Calculus toolbox

The **differential** (or 'derivative') of a function $y = f(x)$ is the gradient of the curve $y = f(x)$ at the point (x,y) .

$$y = f(x)$$

Definition of the derivative →

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x}\right)$$

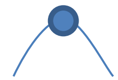
Alternative forms of derivatives

$$f'(x) \equiv \frac{df}{dx} \qquad f''(x) \equiv \frac{d^2 f}{dx^2} \qquad f^{(n)}(x) \equiv \frac{d^n f}{dx^n}$$

$$\dot{x} \equiv \frac{dx}{dt} \qquad \ddot{x} \equiv \frac{d^2 x}{dt^2}$$

Stationary points i.e. zero gradient

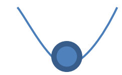
$$\left.\frac{dy}{dx}\right|_{\text{stationary point}} = 0$$



$y = ax^2,\ a < 0$

Maxima

$$\left.\frac{d^2 y}{dx^2}\right|_{\text{stationary point}} < 0$$



$y = ax^2,\ a > 0$

Minima

$$\left.\frac{d^2 y}{dx^2}\right|_{\text{stationary point}} > 0$$

Product Rule

$$y = u(x)v(x)$$

$$y = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule

$$y = \frac{u(x)}{v(x)}$$

$$y = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Chain Rule

$$y = f(z)$$

$$z = g(x)$$

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

Implicit differentiation

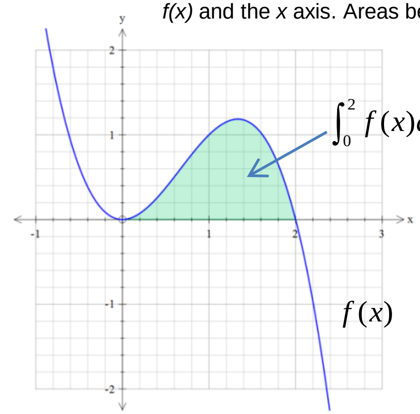
$$\frac{d}{dx} f(y) = \frac{df}{dy} \times \frac{dy}{dx}$$

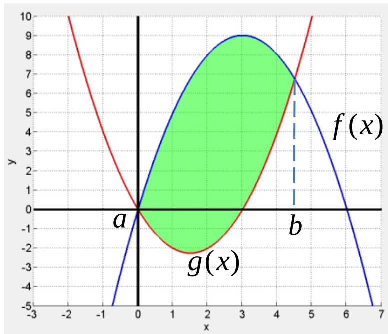
$$\text{e.g. } \frac{d}{dx} y^3 = 3y^2 \frac{dy}{dx}$$

Reciprocity

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$$

A **definite integral** can be used to find the area between a curve $y = f(x)$ and the x axis. Areas below the x axis are *negative*.





Area between two curves

$$A = \int_a^b \left(f(x) - g(x)\right)dx$$

Integration is *anti-differentiation*

Differentiate

$$y \rightarrow \frac{dy}{dx}$$

Integrate

$$\int \frac{dy}{dx} dx$$

$$\int f'(x)dx = f(x) + c$$

$$\frac{dF}{dx} = f(x)$$ *F(x)* is the *anti-differential* of *f(x)*

$$\int_a^b f(x)dx = F(b) - F(a)$$ **Definite integral**

$$\int f(x)dx = F(x) + c$$ **Indefinite integral**

Note integration *creates information* (i.e. the constant of integration)

Integrals mean 'sum rectangular strips of height *f(x)* and width Δx' (in the limit Δx -> zero). This means they have the same properties as sums of discrete numbers.

$$\int \{f(x) + g(x)\}dx = \int f(x)dx + \int g(x)dx$$

$$\int af(x)dx = a\int f(x)dx$$

Ratio of derivative to function

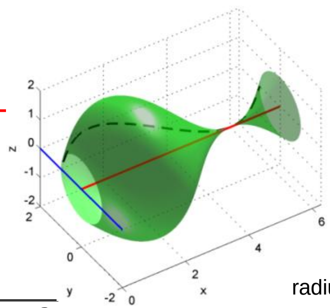
$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

Volumes of revolution

$$V_x = \int_{x=a}^b \pi y^2 dx$$

$$V_y = \int_{y=a}^b \pi x^2 dy$$

$$A_x = 2\pi\int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$ ← surface area



