BPhO 2013 PART 1 AF SOLUTIONS

(a) A chain of resistors, Figure 1.a.i, is composed of *n* units, each consisting of three resistors, each resistor of resistance *R*, Figure 1.a.ii. A unit is attached to the left hand end of the chain in order to increase the number of units from *n* to *(n+1)*.

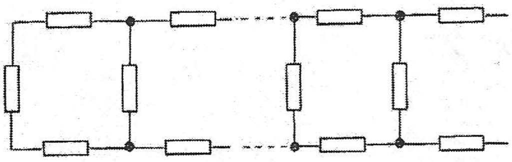


Figure 1.a.i

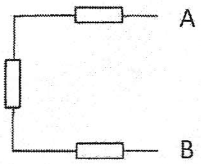


Figure 1.a.ii

(i)

(i) Calculate the resistance (between A and B) across a chain with two units, R_2, and the resistance R_3, across a chain with three units.

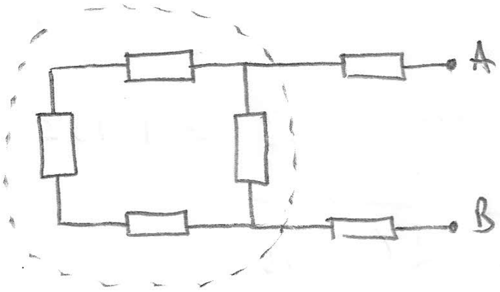
(ii)

(ii) A unit is attached to a long chain. The resistance of the chain, R_T, is not altered by this addition. Determine the resistance of the chain.

[6]

(i)

(a)



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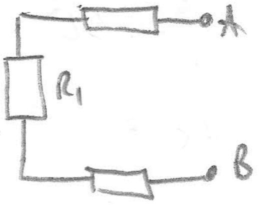
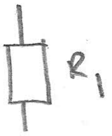


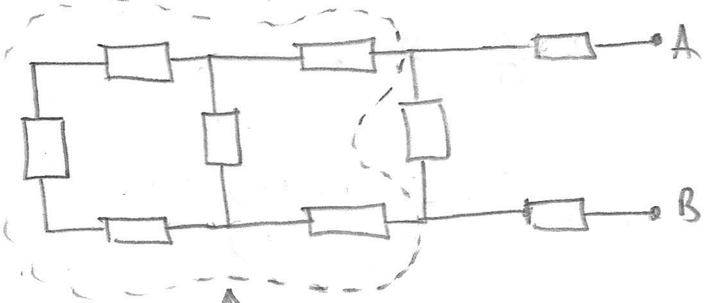


$$R_2 = 2R + R_1$$

$$R_1 = \frac{1}{\frac{1}{3R} + \frac{1}{R}} = \frac{3}{4}R$$

$$\therefore R_2 = \frac{11}{4}R = \boxed{2\tfrac{3}{4}R}$$

(b)



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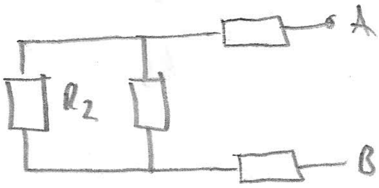
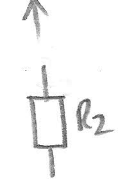


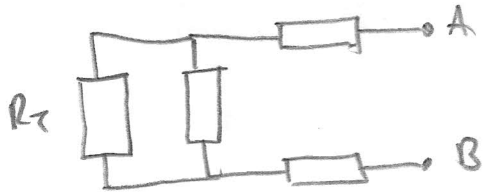


i.e what we computed in a) (i)

$$\therefore R_3 = 2R + \frac{1}{\frac{1}{R} + \frac{1}{R_2}}$$

$$R_3 = 2R + \frac{R}{1 + \frac{4}{11}} = \boxed{2\tfrac{11}{15}R}$$

(ii) Essentially just like (i) (b), but resistance is R_T



ie $R_T = 2R + \dfrac{1}{\frac{1}{R} + \frac{1}{R_T}}$

$R_T = 2R + \dfrac{R}{1 + R/R_T}$

$R_T = 2R + \dfrac{RR_T}{R_T + R}$

$\therefore (R_T + R) R_T = 2R(R_T + R) + RR_T$

$\therefore R_T^2 + RR_T - 2RR_T - 2R^2 - RR_T = 0$

$\therefore R_T^2 - 2RR_T - 2R^2 = 0$

$(R_T - R)^2 - 3R^2 = 0$

$R_T = \pm\sqrt{3}R + R$ $\qquad \sqrt{3} \simeq 1.732$

so taking +ve root, $\boxed{R_T = R(1 + \sqrt{3})}$

$\boxed{R_T \simeq 2.73 R}$

{ ie the termination of the chain sequence }

$R_2 = 2\frac{3}{4} R$ $\qquad 3/4 = 0.75$

$R_3 = 2\frac{11}{15} R$ $\qquad 11/15 = 0.73$

$\vdots$

$R_\infty = 2.73 R$

↑ so probably terminates very quickly.

