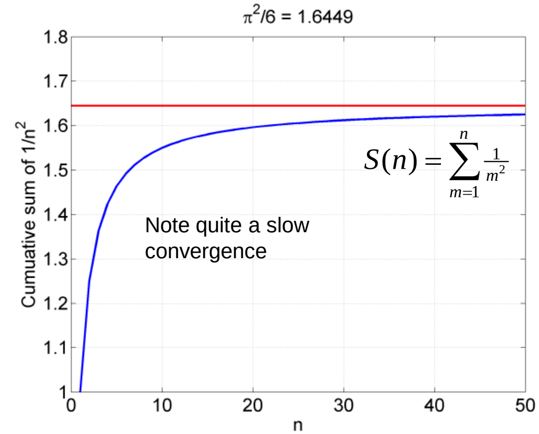
**The Basel problem** was first proposed by Pietro Mengoli in 1650 and first solved by Euler in 1734. A nice solution (not Euler's!) involves the use of a **Fourier series** and applies **Parseval's Theorem.**

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\tfrac{1}{6}\pi^2$$

Note the sum is a special case of the **Riemann zeta function** defined for complex number inputs such that $\mathrm{Re}(z)>1$

$$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$$




Define a **complex Fourier series** for a function defined over interval [ $-L/2, L/2$ ]. The Fourier series will repeat periodically at intervals of $L$.

$$f(x)=\sum_{n=-\infty}^{\infty}a_n e^{i\frac{2n\pi x}{L}}$$

$$a_n=\tfrac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}f(x)e^{-i\frac{2n\pi x}{L}}dx$$

To prove the coefficients result:

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L}f(x)e^{-i\frac{2m\pi x}{L}}dx=\sum_{n=-\infty}^{\infty}a_n\int_{-\frac{1}{2}L}^{\frac{1}{2}L}e^{i\frac{2(n-m)\pi x}{L}}dx$$

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L}e^{i\frac{2(n-m)\pi x}{L}}dx=\begin{cases}0 & n\neq m\\ L & n=m\end{cases}$$

$$\therefore\int_{-\frac{1}{2}L}^{\frac{1}{2}L}f(x)e^{-i\frac{2m\pi x}{L}}dx=La_m$$

$$\therefore a_n=\tfrac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}f(x)e^{-i\frac{2n\pi x}{L}}dx$$

Let's also prove **Parseval's Theorem** for a Fourier series

$$\tfrac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}|f(x)|^2dx=\sum_{n=-\infty}^{\infty}|a_n|^2$$

$$\tfrac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}|f(x)|dx=\tfrac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\left(\sum_{n=-\infty}^{\infty}a_n e^{i\frac{2n\pi x}{L}}\right)\left(\sum_{m=-\infty}^{\infty}a_m^* e^{-i\frac{2m\pi x}{L}}\right)dx$$

$$\tfrac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}|f(x)|dx=\tfrac{1}{L}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}a_n a_m^*\int_{-\frac{1}{2}L}^{\frac{1}{2}L}e^{i\frac{2(n-m)\pi x}{L}}dx$$

$$\tfrac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}|f(x)|dx=\tfrac{1}{L}\sum_{n=-\infty}^{\infty}|a_n|^2L$$

$$\therefore\tfrac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}|f(x)|dx=\sum_{n=-\infty}^{\infty}|a_n|^2$$

Conjugation of complex numbers

$$(a+ib)^*=a-ib$$

$$(a+ib)(a-ib)=a^2+b^2$$

$$\therefore(a+ib)(a+ib)^*=|a+ib|^2$$

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L}e^{i\frac{2(n-m)\pi x}{L}}dx=\begin{cases}0 & n\neq m\\ L & n=m\end{cases}$$

Pietro Mengoli (1626-1686)

Leonhard Euler 1707-1783

To **solve the Basel problem** consider the following function and interval:

$$f(x)=x$$

$$L=2\pi$$

$$f(x)=\sum_{n=-\infty}^{\infty}a_n e^{i\frac{2n\pi x}{L}}$$

$$\cos x=\tfrac{1}{2}\left(e^{ix}+e^{-ix}\right)$$

$$\sin x=\tfrac{1}{2i}\left(e^{ix}-e^{-ix}\right)$$

$$\cos n\pi=(-1)^n,\quad \sin n\pi=0$$

$$a_0=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}x\,dx=0$$

$$a_n=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}xe^{-inx}dx\quad n\neq0$$

$$a_n=\tfrac{1}{2\pi}\left\{\tfrac{1}{-in}\left[xe^{-inx}\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\tfrac{1}{-in}e^{-inx}(1)dx\right\}$$

$$a_n=\tfrac{1}{2\pi}\left\{\tfrac{i\pi}{n}\left[e^{in\pi}+e^{-in\pi}\right]_{-\pi}^{\pi}-\tfrac{i}{n}\left[\tfrac{1}{-in}e^{-inx}\right]_{-\pi}^{\pi}\right\}$$

$$a_n=\tfrac{1}{2\pi}\left\{\tfrac{i\pi}{n}\left[e^{in\pi}+e^{-in\pi}\right]_{-\pi}^{\pi}-\tfrac{1}{n^2}\left[e^{in\pi}-e^{-in\pi}\right]_{-\pi}^{\pi}\right\}$$

$$a_n=\tfrac{1}{2\pi}\left\{\tfrac{2i\pi}{n}\cos n\pi-\tfrac{2i}{n^2}\sin n\pi\right\}$$

$$\therefore a_n=\frac{i(-1)^n}{n}$$

$$\tfrac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}|f(x)|dx=\sum_{n=-\infty}^{\infty}|a_n|^2$$

Parseval's Theorem with inputs:

$$f(x)=x,\ L=2\pi$$

$$a_n=\begin{cases}\dfrac{i(-1)^n}{n} & n\neq0\\ 0 & n=0\end{cases}$$

$$\therefore\tfrac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx=2\sum_{n=1}^{\infty}\tfrac{1}{n^2}$$

$$\therefore\tfrac{1}{2\pi}\left[\tfrac{1}{3}x^3\right]_{-\pi}^{\pi}=2\sum_{n=1}^{\infty}\tfrac{1}{n^2}$$

$$\therefore\tfrac{1}{6\pi}\left(\pi^3-(-\pi)^3\right)=2\sum_{n=1}^{\infty}\tfrac{1}{n^2}$$

$$\therefore\tfrac{2\pi^3}{6\pi}=2\sum_{n=1}^{\infty}\tfrac{1}{n^2}$$

$$\therefore\sum_{n=1}^{\infty}\tfrac{1}{n^2}=\tfrac{1}{6}\pi^2$$

Joseph Fourier 1768-1830

Antoine Parseval (1755-1836)