Length along a curve

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

radius of curvature

$$R = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} \left(\frac{d^2 y}{dx^2}\right)^{-1}$$

Integration by-parts

$$\int (uv)\,dx = u\int v dx - \int \left\{\frac{du}{dx}\left(\int v dx\right)dx\right\}$$

$$\int_a^b (uv)\,dx = \left[u\int v dx\right]_a^b - \int_a^b \left\{\frac{du}{dx}\left(\int v dx\right)\right\}dx$$

Calculus of Variations

$$I = \int_a^b f(y, y', x)dx, \quad y' = \frac{dy}{dx}$$

If *I* is *stationary* (i.e. maxima or minima)

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$ *Euler-Lagrange Equation*

Derivatives and integrals of basic functions .... and some useful identities involving basic functions

For all trigonometric functions, *x* is in **radians**

$\frac{dy}{dx}$	$y=f(x)$	$\int f(x)dx$
$\frac{dy}{dx}=nx^{n-1}$	$y=x^n$	$\int x^n dx=\frac{1}{n+1}x^{n+1}+c,\quad n\neq -1$
$\frac{dy}{dx}=ae^{ax}$	$y=e^{ax}$	$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$
$\frac{dy}{dx}=ab^{ax}\ln b$	$y=b^{ax}=\left(e^{\ln b}\right)^{ax}=e^{ax\ln b}$	$\int b^{ax}dx=\frac{b^{ax}}{a\ln b}+c$
$\frac{dy}{dx}=\frac{1}{x}$	$y=\ln\|x\|$	$\int \ln\|x\|dx=x\ln\|x\|-x+c$
$\frac{dy}{dx}=a\cos ax$	$y=\sin ax$	$\int \sin ax dx=-\frac{1}{a}\cos ax+c$
$\frac{dy}{dx}=-a\sin ax$	$y=\cos ax$	$\int \cos ax dx=\frac{1}{a}\sin ax+c$
$\frac{dy}{dx}=a\sec^2 ax$	$y=\tan ax$	$\int \tan ax dx=-\frac{1}{a}\ln\|\cos ax\|+c$

$\frac{dy}{dx}$	$y=f(x)$	$\int f(x)dx$
$\frac{dy}{dx}=\sec x\tan x$	$y=\sec x$	$\int \sec x dx=\ln\|\sec x+\tan x\|+c$
$\frac{dy}{dx}=-\cot x\times\operatorname{cosec} x$	$y=\operatorname{cosec} x$	$\int \operatorname{cosec} x dx=-\ln\|\operatorname{cosec} x+\cot x\|+c$
$\frac{dy}{dx}=-\operatorname{cosec}^2 x$	$y=\cot x$	$\int \cot x dx=\ln\|\sin x\|+c$
$\frac{dy}{dx}=\frac{a}{\sqrt{1-a^2x^2}}$	$y=\sin^{-1} ax$	$\int \sin^{-1} x dx=\frac{1}{a}\sqrt{1-a^2x^2}+x\sin^{-1} ax+c$
$\frac{dy}{dx}=-\frac{a}{\sqrt{1-a^2x^2}}$	$y=\cos^{-1} ax$	$\int \cos^{-1} x dx=x\cos^{-1} ax-\frac{1}{a}\sqrt{1-a^2x^2}+c$
$\frac{dy}{dx}=-\frac{a}{1+a^2x^2}$	$y=\tan^{-1} ax$	$\int \tan^{-1} ax dx=x\tan^{-1} ax-\frac{1}{2a}\ln\left(a^2x^2+1\right)+c$

$$\sin 2x=2\sin x\cos x$$
$$\cos 2x=\cos^2 x-\sin^2 x$$

$$\sec x=\frac{1}{\cos x}\qquad \operatorname{cosec} x=\frac{1}{\sin x}\qquad \cot x=\frac{1}{\tan x}$$

$$e^{ix}=\cos x+i\sin x$$
$$\cos x=\tfrac{1}{2}\left(e^{ix}+e^{-ix}\right)$$
$$\sin x=\tfrac{1}{2i}\left(e^{ix}-e^{-ix}\right)$$

$$\sin^2 x+\cos^2 x=1$$
$$\tan^2 x+1=\sec^2 x$$
$$\cot^2 x+1=\operatorname{cosec}^2 x$$