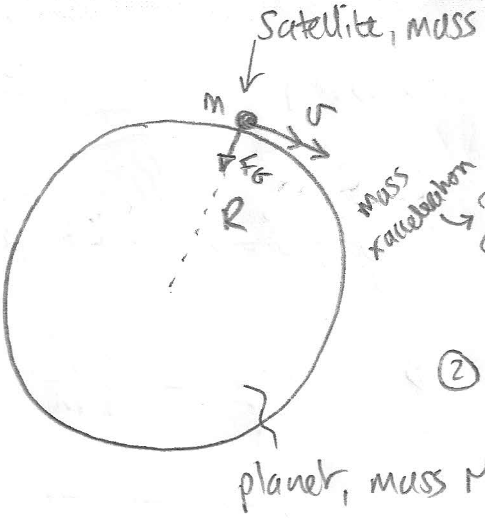
(b)

(i) A satellite is in orbit just above a spherical planet of radius R and uniform density ρ. If the periodic time for each orbit is T, find an expression for ρT^2. Comment on the result.

(ii) A man with a mass of 75 kg stands at the end of a diving board, depressing it by 0.30 m. What would be the period of his motion if he was to jump lightly in rhythm with the harmonic motion of the diving board?

[6]

(i)



Newton II radially:

mass × acceleration

$$\frac{Mv^2}{R} = \frac{GMM}{R^2} \quad (1)$$

F_G

$$(2)\ M = \frac{4}{3}\pi R^3 \rho \qquad v = \frac{2\pi R}{T} \quad (3)$$

combining equations (1), (2), (3):

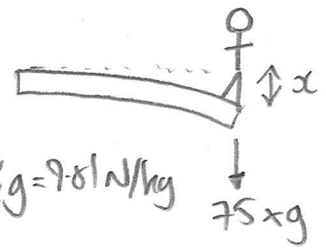
$$\frac{4\pi^2 R^{\cancel{2}}}{T^2} = \frac{G}{\cancel{R}}\frac{4}{3}\pi \cancel{R^3}\rho$$

$$\boxed{\sqrt{\frac{3\pi}{G\rho}} = T}$$

ie $\boxed{\rho T^2 = 3\pi / G}$

So T is independent of planet radius, and ρT^2 is a universal constant (true for all spherical planets).

(ii)



$g = 9.81$ N/kg

Assume diving board is like an elastic, Hookean spring for small displacements.

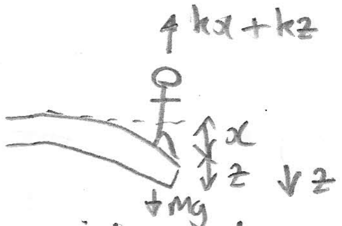
$\therefore$ in eq: $75 \times g = kx$

$\Rightarrow$ Spring constant $k = \dfrac{75 \times 9.81}{0.3}$

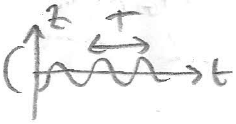
$k = \boxed{2453\ \text{Nm}^{-1}}$

Now if we can neglect the diving board mass, and the man bouncing on the board can be modelled as a mass $M = 75\,kg$ bouncing vertically via a spring of spring constant $k = 2453\,N/m$

$$\boxed{M\ddot{z} = -kz}$$ (Newton II



(z is displacement <u>beyond</u> $x = 0.30\,m$)

$\therefore$ SHM $\quad \ddot{z} = -\left(\frac{2\pi}{T}\right)^2 z$ 

with period T

$$\ddot{z} = -\frac{k}{M} z \quad \Rightarrow \quad \frac{2\pi}{T} = \sqrt{\frac{k}{M}}$$

$$\therefore \quad \boxed{T = 2\pi\sqrt{\frac{M}{k}}} \qquad \therefore \quad T = 2\pi\sqrt{\frac{M}{Mg/x}}$$

[Note $k = mg/x$] $\qquad \therefore \quad \boxed{T = 2\pi\sqrt{\frac{x}{g}}}$

So independent of the mass of the man! ← Although x really is computed from k, x...

$$\therefore \quad T = 2\pi\sqrt{\frac{0.3}{9.81}} = \boxed{1.1\,s}$$

(c)

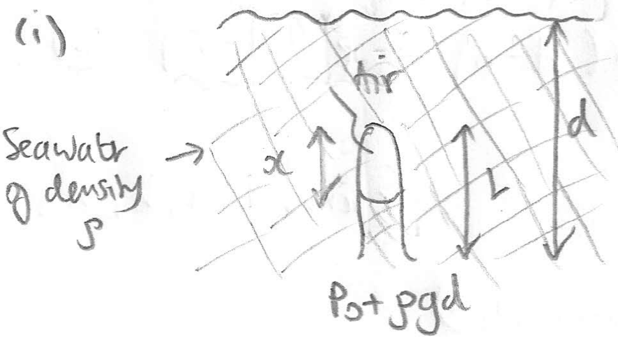
length L

(i) A uniform vertical tube, open at the lower end and sealed at the upper end, is lowered into sea water, trapping air in the tube. When the tube is submerged to a depth of 10.0 m, sea water has exactly filled the lower half of the tube. To what depth must the tube be lowered so that sea water fills 90% of the tube?

