Chain rule of differentiation

$$y(x) = y(z(x))$$
$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

i.e. a 'function of a function' of the form *fg(x)* , where *f* and *g* are both functions *we know how to differentiate*.

e.g $y = \sqrt{x^2+1}$

$z = x^2+1 \quad \therefore \frac{dz}{dx} = 2x$

$y = \sqrt{z} \quad \therefore \frac{dy}{dz} = \frac{1}{2\sqrt{z}}$

$\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \boxed{\frac{x}{\sqrt{x^2+1}}}$

e.g $y = \cos\left(x^2 - \frac{1}{x}\right)$

$z = x^2 - \frac{1}{x} \quad \therefore \frac{dz}{dx} = 2x + \frac{1}{x^2}$

$y = \cos z \quad \therefore \frac{dy}{dz} = -\sin z$

$\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \boxed{-\left(2x + \frac{1}{x^2}\right)\sin\left(x^2 - \frac{1}{x}\right)}$

e.g $y = (x^4 - 3x)^5$

$z = x^4 - 3x \quad \therefore \frac{dz}{dx} = 4x^3 - 3$

$y = z^5 \quad \therefore \frac{dy}{dz} = 5z^4$

$\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \boxed{5(x^4-3x)^4\left(4x^3-3\right)}$

e.g $y = \ln\left|\sqrt{x} + \sin x\right|$

$z = \sqrt{x} + \sin x \quad \therefore \frac{dz}{dx} = \frac{1}{2\sqrt{x}} + \cos x$

$y = \ln|z| \quad \therefore \frac{dy}{dz} = \frac{1}{z}$

$\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \boxed{\frac{\frac{1}{2\sqrt{x}} + \cos x}{\sqrt{x} + \sin x}}$

e.g $y = e^{\sin x}$

$z = \sin x \quad \therefore \frac{dz}{dx} = \cos x$

$y = e^z \quad \therefore \frac{dy}{dz} = e^z$

$\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \boxed{e^{\sin x}\cos x}$

e.g $y = \tan\left(\ln|x|\right)$

$z = \ln|x| \quad \therefore \frac{dz}{dx} = \frac{1}{x}$

$y = \tan z \quad \therefore \frac{dy}{dz} = \sec^2 z$

$\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \boxed{\frac{\sec^2\left(\ln|x|\right)}{x}}$

$$y(x) = y(z(w(v(x))))$$
$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dw} \times \frac{dw}{dv} \times \frac{dv}{dx}$$

The chain rule can be extended for any level of 'nested functions'

e.g. $y = \ln\left|\tan e^{2x}\right|$

$\frac{dy}{dx} = \frac{1}{\tan e^{2x}} \times \sec^2(e^{2x}) \times 2e^{2x}$

$\frac{dy}{dx} = \frac{2e^{2x}\sec^2(e^{2x})}{\tan e^{2x}} = \frac{2e^{2x}\left(1 - \tan^2(e^{2x})\right)}{\tan e^{2x}} = \boxed{2e^{2x}\left(\cot e^{2x} - \tan e^{2x}\right)}$

e.g $y = \sqrt{\sqrt{\sqrt{\sqrt{x}+1}+1}+1}$

$v(x) = \sqrt{x} + 1 \quad \therefore \frac{dv}{dx} = \frac{1}{2\sqrt{x}}$

$w(v) = \sqrt{v} + 1 \quad \therefore \frac{dv}{dv} = \frac{1}{2\sqrt{v}}$

$z(w) = \sqrt{w} + 1 \quad \therefore \frac{dv}{dw} = \frac{1}{2\sqrt{w}}$

$y(z) = \sqrt{z} + 1 \quad \therefore \frac{dv}{dz} = \frac{1}{2\sqrt{z}}$

$\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dw} \times \frac{dw}{dv} \times \frac{dv}{dx} = \frac{1}{16\sqrt{z}\sqrt{w}\sqrt{v}\sqrt{x}} = \frac{1}{16\sqrt{zwvx}}$

$\frac{dy}{dx} = \boxed{\frac{1}{16\sqrt{\sqrt{\sqrt{\sqrt{x}+1}}\left(\sqrt{\sqrt{x}+1}\right)\left(\sqrt{x}+1\right)\sqrt{x}}}}$