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$
$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$
$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

$$\log_b x+\log_b y=\log_b xy\qquad \log_b x-\log_b y=\log_b\frac{x}{y}$$
$$\log_b x^a=a\log_b x\qquad \log_a x=\frac{\log_b x}{\log_b a}$$
$$x=b^{\log_b x}\qquad N^x=10^{x\log_{10}N-\mathrm{floor}(x\log_{10}N)}\times 10^{\mathrm{floor}(x\log_{10}N)}$$

floor() means *round down to the nearest integer*. Use this result to evaluate very large (or very small) numbers of the form N^x in standard form.

More derivatives and integrals of trigonometric functions

$\frac{dy}{dx}$	$y=f(x)$	$\int f(x)dx$
$\frac{dy}{dx}=m\cos mx\cos nx-n\sin mx\sin nx$	$y=\sin mx\cos nx$	$\int\sin mx\cos nx dx=-\frac{\cos(m-n)x}{2(m-n)}-\frac{\cos(m+n)x}{2(m+n)},\quad m^2\neq n^2$
$\frac{dy}{dx}=m\cos mx\sin nx+n\sin mx\cos nx$	$y=\sin mx\sin nx$	$\int\sin mx\sin nx dx=\frac{\sin(m-n)x}{2(m-n)}-\frac{\sin(m+n)x}{2(m+n)},\quad m^2\neq n^2$
$\frac{dy}{dx}=-m\sin mx\cos nx-n\sin nx\cos mx$	$y=\cos mx\cos nx$	$\int\cos mx\cos nx dx=\frac{\sin(m-n)x}{2(m-n)}+\frac{\sin(m+n)x}{2(m+n)},\quad m^2\neq n^2$
$\frac{dy}{dx}=\cos 2x$	$y=\sin x\cos x$	$\int\sin x\cos x dx=-\tfrac{1}{2}\cos^2 x+c$
$\frac{dy}{dx}=\sin 2x$	$y=\sin^2 x$	$\int\sin^2 x dx=\tfrac{1}{2}x-\tfrac{1}{4}\sin 2x+c$
$\frac{dy}{dx}=-2\sin x\cos x$	$y=\cos^2 x$	$\int\cos^2 x dx=\tfrac{1}{2}x+\tfrac{1}{4}\sin 2x+c$
$\frac{dy}{dx}=2\tan x\sec^2 x$	$y=\tan^2 x$	$\int\tan^2 x dx=\tan x-x+c$
$\frac{dy}{dx}=3\sin^2 x\cos x$	$y=\sin^3 x$	$\int\sin^3 x dx=\tfrac{1}{12}\left(\cos 3x-9\cos x\right)+c$
$\frac{dy}{dx}=-3\cos^2 x\sin x$	$y=\cos^3 x$	$\int\cos^3 x dx=\tfrac{1}{12}\left(9\sin x+\sin 3x\right)+c$
$\frac{dy}{dx}=3\tan^2 x\sec^2 x$	$y=\tan^3 x$	$\int\tan^3 x dx=\tfrac{1}{2}\sec^2 x+\ln\left(\cos x\right)+c$

$\sin^2 x=\tfrac{1}{2}(1-\cos 2x)$ $\quad\cos^2 x=\tfrac{1}{2}(1+\cos 2x)$

$\sin^3 x=\tfrac{1}{4}(3\sin x-\sin 3x)$ $\quad\cos^3 x=\tfrac{1}{4}(3\cos x+\cos 3x)$