(ii) A mercury barometer has some air above the mercury, Figure 1.c.i. The top of the barometer is 1.000 m above the level of the mercury in the reservoir. When the tube is vertical the height of the mercury column is 0.700 m. When the tube is inclined at 60° to the vertical, Figure 1.c.ii, the reading of the mercury level is 0.950 m. What is the atmospheric pressure in mm of mercury? [6]

$g = 9.81$ N/kg

(i)

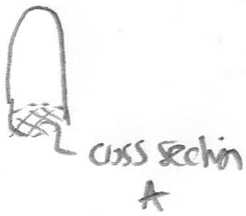


Assume gas in tube is ideal and at same temperature as the sea. ∴ Boyle's law is true

$$\boxed{P_0 L A = (P_0 + \rho g d) x A} \quad *$$

pressure × volume at surface (prior to submergement!)

assume $d \gg L$ so air bubble pressure is $P_0 + \rho g d$ not $P_0 + \rho g (d - L + x)$



$* \Rightarrow P_0 L = x P_0 + \rho g d x$

$\Rightarrow P_0 (L - x) = \rho g d x$

$\Rightarrow \boxed{P_0 = \dfrac{\rho g d x}{L - x}}$

Now when $x = \frac{L}{2}$

$d = 10.0\text{m} = D$

$\Rightarrow \boxed{P_0 = \rho g D}$

(or $P_0 = 10\rho g$ in air case)

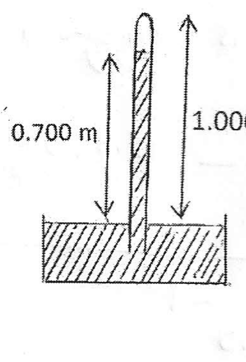
$\dfrac{P_0 (L - x)}{\rho g x} = d \quad \therefore \boxed{d = \dfrac{D(L - x)}{x}}$

Figure 1.c.i

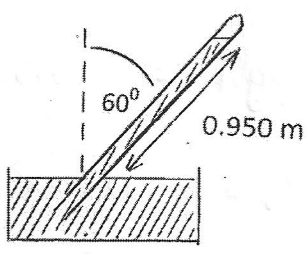
Figure 1.c.ii

So when $x = 0.1L$

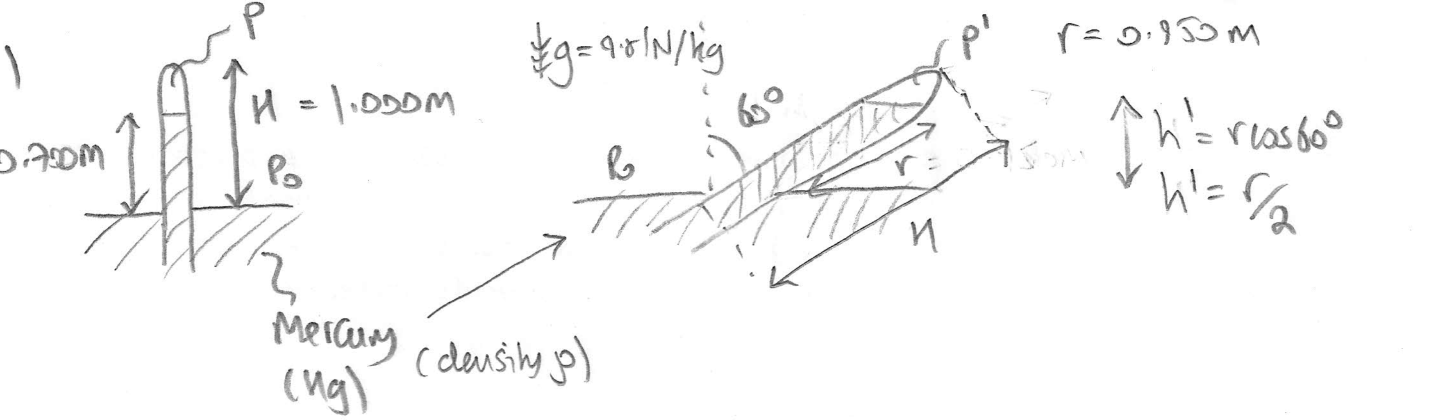
$d = 10.0\text{m} \times \dfrac{0.9}{0.1}$



$\boxed{d = 90.0\text{m}}$



(ii)



Balancing pressures at surface of mercury

$$p + \rho g h = P_0 \qquad (1)$$

$$p' + \rho g h' = P_0 \qquad (2)$$

Boyle's law for gas bubble $\qquad p(H-h) = p'(H-r) \qquad (3)$

$\therefore \quad p = P_0 - \rho g h \qquad$ from (1) $\qquad$ {know: H, h, h', r}

$\therefore \quad p' = (P_0 - \rho g h)\dfrac{(H-h)}{H-r} \qquad$ from (1), (3)

So in (2) $\qquad P_0 = (P_0 - \rho g h)\dfrac{(H-h)}{H-r} + \rho g h'$

$$\Rightarrow \quad \rho g\left(h\frac{H-h}{H-r} - h'\right) = P_0\left(\frac{H-h}{H-r} - 1\right)$$

$$\Rightarrow \quad \boxed{P_0 = \frac{\rho g H\left(\dfrac{h}{H}\dfrac{1-h/H}{1-r/H} - h'/H\right)}{\dfrac{1-h/H}{1-r/H} - 1}}$$

