}$$

o) The Stefan-Boltzmann law says that the emitted power of a spherical "black body", P, is related to the radius, R, and absolute surface temperature, T, as $P \propto R^2T^4$. Wien's "displacement law" says that the wavelength λ_{max} corresponding to the peak value of the emitted power of this spectrum is inversely proportional to the absolute surface temperature. The Sun currently has its peak wavelength as 500 nm.
What will be the new peak wavelength when it becomes a red giant, given its radius will be 200 times larger and its power output 4000 times larger?

[3]

* $P = k R^2 T^4$ Stefan-Boltzmann

* $\lambda_{max} = \dfrac{b}{T}$ Wien



So $$\boxed{\lambda_{max} = \frac{b}{\left(P/kR^2\right)^{\frac{1}{4}}}}$$

$$\lambda_{max,\odot} = bk^{\frac{1}{4}} \frac{R_{\odot}^{\frac{1}{2}}}{P_{\odot}^{\frac{1}{4}}}$$

$$\lambda_{max,RJ} = bk^{\frac{1}{4}} \frac{R_{RJ}^{\frac{1}{2}}}{P_{RJ}^{\frac{1}{4}}}$$

So $$\boxed{\frac{\lambda_{max,RJ}}{\lambda_{max,\odot}} = \left(\frac{R_{RJ}}{R_{\odot}}\right)^{\frac{1}{2}} \Big/ \left(P_{RJ}/P_{\odot}\right)^{\frac{1}{4}}}$$

$$= \frac{200^{\frac{1}{2}}}{4000^{\frac{1}{4}}} = 1.778\ldots$$

$$\therefore \lambda_{max,RJ} = 1.778\ldots \times 500\,\text{nm}$$
$$= \boxed{889\,\text{nm}}$$

☺ A nice "paradigm example".

E

p) One spectral line in hydrogen is caused by photons with an energy of 2.55 eV. The same line is redshifted in the spectrum of a distant galaxy by 5.4 nm. Calculate

(i) the wavelength of the photon,

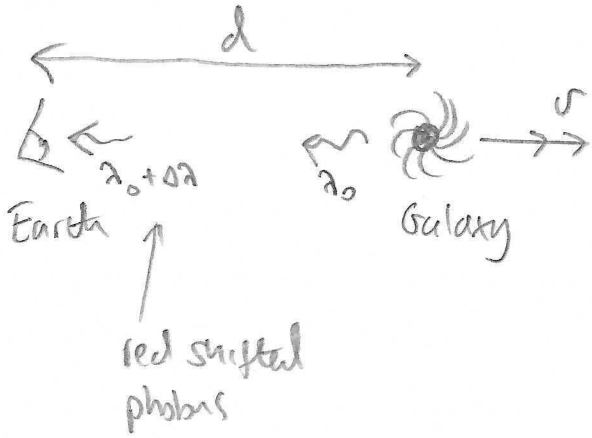
(ii) the speed of recession of the galaxy,

(iii) the distance to the galaxy.

How far away is the galaxy? Give your answer in megaparsecs (Mpc).

The Hubble constant, $H_0 = 70\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$.

[3]



(i) $E = \frac{hc}{\lambda_0}$

$\therefore \boxed{\lambda_0 = \frac{hc}{E}}$

$\therefore \lambda_0 = \frac{6.63\times10^{-34}\times2.998\times10^{8}}{2.55\times1.602\times10^{-19}}$ (m)

$= 4.87\times10^{-7}\,\text{m}$

$= \boxed{487\,\text{nm}}$

(ii) Doppler effect (on axis)

$\boxed{\frac{\Delta\lambda}{\lambda_0} = \frac{v}{c}}$

$\therefore v = \frac{5.4}{487}\times3.00\times10^{8}\,\text{m/s}$

$= \boxed{3.32\times10^{6}\,\text{m/s}}$

(iii) Hubble's law $\boxed{v = H_0 d}$ $\therefore d = \frac{v}{H_0}$

For less compounded rounding errors use $v = \frac{\Delta\lambda}{\lambda_0}c$

$\therefore \boxed{d = \frac{\Delta\lambda}{\lambda_0}\frac{c}{H_0}}$ $\Rightarrow d = \frac{5.4}{487}\frac{\times3.00\times10^{5}\,\text{km/s}}{70\,\text{km/s/MpC}}$

$= \boxed{47.5\,\text{Mpc}}$

d

q) A car drives along a road that has small depressions regularly spaced about 8.0 m apart. When four 80 kg passengers enter the 800 kg car, it sinks down by 1.8 cm. At approximately what speed might travelling in the vehicle become very uncomfortable?