Derivatives and integrals of hyperbolic functions

$\frac{dy}{dx}$	$y=f(x)$	$\int f(x)dx$
$\frac{dy}{dx}=a\sinh ax$	$y=\cosh ax$	$\int \sinh ax dx=\frac{1}{a}\sinh ax+c$
$\frac{dy}{dx}=a\cosh ax$	$y=\sinh ax$	$\int \sinh ax dx=\frac{1}{a}\cosh ax+c$
$\frac{dy}{dx}=a\,\text{sech}^2\,ax$	$y=\tanh ax$	$\int \tanh ax dx=\frac{1}{a}\ln\left(\cosh ax\right)+c$
$\frac{dy}{dx}=-a\,\text{sech}\,ax\times\tanh ax$	$y=\text{sech}\,ax$	$\int \text{sech}\,ax dx=\frac{2}{a}\tan^{-1}\left(e^{ax}\right)+c$
$\frac{dy}{dx}=-a\,\text{cosech}\,ax\times\coth ax$	$y=\text{cosech}\,ax$	$\int \text{cosech}\,ax dx=\frac{1}{a}\ln\left(\tanh\frac{1}{2}ax\right)+c$
$\frac{dy}{dx}=-a\,\text{cosech}^2 ax$	$y=\coth ax$	$\int \coth ax dx=\frac{1}{a}\ln\left(\sinh ax\right)+c$
$\frac{dy}{dx}=\frac{a}{\sqrt{a^2x^2-1}}$	$y=\cosh^{-1}ax$	$\int \cosh^{-1}ax dx=x\cosh^{-1}ax-\frac{1}{a}\sqrt{ax-1}\sqrt{ax+1}+c$
$\frac{dy}{dx}=\frac{a}{\sqrt{a^2x^2+1}}$	$y=\sinh^{-1}ax$	$\int \sinh^{-1}ax dx=x\sinh^{-1}ax-\frac{1}{a}\sqrt{a^2x^2+1}+c$
$\frac{dy}{dx}=\frac{a}{1-a^2x^2}$	$y=\tanh^{-1}ax$	$\int \tanh^{-1}ax dx=x\tanh^{-1}ax+\frac{1}{2a}\ln\left(1-a^2x^2\right)+c$

$$\cosh x=\tfrac{1}{2}\left(e^x+e^{-x}\right)$$
$$\sinh x=\tfrac{1}{2}\left(e^x-e^{-x}\right)$$
$$\tanh x=\frac{\sinh x}{\cosh x}$$

$$\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right)$$
$$\cosh^{-1}x=\ln\left(x+\sqrt{x^2-1}\right)$$
$$\tanh^{-1}x=\tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\cosh^2 x-\sinh^2 x=1$$
$$\text{sech}^2 x+\tanh^2 x=1$$
$$\coth^2 x-\text{cosech}^2 x=1$$

$$\sinh 2x=2\sinh x\cosh x$$
$$\cosh 2x=\cosh^2 x+\sinh^2 x$$

$$\sinh(A\pm B)=\sinh A\cosh B\pm\cosh A\sinh B$$
$$\cosh(A\pm B)=\cosh A\cosh B\pm\sinh A\sinh B$$
$$\tanh(A\pm B)=\frac{\tanh A\pm\tanh B}{1\pm\tanh A\tanh B}$$

$$\cosh^2 x=\tfrac{1}{2}\left(\cosh 2x+1\right)$$
$$\sinh^2 x=\tfrac{1}{2}\left(\cosh 2x-1\right)$$
$$\cosh^3 x=\tfrac{1}{4}\left(3\cosh x+\cosh 3x\right)$$
$$\sinh^3 x=\tfrac{1}{4}\left(-3\sinh x+\sinh 3x\right)$$

Standard integrals

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}, \quad a \in \mathbb{Z}^+$$

i.e. positive integers

$$\int_0^\infty e^{-ax^2}\,dx = \tfrac{1}{2}\sqrt{\tfrac{\pi}{a}}, \quad a > 0$$

$$\int_0^\infty x e^{-ax^2}\,dx = \tfrac{1}{2a}, \quad a > 0$$

$$\int_0^\infty x^n e^{-ax^2}\,dx = \begin{cases} 1\times 3\times 5\times \ldots (n-1)(2a)^{-\frac{n+1}{2}}\sqrt{\tfrac{\pi}{2}} & n>0,\ n \text{ even} \\ 2\times 4\times 6\times \ldots\ldots (n-1)(2a)^{-\frac{n+1}{2}} & n>1,\ n \text{ odd} \end{cases}$$

$$\int_{-\infty}^\infty e^{2bx-ax^2}\,dx = \sqrt{\tfrac{\pi}{a}}e^{\frac{1}{a}b^2},\ a > 0$$

$$\int_0^1 x^p(1-x)^q\,dx = \frac{p!q!}{(p+q+1)!}, \quad p,q \in \mathbb{Z}^+$$

$$\int_0^\infty \cos(ax^2)\,dx = \tfrac{1}{2}\sqrt{\tfrac{\pi}{2a}}, \quad a > 0$$