{Perhaps easier if you sub in numbers early on... but you lose generality}.

let $\rho g H = 1000\,mm\,Hg$ $\qquad$ {This is clearly a pressure}

$$\therefore \quad P_0 = 1000\,mm\,Hg \times \frac{\left(0.7 \times \dfrac{1-0.7}{1-0.950} - 0.950/2\right)}{\dfrac{1-0.7}{1-0.950} - 1} = \boxed{745\ mm\,Hg}$$

Can we make the formula for P_0 a little nicer?

$$\rho g \left(h \frac{H-h}{H-r} - h' \right) = P_0 \left(\frac{H-h}{H-r} - 1 \right)$$

$$\rho g \left(h(H-h) - h'(H-r) \right) = P_0 \left(H - h - H + r \right)$$

$$\therefore \quad P_0 = \frac{\rho g H}{r-h} \left(\frac{h}{H}(H-h) - \frac{h'}{H}(H-r) \right)$$

$h' = r\cos\theta$ $\qquad$ ($\theta = 60^\circ$ in our case)

$$P_0 = \rho g H \frac{h}{H} \left(\frac{H-h}{r-h} - \frac{r\cos\theta}{h} \frac{H-r}{r-h} \right)$$

$$\boxed{P_0 = \rho g H \left(\frac{h}{H}\right) \left(\frac{H - h - r\cos\theta (H-r)/h}{r-h} \right)} \quad *$$

Not sure this is any nicer than

$$\boxed{P_0 = \frac{\rho g H \left(\frac{h}{H} \frac{1 - h/H}{1 - r/H} - h'/H \right)}{\frac{1-h/H}{1-r/H} - 1}}$$

* check it works: $P_0 = 1000\text{ mmHg} \times \frac{0.7}{1.0} \left(\frac{0.3 - 0.950\cos 60^\circ (0.050)/0.7}{0.950 - 0.7} \right)$

$= 750\text{ mmHg}$ ✓

Hmm. Dodgy $\sin\theta > 0$. So $x > 0$. But $t_{max} = \frac{\pi}{10}$

(d) A car travels along a horizontal road starting at time $t = 0$, and finishing at $t = \pi/10$. At time t it has travelled a distance x, has a speed v and an acceleration f given by

$x = A\sin(5t)$, $v = 5A\cos(5t)$ and $f = -25A\sin(5t)$, where A is a constant.

Determine the average speed, v_{AV}, and average acceleration, f_{AV}.

[3]

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}}$$

$$= \frac{A\sin(5 \times \pi/10)}{\pi/10} = \frac{A\sin\frac{\pi}{2}}{\pi/10} = \boxed{\frac{10A}{\pi}}$$

$$\text{Average acceleration} = \frac{\text{Velocity change}}{\text{time taken}}$$

$$= \frac{5A\cos(5 \times \pi/10) - 5A\cos(0)}{\pi/10}$$

$$= \frac{0 - 5A}{\pi/10} = \boxed{-\frac{50A}{\pi}}$$

(e) One gram of hydrogen atoms is separated into electrons and protons. The electrons are deposited on the Moon, the protons remaining on the Earth. What, numerically, is the force that results? The Earth – Moon distance is R_{EM} = 3.84 x 10^8 m.

[5]

Force (by Coulomb's law) is $\boxed{\dfrac{n^2 e^2}{4\pi\varepsilon_0 R_{EM}^2}}$

# of protons (or electrons) is n

Ignoring the mass of the electron, mass of a Hydrogen atom $\approx M_p = 1.673 \times 10^{-27}$ kg

$\therefore \; n = \dfrac{10^{-3}\text{ kg}}{1.673\times10^{-27}\text{ kg/proton}} = \boxed{5.98\times10^{23}}$

[MS says that N_A H_2 molecules is 2.00g

$\therefore$ 1.00g has $\frac{1}{2}\times N_A$ molecules or N_A protons]

ie $n = N_A = 6.02\times10^{23}$, not 5.98×10^{23}

so force is $\approx \dfrac{5.98^2 \times (1.60\times10^{-19})^2}{4\pi\times8.85\times10^{-12}\times(3.84\times10^{8})^2}\times10^{46}$

$\approx 5.98^2 \times 15.6$ N

$= \boxed{558\text{ N}}$ (!!)

[MS is incorrect here I think. $\frac{1}{2}N_A$ molecules of H_2 in 1g, but this is N_A protons and N_A electrons.

$\therefore$ charge is $N_A e$ not $N_A e/2$

MS is $\left(\dfrac{6.02}{2}\right)^2 \times 15.6\text{ N} = 141\text{ N}$]

(f) Determine the half life of uranium given that 3.23 x 10^{-7}g of radium is found per gram of uranium in ancient minerals. The half life of radium is 1,600 years. The atomic weights of uranium and radium are 238 and 226 respectively. All the radium arises from the uranium. R

u

[5]

let N be # of radioactive atoms:

$$\boxed{\frac{dN_u}{dt} = -\lambda_u N_u} \qquad \boxed{\frac{dN_R}{dt} = -\lambda_R N_R}$$

Assume amount of radium is a constant

$\therefore$ rate of production of radium (from Uranium)