[3]

* This is a resonance problem. When the time between bumps ≈ period of oscillation of the car's suspension ⇒ maximum amplitude of vibrations

* Model the car suspension as a Hookean spring

Assume 'sink' of 1.8 cm is relative to the unoccupied car. ∴ Spring constant $k = \dfrac{4 \times 80 \times 9.81}{1.8 \times 10^{-2}}$ N/m

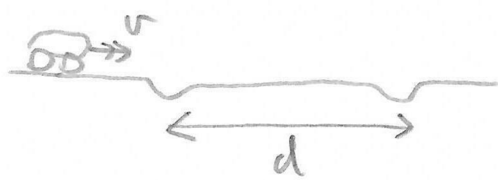
$= \boxed{1.74 \times 10^5 \text{ N/m}}$

* Period of SHM in a mass-spring system is

$$\boxed{T = 2\pi\sqrt{\frac{M}{k}}}$$

This is now the total mass of car + passengers

* Time between bumps is $\boxed{t = \dfrac{d}{v}}$



So if $t = T \Rightarrow \dfrac{d}{v} = 2\pi\sqrt{\dfrac{M}{k}}$

$$\therefore \boxed{v = \frac{1}{2\pi} d \sqrt{\frac{k}{M}}}$$

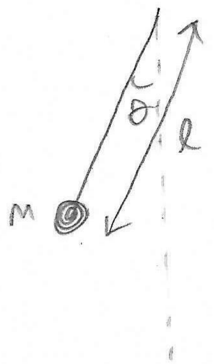
$$= \frac{1}{2\pi} \times 8 \sqrt{\frac{1.74 \times 10^5}{4 \times 80 + 800}} \text{ m/s}$$

$$= \boxed{15.9 \text{ m/s}}$$

↑ Don't forget the car mass

r) A pendulum clock is controlled by the swing of a simple pendulum (a mass on the end of a light rod) and is intended to have a period of 1.00 seconds. However, the clock runs slow by ten minutes each day. What percentage change should be made in the length of the pendulum?

[3]



Period of pendulum is

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

(for small oscillations s.t. $\sin\theta \approx \theta$ in radians)

T is too long by ΔT. $T_0 = 1.00\,s$ is desired

let $l = l_0 + \delta l$ where δl is the change which reduces T to T_0

So $$T_0 + \Delta T = 2\pi\sqrt{l_0/g}$$

$$T_0 = 2\pi\sqrt{\frac{l_0+\delta l}{g}}$$

Now $$\frac{\Delta T}{T_0} = \frac{10\times 60}{24\times 3600} = \frac{1}{144}.$$

$$1 + \frac{\Delta T}{T_0} = \frac{\frac{2\pi}{\sqrt{g}}\sqrt{l_0}}{\frac{2\pi}{\sqrt{g}}\sqrt{l_0+\delta l}}$$

$$1 + \frac{\Delta T}{T_0} = \frac{1}{\sqrt{1+\frac{\delta l}{l_0}}}$$

MS anticipates a binomial expansion of $\left(\sqrt{1+\frac{\delta l}{l_0}}\right)^{-1}$ to $1 - \frac{1}{2}\frac{\delta l}{l_0}$, but we don't need to!

$$\left[\Rightarrow \frac{\delta l}{l_0} \approx -2\frac{\Delta T}{T_0} = -\frac{1}{72} \simeq -\boxed{1.4\%}\right]$$

$$\boxed{\frac{\delta l}{l_0} = \left(1+\frac{\Delta T}{T_0}\right)^{-2} - 1}$$

Exact.

$$= \left(1+\frac{1}{144}\right)^{-2} - 1$$

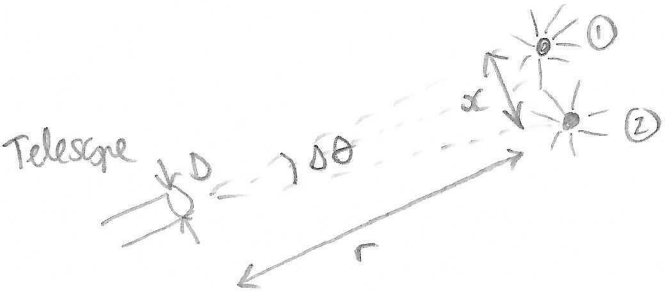
$$= \boxed{-0.014}$$

to 2 sf same as approx.

so <u>shorten</u> l_0 by 1.4%

s) A binary star system is 2140 light years away and consists of two stars like the Sun. The average separation between the stars is 0.00593 light years.
Determine the diameter of the telescope needed to resolve them if using a visible wavelength of 550 nm.