$$\int_0^\infty \sin(ax^2)\,dx = \tfrac{1}{2}\sqrt{\tfrac{\pi}{2a}}, \quad a > 0$$

$$\int_0^\infty \frac{\sin x}{x}\,dx = \tfrac{1}{2}\pi$$

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \tfrac{1}{2}\pi$$

$$\int_0^\infty \frac{1}{(1+x)x^a}\,dx = \frac{\pi}{\sin a\pi}, \quad (0 < a < 1)$$

$$\int \frac{1}{a+bx}\,dx = \tfrac{1}{b}\ln(a+bx) + c$$

$$\int \frac{1}{(a+bx)^2}\,dx = \frac{-1}{b(a+bx)} + c$$

$$\int \frac{1}{x(a+bx)}\,dx = -\tfrac{1}{a}\ln\left(\frac{a+bx}{x}\right) + c$$

$$\int \frac{1}{a^2+b^2x^2}\,dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + c$$

$$\int \frac{1}{x^2-a^2}\,dx = \tfrac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{x}{(x^2\pm a^2)^n}\,dx = \frac{-1}{2(n-1)(x^2\pm a^2)^{n-1}} + c$$

$$\int \frac{1}{x(x^n+a)}\,dx = \frac{1}{an}\ln\left|\frac{x^n}{x^n+a}\right| + c$$

$$\int \frac{x}{x^2\pm a^2}\,dx = \tfrac{1}{2}\ln\left|x^2\pm a^2\right| + c$$

$$\int \frac{1}{\sqrt{x^2\pm a^2}}\,dx = \ln\left|x+\sqrt{x^2\pm a^2}\right| + c$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \tfrac{1}{a}\sec^{-1}\left(\tfrac{1}{a}x\right) + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\tfrac{1}{a}x\right) + c$$

$$\int \frac{x}{\sqrt{x^2\pm a^2}}\,dx = \sqrt{x^2\pm a^2} + c$$

$$\int x\ln ax\,dx = \tfrac{1}{2}x^2\left(\ln ax - \tfrac{1}{2}\right) + c$$

$$\int \frac{x^n}{(1+tx)^{n+2}}\,dx = \tfrac{1}{n+1}\left(\frac{x}{1+tx}\right)^{n+1} + c$$

Integration by substitution

Integral includes	Try substitution
$(ax+b)^n$	$u = ax+b$
$\sqrt[n]{ax+b}$	$u^n = ax+b$
$a-bx^2$	$a\sin^2 u = bx^2$
$a+bx^2$	$a\tan^2 u = bx^2$
bx^2-a	$a\sec^2 u = bx^2$
e^x	$u = e^x$
$\ln(ax+b)$	$e^u = ax+b$

e.g. $I = \int \frac{1}{1+x^2}\,dx$

$x = \tan u \quad \therefore u = \tan^{-1} x$

$\frac{dx}{du} = \sec^2 u \quad \therefore\ dx = \sec^2 u\,du$

$1+x^2 = 1+\tan^2 u = \sec^2 u$

$I = \int \frac{1}{1+x^2}\,dx = \int \frac{\sec^2 u}{\sec^2 u}\,du$

$I = \int du = u + c$

$I = \tan^{-1} x + c$

Special functions

$$\mathrm{erf}\,(x) = \tfrac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$ Error function

$$C(x) = \int_0^x \cos\left(\tfrac{1}{2}\pi t^2\right) dt$$ Fresnel integrals

$$S(x) = \int_0^x \sin\left(\tfrac{1}{2}\pi t^2\right) dt$$

$$\mathrm{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} dt, \quad x > 0$$ Exponential integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \; x > 0$$ Gamma function