$= -\frac{dN_u}{dt}$, which also = rate of loss of radium, $-\frac{dN_R}{dt}$

ie $$\boxed{\lambda_u N_u = \lambda_R N_R}$$

$$\Rightarrow \quad \frac{\ln 2}{T_u} N_u = \frac{\ln 2}{T_R} N_R$$

T is $\frac{1}{2}$ life

$$\therefore \quad \boxed{\frac{N_u}{N_R} T_R = T_u}$$

Unified atomic mass unit
1.660 x10^{-27} kg

In 1g of U: $238u \times N_u = 10^{-3}$ kg

In 3.23 x10^{-7}g of R: $226u \times N_R = 3.23\times10^{-10}$ kg

$$\therefore \quad \frac{238}{226}\frac{N_u}{N_R} = \frac{1}{3.23\times10^{-7}} \quad \therefore \quad \frac{N_u}{N_R} = \frac{226}{238}\frac{1}{3.23\times10^{-7}}$$

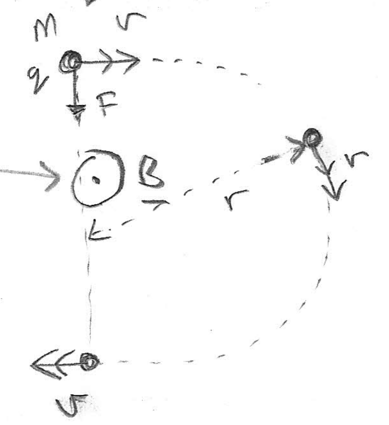
$$\therefore \quad T_u = 1600 \text{ years} \times \frac{226}{238} \times \frac{1}{3.23\times10^{-7}}$$

$$\boxed{T_u = 4.70\times10^{8} \text{ years}}$$

(g) A mixed beam of deuterons (an isotope of hydrogen, $^2H^+$) and protons, which have been accelerated through 1.00×10^4 V, enter a uniform magnetic field of 0.500 T in a direction at right angles to the field. Calculate the separation of the proton beam from the deuteron beam when each has described a semicircle in the field.

[6]

proton beam is a current



Left hand rule or $q\underline{v}\times\underline{B}$ is a radially inward force $\underline{F}$

Newton II radially:

$$\frac{mv^2}{r} = Bqv$$

$$\therefore r = \frac{mv^2}{Bqv}$$

$$\therefore \boxed{r = \frac{mv}{Bq}}$$

let m_D, m_p be deuteron and proton masses

$q = e = 1.6\times10^{-19}$C for both.

$m_D \approx 2m_p$

Now assuming classical physics

$$\frac{1}{2}mv^2 = qV \qquad \therefore \boxed{v = \sqrt{\frac{2qV}{m}}}$$

$$\therefore r = \frac{m}{Bq}\sqrt{\frac{2qV}{m}}$$

$$\boxed{r = \sqrt{\frac{2Vm}{B^2q}}}$$

$\therefore$ Separation after a semicircular path is:

$$\delta = 2r_D - 2r_p$$

$$\boxed{\delta = 2\sqrt{\frac{2V}{B^2e}}\left(\sqrt{m_D} - \sqrt{m_p}\right)}$$

{Not given, but $m_D = 3.344\times10^{-27}$ kg, $m_p = 1.673\times10^{-27}$ kg ↑ close to ×2 (3.346×10^{-27}) }

$$\therefore \delta = 2\sqrt{\frac{2\times1.00\times10^4}{0.500^2\times1.60\times10^{-19}}}\left(\sqrt{3.344} - \sqrt{1.673}\right)\times10^{-27/2} \text{ (m)}$$

$\approx (\sqrt{2}-1)\sqrt{1.673} = 0.536$

$\delta = 4.47 \text{ cm} \times 0.535$

$\boxed{\delta \approx 2.4\text{cm}}$

(h) A sound source, frequency f and velocity u, is moving along a straight line towards an observer who is approaching the sound source with velocity v. Determine the frequency f_O heard by the observer if the speed of sound is c.

A moving sound source, S, has velocity of 15.0 m s^{-1} and frequency 200 Hz. An observer P, speed 18.0 ms^{-1}, and S are approaching a point Q along paths inclined at 30° to each other, Figure 1.h. What frequency is heard by the observer when S and P are equidistant from Q? The speed of sound is 331 ms^{-1}.

[8]

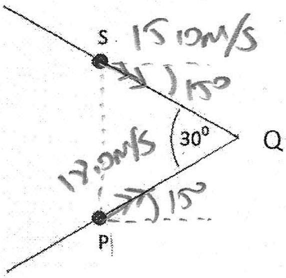
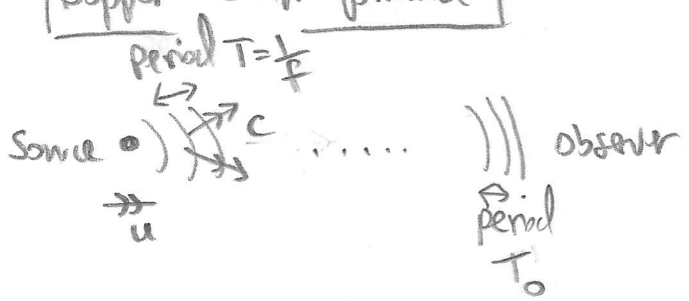


