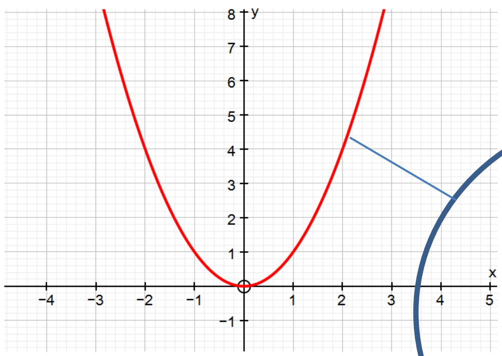
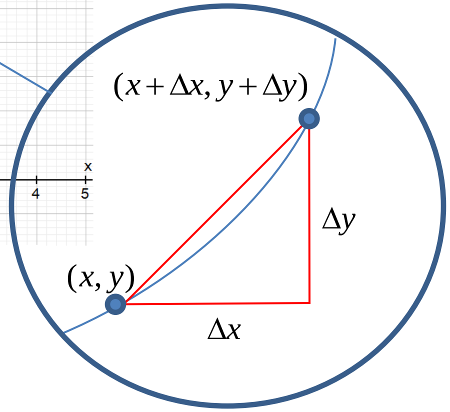
Differentiation of x^2 and x^3 from first principles





$$y = x^2$$

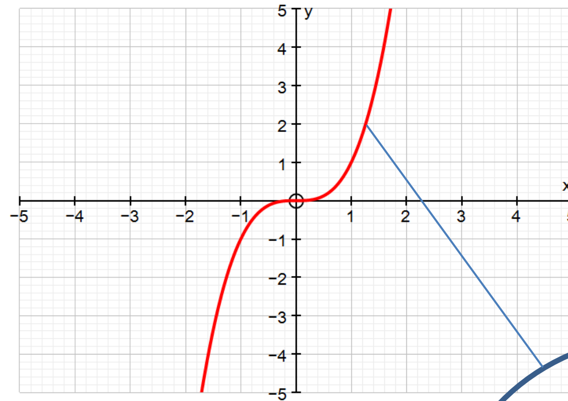
$$y + \Delta y = (x + \Delta x)^2$$

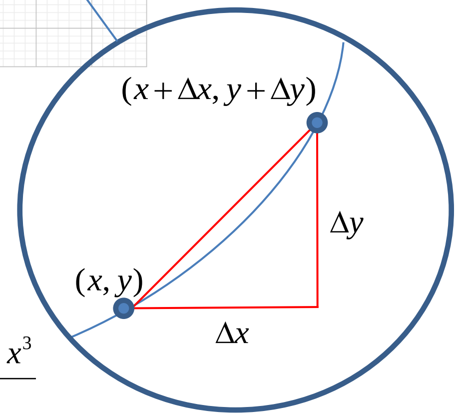
$$\therefore \frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}(2x + \Delta x) = \boxed{2x}$$





$$y = x^3$$

$$y + \Delta y = (x + \Delta x)^3$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^3 - x^3}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$