Product rule

$$y(x) = u(x)v(x)$$
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

We can justify the product rule by considering a small finite change in variable *x*, and the consequential changes in functions *u* and *v*

$$y = uv$$
$$y + \Delta y = (u+\Delta u)(v+\Delta v)$$
$$\therefore \Delta y = (u+\Delta u)(v+\Delta v) - uv$$
$$\Delta y = uv + v\Delta u + u\Delta v + \Delta u\Delta v - uv$$
$$\frac{\Delta y}{\Delta x} = v\frac{\Delta u}{\Delta x} + (u+\Delta u)\frac{\Delta v}{\Delta x}$$
$$\therefore \frac{dy}{dx} = \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Note we assume

$$\lim_{\Delta x\to 0}\left((u+\Delta u)\frac{\Delta v}{\Delta x}\right) = u\frac{dv}{dx}$$

$$y = x^2 e^{-x}$$
$$u = x^2 \quad \therefore \frac{du}{dx} = 2x$$
$$v = e^{-x} \quad \therefore \frac{dv}{dx} = -e^{-x}$$
$$\therefore \frac{dy}{dx} = x^2\left(-e^{-x}\right) + e^{-x}(2x)$$
$$\frac{dy}{dx} = -x^2e^{-x} + 2xe^{-x} = xe^{-x}(2-x)$$

$$y = \sqrt{x}\sin x$$
$$u = \sqrt{x} \quad \therefore \frac{du}{dx} = \tfrac{1}{2\sqrt{x}}$$
$$v = \sin x \quad \therefore \frac{dv}{dx} = \cos x$$
$$\therefore \frac{dy}{dx} = \sqrt{x}\times(\cos x) + \sin x \times\left(\tfrac{1}{2\sqrt{x}}\right)$$
$$\frac{dy}{dx} = \frac{2x\cos x + \sin x}{2\sqrt{x}}$$

$$y = x^4 \ln x$$
$$\frac{dy}{dx} = x^4\left(\tfrac{1}{x}\right) + \left(4x^3\right)\ln x$$
$$\frac{dy}{dx} = x^3 + 4x^3\ln x = x^3(1+4\ln x)$$

$$y = (x+5)^2(x-3)^3$$
$$\frac{dy}{dx} = 2(x+5)(x-3)^3 + 3(x+5)^2(x-3)^2$$
$$\frac{dy}{dx} = (x+5)(x-3)^2(2x-6+3x+15)$$
$$\frac{dy}{dx} = (x+5)(x-3)^2(5x+9)$$

Quotient rule

$$y(x) = \frac{u(x)}{v(x)} = uv^{-1}$$

By the product rule

$$\frac{dy}{dx} = u\frac{d}{dx}v^{-1} + v^{-1}\frac{du}{dx}$$
$$= u\left(-v^{-2}\frac{dv}{dx}\right) + v^{-1}\frac{du}{dx}$$
$$= -uv^{-2}\frac{dv}{dx} + vv^{-2}\frac{du}{dx}$$
$$\therefore \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$y = \frac{x^2+1}{x^3+1}$$
$$u = x^2+1$$
$$v = x^3+1$$
$$\frac{dy}{dx} = \frac{2x\left(x^3+1\right) - 3x^2\left(x^2+1\right)}{\left(x^3+1\right)^2}$$
$$\frac{dy}{dx} = \frac{2x - x^4 - 3x^2}{\left(x^3+1\right)^2}$$

$$y = \tan x = \frac{\sin x}{\cos x}$$
$$u = \sin x$$
$$v = \cos x$$
$$\frac{dy}{dx} = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$
$$\frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$$

$$y = \frac{1}{\ln x}$$
$$u = 1$$
$$v = \ln x$$
$$\frac{dy}{dx} = \frac{\ln x(0) - 1\left(\tfrac{1}{x}\right)}{(\ln x)^2}$$
$$\frac{dy}{dx} = -\frac{1}{x(\ln x)^2}$$

$$y = \frac{\ln\left(1+x^2\right)}{\ln\left(1+x^3\right)}$$
$$\frac{dy}{dx} = \frac{\ln\left(1+x^3\right)\left(\frac{2x}{1+x^2}\right) - \ln\left(1+x^2\right)\left(\frac{3x^2}{1+x^3}\right)}{\left(\ln\left(1+x^3\right)\right)^2}$$