Figure 1.h

$\boxed{\text{Doppler shift formula:}}$



Time between wave crests at observer is:

$$T_o = T - \frac{uT}{c}$$

$$\therefore \frac{1}{f_o} = \frac{1}{f}\left(1 - \frac{u}{c}\right)$$

$$\therefore \boxed{f_o = \frac{f}{1 - u/c}}$$

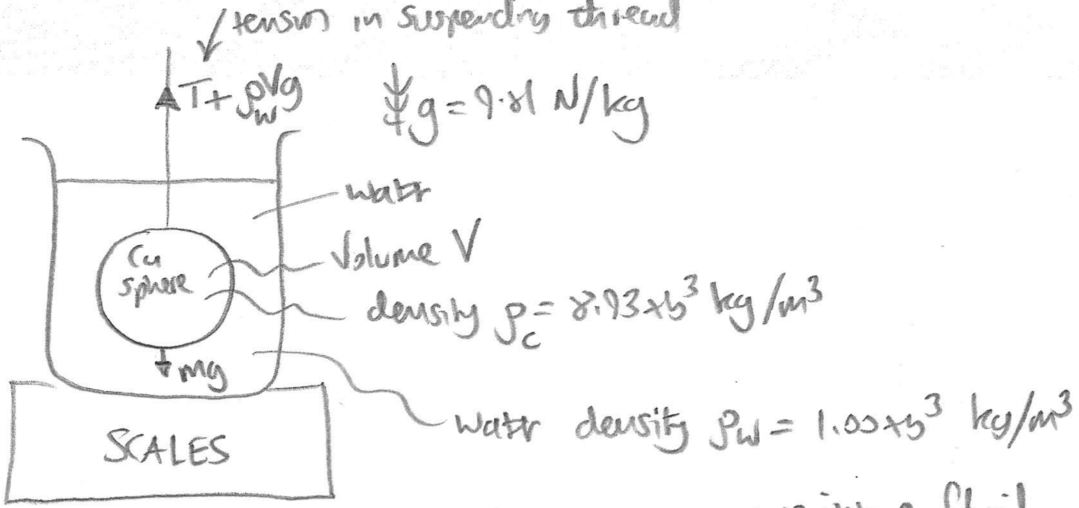
Now in the situation described, P approaches S upwards at a velocity of $18.0 \sin 15°$ m/s and S approaches P downwards at $15.0 \sin 15°$ m/s. $\therefore$ Relative to P, S approaches at $\boxed{u = (18+15)\sin 15°}$ m/s

$$\therefore f_o = \frac{200\,\text{Hz}}{1 - \frac{(18+15)\sin 15°}{331}} = \boxed{205 \text{ Hz}}$$

(i) A beaker, containing some water, has a total mass of 0.300 kg. The beaker rests on a weighing scale. A 250 g copper sphere, density 8.93×10^3 kg m^{-3}, is suspended so that it is completely immersed in the water, but does not touch the bottom of the beaker. What is the reading on the weighing scale in newtons? The density of water is 1.00×10^3 kg m^{-3}.

M

[3]



$g = 9.81$ N/kg

* Volume $V = m/\rho_c$

* Upthrust of water on sphere $= \underbrace{\rho_W V g}_{\text{weight of fluid displaced by sphere}} = \dfrac{\rho_W m g}{\rho_c}$

* By Newton III, the force ("downthrust") of sphere on the water is $\therefore \rho_W mg/\rho_c$

$\therefore$ Weight on the scales is = original weight $W_0 + \rho_W mg/\rho_c$

ie

$$\boxed{W = W_0 + \rho_W mg/\rho_c}$$

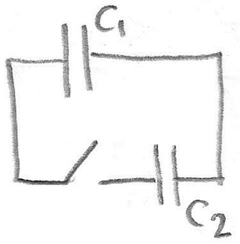
$$W = \left(0.300 + \frac{1.00 \times 10^3}{8.93 \times 10^3} \times 0.250\right) \times 9.81$$

$$= \boxed{3.21\ N}$$

(j) A capacitor, capacitance C_1, with a charge Q_0, is connected in a closed (loop) series circuit with an uncharged capacitor, capacitance C_2 and a switch, which is initially open. Compare the energy stored in the capacitors before and after the switch is closed by considering the potential across each capacitor. What can one conclude?

[8]

BEFORE

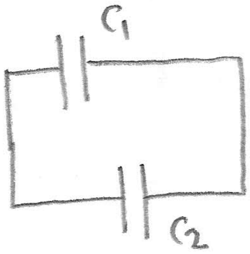


Energy $E_B = \frac{1}{2} C_1 V^2$ $\qquad Q_0 = C_1 V \therefore V = Q_0/C_1$

So $E_B = \frac{1}{2} C_1 Q_0^2/C^2$

$$\boxed{E_B = \frac{1}{2} Q_0^2/C_1}$$

After



Now if no more current flows $V_1 = V_2 \therefore \boxed{\frac{Q_1}{C_1} = \frac{Q_2}{C_2}}$