[2]



Since $r \gg x$

$$r\Delta\theta \simeq x$$

$$\Delta\theta \simeq \frac{x}{r}$$

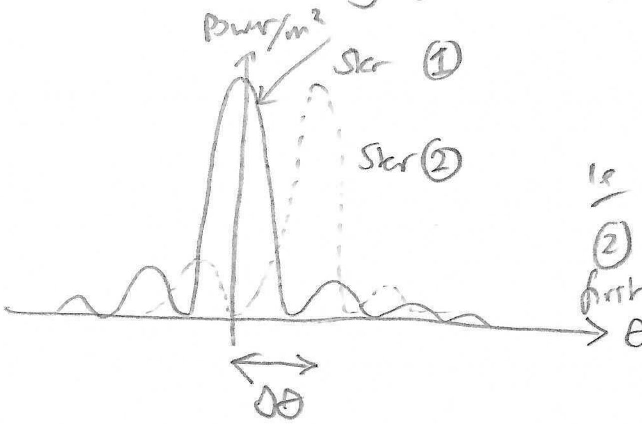
$$\simeq \frac{0.00593}{2140} \text{ radians}$$

$$\simeq \boxed{2.77 \times 10^{-6} \text{ radians}}$$

To 'just' resolve the stars, assume the Rayleigh criterion ie a diffraction limit

$$\boxed{\Delta\theta \simeq 1.22\lambda/D}$$

for a spherical aperture of diameter D.



ie peak of 2 is at first minima of star 1.

[if thin slits, $\Delta\theta \simeq \lambda/D$]

So to resolve the stars, $D > \lambda/\Delta\theta \times 1.22$

$$D > \frac{550 \times 10^{-9}\ \text{m}}{2.77 \times 10^{-6}} \times 1.22$$

$$\boxed{D > 0.24\text{m}}$$

which sounds like something an amateur astronomer could achieve with a 25cm telescope.

t) A glass prism of refracting angle 75.0° is shown in **Fig. 4** has a refractive index of $n = 1.40$.

(i) For what range of incident angles will light from air that is incident on face **AB** emerge from face **AC**?

(ii) Show your result on a diagram.

[3]

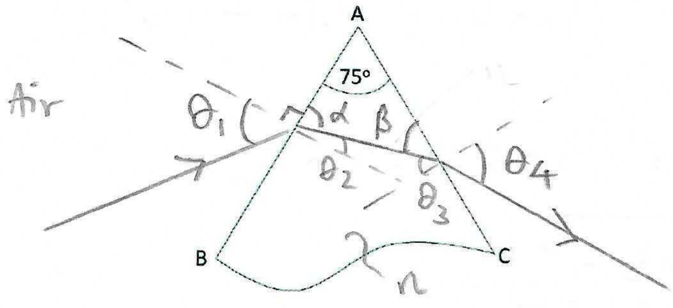
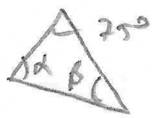


Figure 4: Glass prism with an apex angle of 75°.

Normals: $90^\circ = \alpha + \theta_2$ ① $\quad 75^\circ + \alpha + \beta = 180^\circ$ ③

$90^\circ = \beta + \theta_3$ ②



Snell: $\sin\theta_1 = n\sin\theta_2$ ④

$\sin\theta_4 = n\sin\theta_3$ ⑤

So $180^\circ = \alpha + \beta + \theta_2 + \theta_3$ ① + ②

$75^\circ + \cancel{\alpha + \beta} = \cancel{\alpha + \beta} + \theta_2 + \theta_3$ using ③

$\therefore 75^\circ = \theta_2 + \theta_3$ $\quad \therefore \theta_3 = 75^\circ - \theta_2$

$\therefore \sin\theta_4 = n\sin(75^\circ - \theta_2)$

$$\boxed{\theta_4 = \sin^{-1}\left(n\sin\left(75^\circ - \sin^{-1}\left(\frac{1}{n}\sin\theta_1\right)\right)\right)}$$

$\frac{1}{n}\sin\theta_1$ will < 1 for $0 \le \theta_1 \le 90^\circ$ so expect rays to emerge from AC for all angles of incidence.