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = n!$$ Note *Stirling's Approximation* $n! \approx \sqrt{2\pi n}\, n^{n+\frac{1}{2}} e^{-n}$

$$\Im(\omega) = \int_{-\infty}^{-\infty} f(t) e^{-2\pi i \omega t} dt$$ Fourier transform

$$f(t) = \int_{-\infty}^{-\infty} \Im(\omega) e^{2\pi i \omega t} d\omega$$ Inverse Fourier transform

$$L(s) = \int_0^{-\infty} f(t) e^{-st} dt$$ Laplace transform

Fourier series

$$x(t) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right) \right\}$$

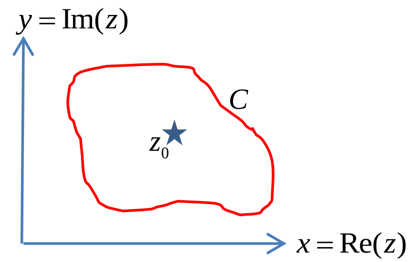
$$a_n = \frac{2}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} x(t) \cos\left(\frac{2n\pi t}{T}\right) dt$$

$$b_n = \frac{2}{T} \int_{-\frac{1}{2}T}^{\frac{1}{2}T} x(t) \sin\left(\frac{2n\pi t}{T}\right) dt$$

Contour integration of complex variables

$$f(z) = u(x,y) + iv(x,y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ *Cauchy-Riemann equations*



$$\oint_C f(z) dz = 2\pi i \sum_k R_k$$ Cauchy (Residue) theorem

e.g. $f(z) = \dfrac{1}{(z - z_0)^m}$ has a *pole* at $z = z_0$ of *order* m

$$R_k = a_{-1}^{(k)}$$ Residue – i.e. the -1[th] term of the *Laurent expansion*

$$f(z) = \sum_{n=-m}^{\infty} a_n^{(k)} (z - z_0^{(k)})^n$$ Laurent expansion of f(z) about poles z_0 of order m

$$R_k = \lim_{z \to z_0^{(k)}} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left(\left(z - z_0^{(j)}\right)^m f(z) \right) \right\}$$ Computing the Residue

$$\iint_A \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) dx dy = \oint_C (p dy - q dx)$$

Green's Theorem in a plane, for functions *p,q* which have continuous derivatives within area *A* bounded by contour *C*

Example:

$$\int_0^\infty \frac{2x \sin x}{1+x^2} dx = \frac{\pi}{e}$$

Consider this complex contour integral. Simple poles (i.e. m = 1) at $z_0 = -i, i$

$$I = \oint_C \frac{z e^{iz}}{1+z^2} dz = \oint_C \frac{z e^{iz}}{(z+i)(z-i)} dz$$

Residues are:

$$R_\pm = \lim_{z_0 \to \pm i} \left\{ \left((z - z_0) f(z) \right) \right\}$$

$$R_\pm = \begin{cases} \dfrac{ie^{-1}}{i+i} & z_0 = i \\ \dfrac{-ie}{-i-i} & z_0 = -i \end{cases} = \begin{cases} \tfrac{1}{2} e^{-1} & z_0 = i \\ \tfrac{1}{2} e & z_0 = -i \end{cases}$$

Now consider a semi-circular contour of radius *R* in the upper half plane of the Argand diagram

$$I = \int_{-R}^{R} \frac{x(\cos x + i \sin x)}{1+x^2} dx + \int_{\theta=0}^{\pi} \frac{Re^{iR(\cos\theta + i\sin\theta)}}{1+R^2 e^{2i\theta}} iRe^{i\theta} d\theta$$

$$z = Re^{i\theta}$$

As *R* become infinite the second integral tends to zero. Hence, using the residue theorem and considering imaginary parts:

$$\int_{-\infty}^{\infty} \frac{ix \sin x}{1+x^2} dx = 2\pi i \times \tfrac{1}{2} e^{-1}$$ Contour only encloses the pole at *+i*

$$\int_0^\infty \frac{2x \sin x}{1+x^2} dx = \frac{\pi}{e}$$ Since the integrand is even

Vector calculus

Cartesian coordinate unit vectors

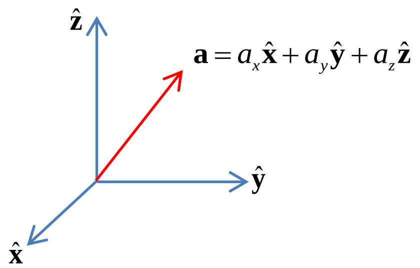
$\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}=\hat{\mathbf{x}}\cdot\hat{\mathbf{z}}=\hat{\mathbf{y}}\cdot\hat{\mathbf{z}}=0$

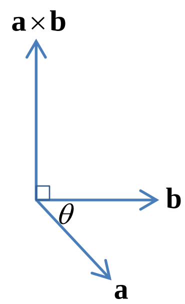
$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}=\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}=1$