By conservation of charge $Q_1 + Q_2 = Q_0$

So $Q_1 = \frac{C_1}{C_2}(Q_0 - Q_1)$

$\therefore Q_1 (1 + C_1/C_2) = \frac{C_1}{C_2} Q_0$

$$\therefore \boxed{Q_1 = \frac{C_1 Q_0}{C_2 + C_1}}$$

$\therefore Q_2 = Q_0 - Q_1$

$Q_2 = Q_0 \left(1 - \frac{C_1}{C_2 + C_2}\right)$

$$\boxed{Q_2 = \frac{C_2 Q_0}{C_2 + C_1}}$$

Total energy stored is now:

$$E_A = \frac{1}{2} Q_1^2/C_1 + \frac{1}{2} Q_2^2/C_2 = \frac{1}{2} Q_0^2 \left(\frac{1}{C_2 + C_1} + \frac{1}{C_2 + C_1}\right) = \boxed{\frac{1}{2} \frac{Q_0^2}{C_1 + C_2}}$$

So this means some energy is lost due to heating:

$$\Delta E = E_B - E_A$$

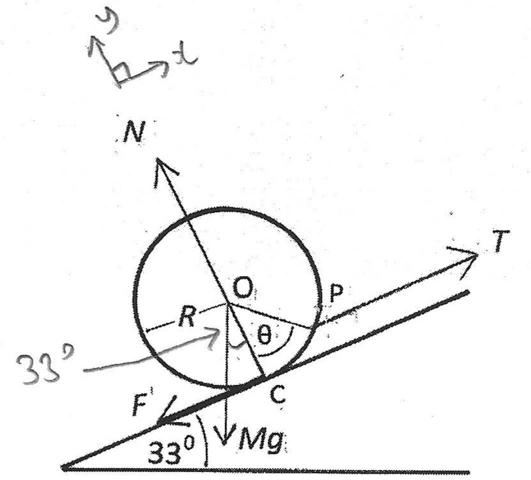
$$\boxed{\Delta E = \frac{Q_0^2}{2}\left(\frac{1}{C_1} - \frac{1}{C_1+C_2}\right)}$$

[For completeness: $V_1 = \dfrac{Q_1}{C_1} = \dfrac{Q_0}{C_1+C_2}$

$V_2 = \dfrac{Q_2}{C_2} = \dfrac{Q_0}{C_1+C_2}$]

ie capacitors in // add

(k)



$\downarrow$ 9.81 N/kg $= g$

Figure 1.k

A uniform sphere, radius R and mass M = 5.00 kg, is pulled up an inclined plane, inclination 33.0° to the horizontal, by a string of tension T, which is attached to a point P on its surface, making an angle θ with the line joining the centre of the sphere, O, and its contact point with the plane, C. The string is parallel to the plane. The coefficient of friction between the sphere and the plane μ = 0.420. The sphere is about to slide up the plane. The frictional force is F and the normal reaction is N, Figure 1.k.

ie limiting friction, (just) in equilibrium

Determine, numerically:

(i) T, by resolving the forces along and perpendicular to the slope ($//x$, $//y$)

(ii) θ.

[10]

(i) Newton II: $//x$: $0 = T - Mg\sin 33° - F$ (1)

$//y$: $0 = N - Mg\cos 33°$ (2)

Friction: $F = \mu N$ (3)

$\therefore$ $\boxed{F = \mu Mg\cos 33°}$

$\therefore$ $\boxed{T = Mg\sin 33° + \mu Mg\cos 33°}$

$\therefore$ $T = 5.00 \times 9.81\left(\sin 33° + 0.420\cos 33°\right)$

$= \boxed{44.0\ \text{N}}$

(ii) Taking $\overset{+}{\circlearrowleft}$ moments about O

$0 = TR\cos\theta - FR$ $\therefore$ $\cos\theta = F/T$

$\therefore$ $\theta = \cos^{-1}\left(\dfrac{\mu\cos 33°}{\sin 33° + \mu\cos 33°}\right) = \boxed{66.9°}$

16

This is very similar to Q1 of "200 puzzling physics problems" (Gnädig et al). "Snails of pursuit".

(I) Three boats start at time $t = 0$ from the corners of an equilateral triangle, of side 50 km, and maintain constant speeds of 30 km hr^{-1} during the subsequent motion. They each maintain a heading, clockwise, towards the neighbouring boat. They all eventually meet at P.

oops, I've opted for anticlockwise – as does the MS!!

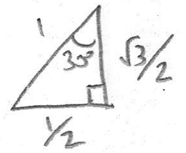
Determine:

(i) qualitatively, the evolution of the triangle formed by the three boats

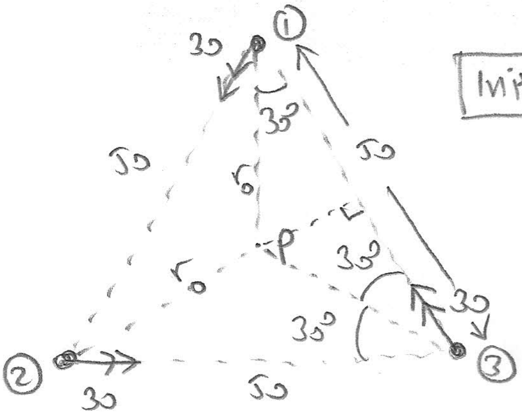
(ii) the velocity components of the three boats in the direction of P, as a function of time, t, and in the perpendicular directions

(iii) the time, t_M, at which they all meet

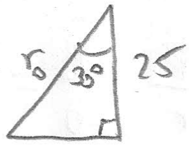
(iv) the distance travelled by each boat, D.