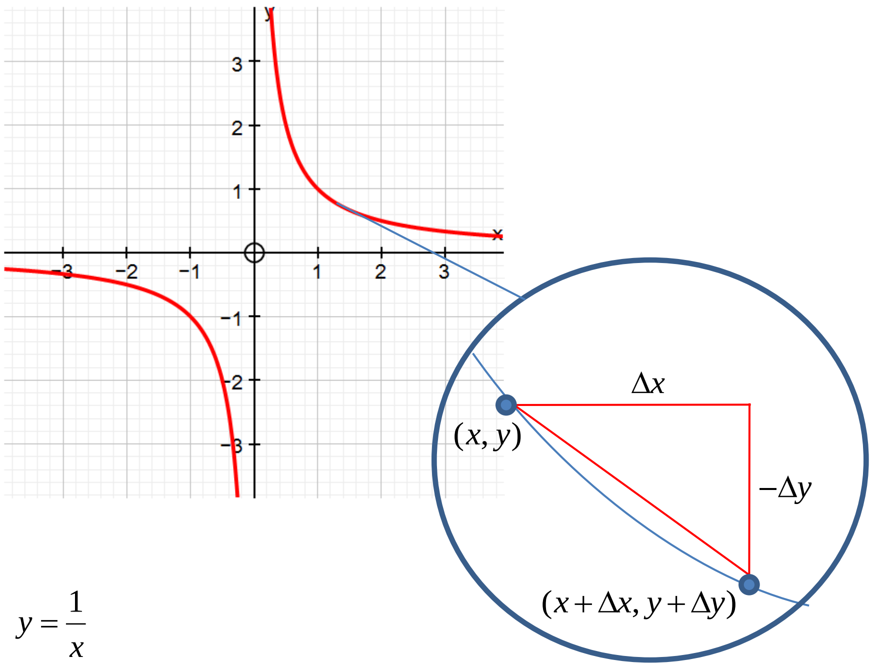
$$\frac{\Delta y}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(3x^2 + 3x\Delta x + (\Delta x)^2\right) = \boxed{3x^2}$$

From this pattern we can infer the following general result for the differentiation of polynomials (which is actually true for any n)

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}}$$

Differentiation of x^{-1} and x^{-2} from first principles



$$y = \frac{1}{x}$$

$$y + \Delta y = \frac{1}{x + \Delta x}$$

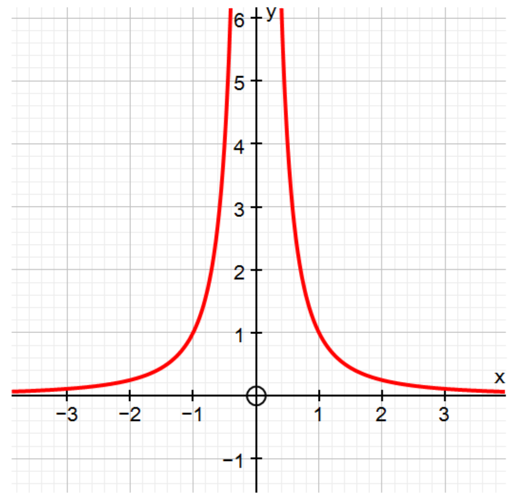
$$\therefore \Delta y = \frac{1}{x + \Delta x} - \frac{1}{x}$$

$$\Delta y = \frac{-\Delta x}{x^2 + x\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{-1}{x^2 + x\Delta x}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{-1}{x^2 + x\Delta x}\right) = \boxed{-\frac{1}{x^2}}$$

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}x^{-1} = -x^{-2}$$
$$\frac{d}{dx}x^{-1} = -\frac{1}{x^2}$$



$$y = \frac{1}{x^2}$$

$$y + \Delta y = \frac{1}{(x + \Delta x)^2}$$

$$\therefore \Delta y = \frac{1}{(x + \Delta x)^2} - \frac{1}{x^2}$$

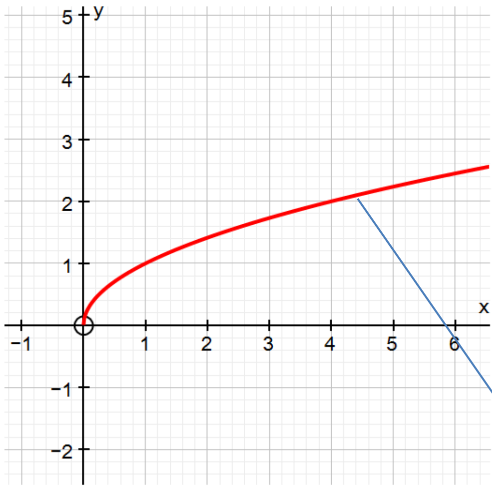
$$\Delta y = \frac{x^2 - x^2 - 2x\Delta x - \Delta x^2}{x^2(x + \Delta x)^2}$$