$\hat{\mathbf{x}}\times\hat{\mathbf{y}}=\hat{\mathbf{z}}$

$\hat{\mathbf{y}}\times\hat{\mathbf{z}}=\hat{\mathbf{x}}$

$\hat{\mathbf{z}}\times\hat{\mathbf{x}}=\hat{\mathbf{y}}$





Vector scalar (dot) and cross products

$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$

$|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$

$\mathbf{a}=a_x\hat{\mathbf{x}}+a_y\hat{\mathbf{y}}+a_z\hat{\mathbf{z}}$

$\mathbf{b}=b_x\hat{\mathbf{x}}+b_y\hat{\mathbf{y}}+b_z\hat{\mathbf{z}}$

$\mathbf{a}\times\mathbf{b}=\hat{\mathbf{x}}\left(a_yb_z-a_zb_y\right)+\hat{\mathbf{y}}\left(a_zb_x-a_xb_z\right)+\hat{\mathbf{z}}\left(a_xb_y-a_yb_x\right)$

$\mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}$

$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$$\nabla f=\frac{\partial f}{\partial x}\hat{\mathbf{x}}+\frac{\partial f}{\partial y}\hat{\mathbf{y}}+\frac{\partial f}{\partial z}\hat{\mathbf{z}}$$ Gradient

$$\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$ Laplacian

$dV=dxdydz$ Volume element

$\mathbf{A}=A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}}$ Generic vector

$$\nabla\cdot\mathbf{A}=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$$ Divergence

$$\nabla\times\mathbf{A}=\hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$ Curl

$\mathbf{r}=x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}$ $\quad\dot{\mathbf{r}}=\dot{x}\hat{\mathbf{x}}+\dot{y}\hat{\mathbf{y}}+\dot{z}\hat{\mathbf{z}}$ $\quad\ddot{\mathbf{r}}=\ddot{x}\hat{\mathbf{x}}+\ddot{y}\hat{\mathbf{y}}+\ddot{z}\hat{\mathbf{z}}$ Displacement, velocity and acceleration

Identities involving the vector derivative operator

$\nabla(fg)=f\nabla g+g\nabla f$

$\nabla\cdot(f\mathbf{A})=f\nabla\cdot\mathbf{A}+\mathbf{A}\cdot\nabla f$

$\nabla\times(f\mathbf{A})=f\nabla\times\mathbf{A}+\nabla f\times\mathbf{A}$

$\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\nabla\times\mathbf{B})+(\mathbf{A}\cdot\nabla)\mathbf{B}+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}$

$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$

$\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$

$\nabla\cdot\nabla f=\nabla^2 f$

$\nabla\times\nabla f=\mathbf{0}$

$\nabla\cdot(\nabla\times\mathbf{A})=0$

$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$

$\int_V(\nabla\cdot\mathbf{A})\,dV=\oint_S\mathbf{A}\cdot d\mathbf{s}$ Gauss's Theorem ('Divergence theorem'). Surface *S* bounds volume *V*

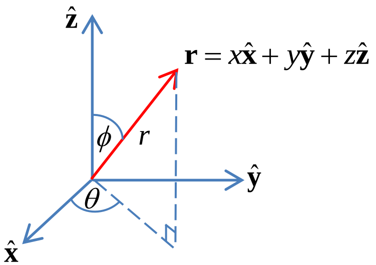
$\int_S(\nabla\times\mathbf{A})\cdot d\mathbf{s}=\oint_L\mathbf{A}\cdot d\mathbf{l}$ Stokes' Theorem. Surface *S* extends from loop *L*.