↓ PTO

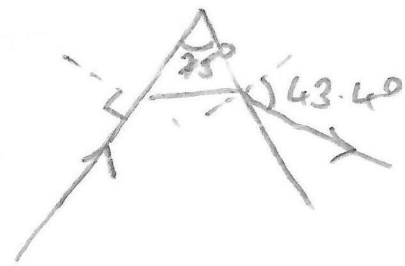
* largest θ_4 is clearly when θ_1 is largest, so set $\theta_1 = 90^\circ \quad \therefore \quad \sin\theta_1 = 1$

$$\therefore \quad \sin^{-1}\left(\frac{1}{n}\sin\theta_1\right) = \sin^{-1}\left(\frac{1}{1.4}\right) = \boxed{45.6^\circ}$$

$$\therefore \quad \theta_4 = \sin^{-1}\left(1.4\sin\left(75^\circ - 45.6^\circ\right)\right)$$

$$= \sin^{-1}\left(1.4\sin(29.4^\circ)\right)$$

$$= \boxed{43.4^\circ}$$



So expect max θ_4 to be 43.4°

[when $\theta_1 = 0$, $\theta_4 = \sin^{-1}\left(1.4\sin 75^\circ\right)$

$$= \sin^{-1}(1.35)$$ ie no real solution

so range of θ_1 that give rise to real θ_4 is when $n\sin\left(75^\circ - \sin^{-1}\left(\frac{1}{n}\sin\theta_1\right)\right) \leq 1$

$$75^\circ - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right) \leq \sin^{-1}\left(\frac{1}{n}\right)$$

$$75^\circ - \sin^{-1}\left(\frac{1}{n}\right) \leq \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)$$

$$\theta_1 \geq \sin^{-1}\left(n\sin\left(75^\circ - \sin^{-1}\left(\frac{1}{n}\right)\right)\right)$$

$$\geq \sin^{-1}\left(1.4\sin\left(75^\circ - \sin^{-1}\left(\frac{1}{1.4}\right)\right)\right)$$

$$\geq \boxed{43.4^\circ}$$ A nice result!]

* Trig fest! *   A somewhat epic question....

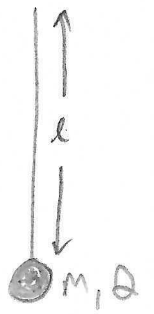
MUCH more structure needed – many algebraic pitfalls
... But overall a classic!!

u) A particle **A**, of mass m carrying a charge of Q is suspended by an insulating thread of length ℓ. Another particle **B**, of negligible mass but of positive charge $+q$ is brought towards **A**, which is repelled. When **B** arrives at the point previously occupied by **A**, the system is in (neutral) equilibrium.

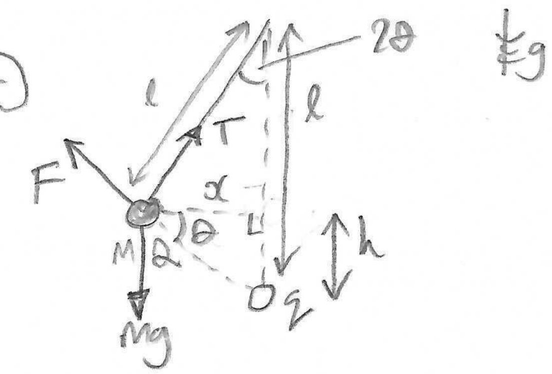
Calculate the work done in terms of $m, g, \ell, q, Q,$ and k, where $k = \dfrac{1}{4\pi\epsilon_0}$.

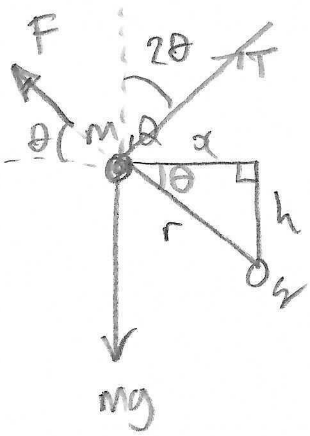
[6]