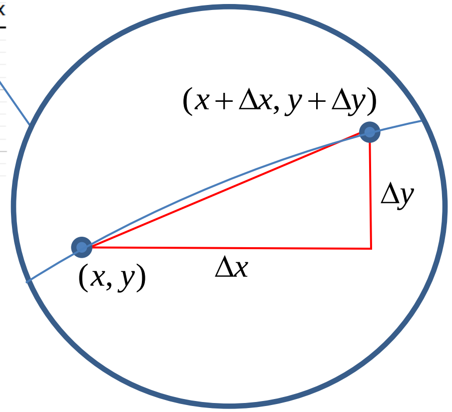
$$\frac{\Delta y}{\Delta x} = \frac{-2x - \Delta x}{x^2(x + \Delta x)^2}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{-2x - \Delta x}{x^2(x + \Delta x)^2}\right) = -\frac{-2x}{x^2(x)^2} = \boxed{-\frac{2}{x^3}}$$

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}x^{-2} = -2x^{-3}$$
$$\frac{d}{dx}x^{-2} = -\frac{2}{x^3}$$

Differentiation of $\sqrt{x}$ and $\dfrac{1}{\sqrt{x}}$ from first principles





$$y = \sqrt{x}$$

$$y + \Delta y = \sqrt{x+\Delta x}$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{\left(\sqrt{x+\Delta x} - \sqrt{x}\right)}{\Delta x} \times \frac{\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}{\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}$$

Note this nice "multiply by 1" trick

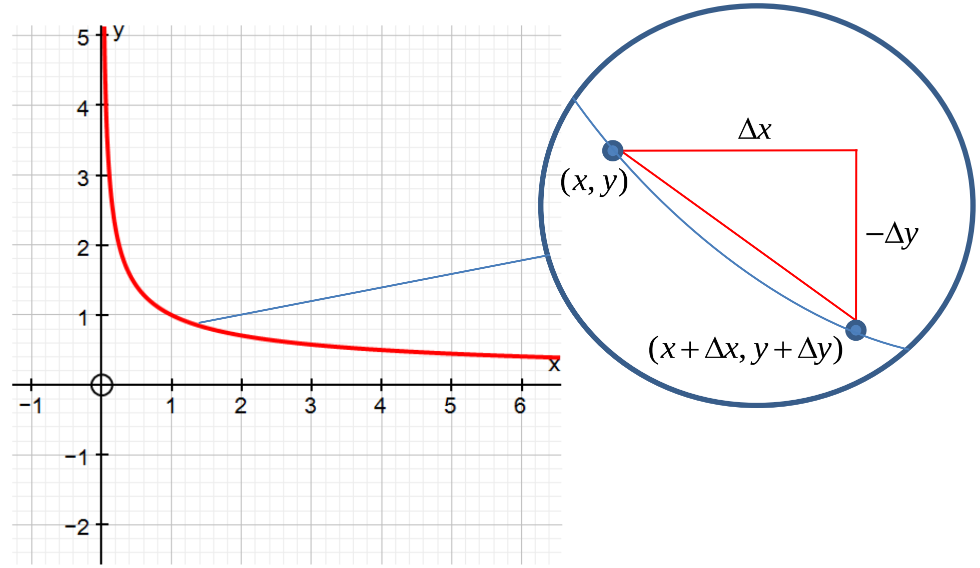
$$\frac{\Delta y}{\Delta x} = \frac{x + \Delta x - \sqrt{x}\sqrt{x+\Delta x} + \sqrt{x}\sqrt{x+\Delta x} - x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)}$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{1}{\sqrt{x+\Delta x} + \sqrt{x}}\right) = \boxed{\frac{1}{2\sqrt{x}}}$$

$$\boxed{\begin{aligned} &\frac{d}{dx}x^n = nx^{n-1} \\ &\frac{d}{dx}x^{\frac{1}{2}} = \tfrac{1}{2}x^{-\frac{1}{2}} \\ &\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} \end{aligned}}$$



$$y = \frac{1}{\sqrt{x}}$$

$$y + \Delta y = \frac{1}{\sqrt{x+\Delta x}}$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x\sqrt{x+\Delta x}} - \frac{1}{\Delta x\sqrt{x}}$$