[8]



Initial condition

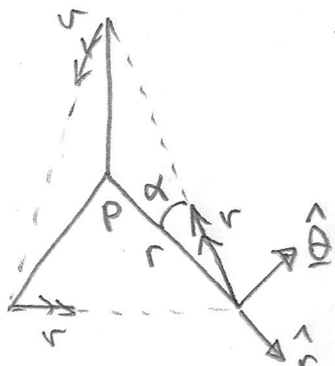


$r_0 \cos 30° = 25$

So $r_0 = \dfrac{25}{\sqrt{3}/2} = \boxed{\dfrac{50}{\sqrt{3}}}$ km

(i) As the boats sail, they will preserve the triangular shape, but this will rotate and reduce in size r.

Consider polar coordinates, centred at P for the boats:



Velocity $\boxed{\underline{v} = \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}}}$ of any of the boats

$\boxed{\dot{r} = -v\cos\alpha}$ $\alpha = 30°$

and $\boxed{r\dot{\theta} = v\sin\alpha}$ *

So the boats will meet at P after

$t_M = \dfrac{r_0}{\underbrace{15\sqrt{3}}_{|\dot{r}|}} = \dfrac{50}{\sqrt{3}\times 15\sqrt{3}} = \boxed{1\tfrac{1}{9}\text{ hours}}$

[* ie to answer (ii) $\underline{v} = -15\sqrt{3}\,\hat{\underline{r}} + 15\,\hat{\underline{\theta}}$ (km/h)]

So if boats are travelling at 30 km/h they travel $\boxed{D = 33.3\text{ km}}$

$\boxed{\text{Extra!}}$

$$\dot{r} = -v\cos\alpha$$
$$r\dot{\theta} = v\sin\alpha$$

$$\therefore \quad \frac{\dot{r}}{r\dot{\theta}} = -\cot\alpha$$

Now $$\frac{\dot{r}}{\dot{\theta}} = \frac{dr}{dt} \times \frac{dt}{d\theta} = \frac{dr}{d\theta}$$

$$\therefore \quad \frac{1}{r}\frac{dr}{d\theta} = -\cot\alpha$$

$$\Rightarrow \quad \frac{1}{r}dr = -\cot\alpha \, d\theta$$

$$\Rightarrow \quad \int_{r_0}^{r} \frac{1}{r} dr = -\cot\alpha \int_{\theta_0}^{\theta} d\theta$$

$$\Rightarrow \quad \ln\left(\frac{r}{r_0}\right) = -\cot\alpha\,(\theta - \theta_0)$$

$$\Rightarrow \quad \boxed{r(\theta) = r_0 e^{-(\theta-\theta_0)\cot\alpha}}$$

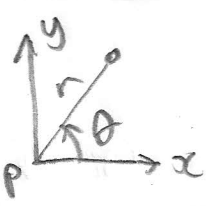
Logarithmic Spiral.

Now $$\dot{r} = -v\cos\alpha$$

$$\therefore \quad \boxed{r(t) = r_0 - vt\cos\alpha}$$

So plotting in terms of time:

$$\boxed{\theta = \theta_0 - \ln\left(\frac{r_0 - vt\cos\alpha}{r_0}\right)\tan\alpha}$$



$y = r\sin\theta$
$x = r\cos\theta$

Initial conditions: $r = r_0 = 50/\sqrt{3}$

1. $\theta_0 = \pi/2$
2. $\theta_0 = -\pi/6$
3. $\theta_0 = \frac{7}{6}\pi$

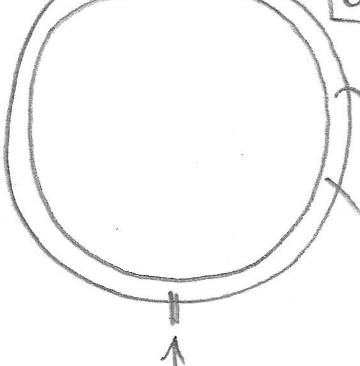
(m) A bicycle tyre has a volume of $1.20 \times 10^{-3}\ m^3$ when fully inflated. The barrel of the bicycle pump has a working volume of $9.0 \times 10^{-5}\ m^3$. How many strokes of the pump are needed to completely inflate the flat tyre to a total pressure of 3.0×10^5 Pa? The atmospheric pressure is 1.00×10^5 Pa. Assume the air is pumped in slowly, so that the temperature remains constant.

[4]

* Assume ideal gases, at constant temperature

[Assume N strokes of pump, each pumping n moles of gas]

Bike tyre



Nn moles of air

pressure p, temperature T

$$\boxed{pV = NnRT} \quad ① \qquad \text{Bike tyre}$$

pump

For pump: $\boxed{P_a V_p = nRT}$ ②

So ①/② $\quad \dfrac{pV}{P_a V_p} = N$

$\therefore$ # strokes $\quad N = \dfrac{(3.0\times10^{5})(1.20\times10^{-3})}{(1.00\times10^{5})(9.0\times10^{-5})}$

$= \boxed{40}$

(19)

AF May 2019