BEFORE



AFTER





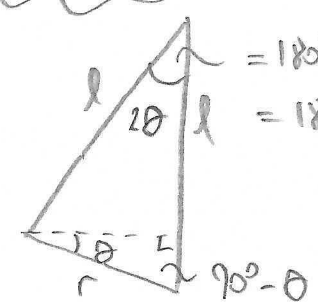
Energy of system "After", relative to "Before" is

$$\boxed{E = mgh + \frac{qQ}{4\pi\varepsilon_0 r}}$$

This is what we want. We ∴ need to find h, r

Coulomb's law:

$$\boxed{F = \frac{qQ}{4\pi\varepsilon_0 r^2}}$$



$\alpha = 180^\circ - 2(90^\circ - \theta)$
$= 180^\circ - 180^\circ + 2\theta$

Isosceles triangle

NII on mass m

$\uparrow$: $0 = T\cos 2\theta + F\sin\theta - mg$ ①

$\rightarrow$: $0 = T\sin 2\theta - F\cos\theta$ ②

So $T\sin 2\theta = F\cos\theta$

$T\cos 2\theta = mg - F\sin\theta$

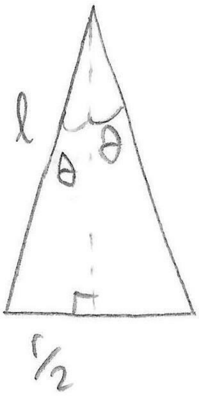
$$\therefore \quad \boxed{\tan 2\theta = \frac{F\cos\theta}{mg - F\sin\theta}}$$

!

Now $\boxed{r\sin\theta = h}$ * and also $\boxed{h = l - l\cos 2\theta}$.

GOAL * want to eliminate θ * $\rightarrow r = \dots, h = \dots$

PTO



So $\frac{r}{2} = l\sin\theta$ $\quad\therefore\quad \boxed{\sin\theta = \frac{r}{2l}}$ *

Also from geometry $h = l(1-\cos 2\theta)$

$\therefore\quad \boxed{\cos 2\theta = 1 - \frac{h}{l}}$

* useful!

Now $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\therefore\quad \frac{2\tan\theta}{1-\tan^2\theta} = \frac{1}{\frac{mg}{F\cos\theta} - \tan\theta}$

$\therefore\quad 1-\tan^2\theta = \left(\frac{mg}{F\cos\theta} - \tan\theta\right) \times 2\tan\theta$

$1-\tan^2\theta = \frac{2mg\tan\theta}{F\cos\theta} - 2\tan^2\theta$

$1+\tan^2\theta = \frac{2mg\sin\theta}{F\cos^2\theta}$

Now $1+\tan^2\theta = \frac{1}{\cos^2\theta}$

so $1 = \frac{2mg\sin\theta}{F}$

$\therefore\quad \boxed{F = 2mg\sin\theta}$

$\therefore\quad \frac{qQ}{4\pi\varepsilon_0 r^2} = \cancel{2}mg\frac{r}{\cancel{2}l}$

$\therefore\quad \frac{qQl}{4\pi\varepsilon_0 mg} = r^3$

$\therefore\quad \boxed{r = \left(\frac{qQl}{4\pi\varepsilon_0 mg}\right)^{1/3}}$

Since $h = r\sin\theta$ and $\sin\theta = \frac{r}{2l}$

$\Rightarrow \boxed{h = \frac{r^2}{2l}}$ ← useful result e.g in Newton's rings, "weapon of doom"...

$\therefore\quad \boxed{h = \frac{1}{2l}\left(\frac{qQl}{4\pi\varepsilon_0 mg}\right)^{2/3}}$

$\therefore\quad E = \frac{mg}{2l}\left(\frac{qQl}{4\pi\varepsilon_0 mg}\right)^{2/3} + \frac{qQ}{4\pi\varepsilon_0}\left(\frac{4\pi\varepsilon_0 mg}{qQl}\right)^{1/3}$

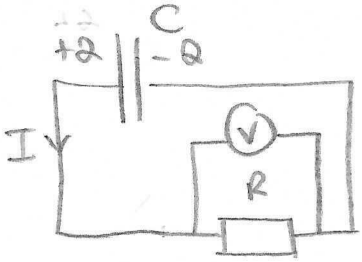
↓ PTO

$$E = \left(qQ \times \frac{1}{4\pi\varepsilon_0} \right)^{2/3} \left(\frac{1}{2} + 1 \right) \left(\frac{mg}{\ell} \right)^{1/3}$$

$$\boxed{E = \frac{3}{2} \left(\frac{q^2 Q^2}{16\pi^2 \varepsilon_0^2} \frac{mg}{\ell} \right)^{1/3}}$$

v) A capacitor of value 1.0 F discharges through a device whose resistance R varies linearly with applied potential difference, V, so that $R = AV + B$, where A and B are constants. The resistance of the device has a value of $10.0\,\Omega$ when $V = 6.0\,\text{V}$, and $4.0\,\Omega$ when $V = 0.06\,\text{V}$.
The capacitor is initially charged to a potential of 6.0 V. V_0
Determine how long it takes for the capacitor to discharge to 1% of this initial value.