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x} - \sqrt{x+\Delta x}}{\Delta x\sqrt{x}\sqrt{x+\Delta x}}$$

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x} - \sqrt{x+\Delta x}}{\Delta x\sqrt{x}\sqrt{x+\Delta x}} \times \frac{\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}{\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}$$

Note this nice "multiply by 1" trick

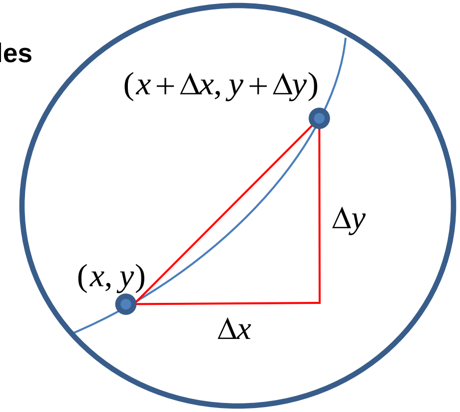
$$\frac{\Delta y}{\Delta x} = \frac{x - x - \Delta x}{\Delta x\left(x\sqrt{x+\Delta x} + \sqrt{x}\left(x+\Delta x\right)\right)}$$

$$\frac{\Delta y}{\Delta x} = \frac{-1}{x\sqrt{x+\Delta x} + \sqrt{x}\left(x+\Delta x\right)}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{-1}{x\sqrt{x+\Delta x} + \sqrt{x}\left(x+\Delta x\right)}\right) = \boxed{-\frac{1}{2x\sqrt{x}}}$$

$$\boxed{\begin{aligned} &\frac{d}{dx}x^n = nx^{n-1} \\ &\frac{d}{dx}x^{-\frac{1}{2}} = -\tfrac{1}{2}x^{-\frac{3}{2}} \\ &\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{2x\sqrt{x}} \end{aligned}}$$

Differentiation of x^n from first principles



$y = x^n$

n is a positive integer

$y + \Delta y = (x+\Delta x)^n$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

Binomial expansion

$$\frac{\Delta y}{\Delta x} = \frac{\overbrace{\binom{n}{0}x^n(\Delta x)^0 + \binom{n}{1}x^{n-1}(\Delta x)^1 + ... + \binom{n}{n}x^0(\Delta x)^n}^{\text{Binomial expansion}} - x^n}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \binom{n}{2}x^{n-1}(\Delta x)^1 + ... + \binom{n}{n}x^0(\Delta x)^{n-1}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(nx^{n-1} + \binom{n}{2}x^{n-1}(\Delta x)^1 + ... + \binom{n}{n}x^0(\Delta x)^{n-1}\right) = nx^{n-1}$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

n is a positive integer

An even more general proof can be formed from the *Generalized Binomial Theorem*

$y = x^n$

$y + \Delta y = (x+\Delta x)^n$

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{x^n\left(1+\frac{\Delta x}{x}\right)^n - x^n}{\Delta x}$$

Since $\left|\frac{\Delta x}{x}\right| \ll 1$ ← This must be true since we will be considering the limit $\Delta x \to 0$

$$\left(1+\frac{\Delta x}{x}\right)^n = 1 + n\frac{\Delta x}{x} + \frac{n(n-1)}{2!}\left(\frac{\Delta x}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{\Delta x}{x}\right)^3 + ...$$

$$x^n\left(1+\frac{\Delta x}{x}\right)^n - x^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)x^{n-2}}{2!}(\Delta x)^2 + ... - x^n$$

$$\frac{x^n\left(1+\frac{\Delta x}{x}\right)^n - x^n}{\Delta x} = nx^{n-1} + \frac{n(n-1)x^{n-2}}{2!}\Delta x + \frac{n(n-1)(n-2)x^{n-3}}{3!}(\Delta x)^2 + ...$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = nx^{n-1}$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

So the result now works for negative and fractional n (i.e. we can differentiate reciprocals of polynomials and surds as well).