$\oint_S(f\nabla g)\cdot d\mathbf{s}=\int_V\left(f\nabla^2 g+\nabla f\cdot\nabla g\right)dV$

$\oint_S(f\nabla g-g\nabla f)\cdot d\mathbf{s}=\int_V\left(f\nabla^2 g-g\nabla^2 f\right)dV$

Green's Theorems

Vector calculus using cylindrical and spherical polar coordinates



$$x = r\sin\phi\cos\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\phi$$

Conversion between Cartesian and polar coordinates

Polar coordinate unit vectors as Cartesian coordinates

$$\hat{\mathbf{r}} = \sin\phi\cos\theta\hat{\mathbf{x}} + \sin\phi\sin\theta\hat{\mathbf{y}} + \cos\phi\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\phi}} = \cos\phi\cos\theta\hat{\mathbf{x}} + \cos\phi\sin\theta\hat{\mathbf{y}} - \sin\phi\hat{\mathbf{z}}$$

Spherical polar coordinates

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r\sin\phi}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$dV = r^2\sin\phi\, dr\, d\theta\, d\phi$$

$$\mathbf{A} = A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial\left(r^2A_r\right)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial A_\theta}{\partial\theta} + \frac{1}{r\sin\phi}\frac{\partial\left(\sin\phi A_\phi\right)}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \frac{\hat{\mathbf{r}}}{r\sin\phi}\left(\frac{\partial\left(\sin\phi A_\theta\right)}{\partial\phi} - \frac{\partial A_\phi}{\partial\theta}\right) + \frac{\hat{\boldsymbol{\theta}}}{r}\left(\frac{\partial\left(rA_\phi\right)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right) + \frac{\hat{\boldsymbol{\phi}}}{r}\left(\frac{1}{\sin\phi}\frac{\partial A_r}{\partial\theta} - \frac{\partial\left(rA_\theta\right)}{\partial r}\right)$$

$$\frac{d\hat{\mathbf{r}}}{dt} = \sin\phi\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{\phi}\hat{\boldsymbol{\phi}} \qquad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\left(\sin\phi\hat{\mathbf{r}} + \cos\phi\hat{\boldsymbol{\phi}}\right) \qquad \frac{d\hat{\boldsymbol{\phi}}}{dt} = -\dot{\phi}\hat{\mathbf{r}} + \cos\phi\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$\dot{\mathbf{r}} = \frac{d}{dt}\left(r\hat{\mathbf{r}}\right) = \dot{r}\hat{\mathbf{r}} + r\sin\phi\dot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$$

$$\ddot{\mathbf{r}} = \frac{d^2}{dt^2}\left(r\hat{\mathbf{r}}\right) = \left(\ddot{r} - r\dot{\phi}^2 - r\sin^2\phi\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2r\cos\phi\dot{\theta}\dot{\phi} + 2\sin\phi\dot{\theta}\dot{r} + r\sin\phi\ddot{\theta}\right)\hat{\boldsymbol{\theta}} + \dots$$
$$\dots + \left(2\dot{r}\dot{\phi} + r\ddot{\phi} - r\sin\phi\cos\phi\dot{\theta}^2\right)\hat{\boldsymbol{\phi}}$$

Cartesian coordinates

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$ Gradient

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$ Laplacian

$$dV = dx\,dy\,dz$$ Volume element

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$ Generic vector

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$ Divergence

$$\nabla\times\mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$ Curl

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad \dot{\mathbf{r}} = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} + \dot{z}\hat{\mathbf{z}} \qquad \ddot{\mathbf{r}} = \ddot{x}\hat{\mathbf{x}} + \ddot{y}\hat{\mathbf{y}} + \ddot{z}\hat{\mathbf{z}}$$ Displacement, velocity and acceleration

Cylindrical polar coordinates

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{r}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial\theta^2} + \frac{\partial^2 A_z}{\partial z^2}$$

$$dV = r\,dr\,d\theta\,dz$$

$$\mathbf{A} = A_r\hat{\mathbf{x}} + A_\theta\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial\left(rA_r\right)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right) + \hat{\boldsymbol{\theta}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \frac{\hat{\mathbf{z}}}{r}\left(\frac{\partial\left(rA_\theta\right)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)$$

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}} \qquad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\hat{\mathbf{r}}$$

$$\dot{\mathbf{r}} = \frac{d}{dt}\left(r\hat{\mathbf{r}}\right) = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{z}\hat{\mathbf{z}}$$

$$\ddot{\mathbf{r}} = \frac{d^2}{dt^2}\left(r\hat{\mathbf{r}}\right) = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} + \ddot{z}\hat{\mathbf{z}}$$

Note alternative representations of time derivatives of polar angle

$$\omega = \dot{\theta} = \frac{d\theta}{dt}$$
$$\dot{\omega} = \ddot{\theta} = \frac{d^2\theta}{dt^2}$$