[6]



$V = IR$

$V = I(AV+B)$

$\therefore \boxed{I = \dfrac{V}{AV+B}}$ ①

$\boxed{I = -\dfrac{dQ}{dt} \quad Q = CV}$ where Q is capacitor charge (separation)

So $I = -C\dfrac{dV}{dt}$ ② $\therefore$ equating ① and ②

$$-C\frac{dV}{dt} = \frac{V}{AV+B}$$

$$\int_{V_0}^{V} \frac{AV+B}{V}\,dV = -\frac{1}{C}\int_0^t$$

$$\int_{V_0}^{V} \left(A + \frac{B}{V}\right)dV = -t/C$$

$$A(V-V_0) + B\ln\left(\frac{V}{V_0}\right) = -t/C$$

$$AV_0\left(\frac{V}{V_0}-1\right) + B\ln\left(\frac{V}{V_0}\right) = -t/C$$

↓ PTO

$$\boxed{\therefore t = ACV_0\left(1-\frac{V}{V_0}\right) + BC\ln\left(\frac{V_0}{V}\right)}$$

$10 = 6A + B$

$4 = 0.06A + B$

$\therefore 6 = 5.94A$

$\boxed{\dfrac{6}{5.94} = A}$ (ΩV^{-1})

$\therefore B = 10 - 36/5.94$

$= \boxed{130/33}\ \Omega$

when $\frac{Q}{Q_0} = 0.01 \Rightarrow \frac{V}{V_0} = 0.01$ since $Q = CV$.

$$\therefore t = \frac{100}{99} \times 1.0 \times 6.0 (1 - 0.01) + \frac{130}{33} \times 1.0 \ln 100 \quad (s)$$

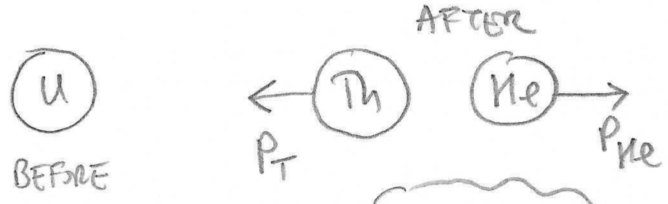
$\frac{6}{5.94}$

$t = ACV_0 \left(1 - \frac{V}{V_0}\right) + BC \ln\left(\frac{V_0}{V}\right)$

$$= 6 + 18.14$$

$$= \boxed{24.1\ s}$$

In frame of stationary U nucleus :



w) The $^{238}_{92}U$ decays according to

$KE = P^2/2M$

P is momentum

$$^{238}_{92}U \rightarrow {}^{234}_{90}Th + {}^{4}_{2}He$$

Determine the kinetic energy of the emitted α−particle in MeV.

Energy & momentum must be conserved

$c = 2.998 \times 10^{8}\ m\,s^{-1}$
Mass of the $^{238}_{92}U$ nucleus is 3.85395×10^{-25} kg
Mass of the $^{234}_{90}Th$ nucleus is 3.78737×10^{-25} kg
Mass of the α−particle is 6.64807×10^{-27} kg

So Th must have some KE.

[5]

So classical physics

{Assume recoil speeds << c}

Energy conservation:

$$M_U c^2 = M_T c^2 + M_{He} c^2 + P_T^2/2M_T + \frac{P_H^2}{2M_{He}}$$

This is what we want.

Momentum conservation

$P_T = P_{He}$ ← Since in zero momentum frame.

$$\therefore \frac{P_{He}^2}{2}\left(\frac{1}{M_{He}} + \frac{1}{M_T}\right) = (M_U - M_T - M_{He})c^2$$

$$\therefore \boxed{\frac{P_{He}^2}{2M_{He}} = \frac{(M_U - M_T - M_{He})c^2}{1 + M_{He}/M_T}}$$

KE of α particle

$$= \frac{\left(3.85395 - 3.78737 - \frac{6.64807}{100}\right) \times 10^{-25} \times (3.00\times10^8)^2}{1 + \frac{6.64807}{100 \times 3.78737}}$$

$$= 8.77 \times 10^{-13}\ J = \boxed{5.48\ MeV}$$

$[1\,MeV = 1.602 \times 10^{-13}\ J]$