Calculus with exponentials and logarithms

The exponential function is defined

$$y = b^{ax}$$
$$b > 0$$

b is the *base* and *ax* is the *exponent*.

The inverse function is the *logarithm*

$$x = \tfrac{1}{a}\log_b y$$
$$f(x) = b^{ax}$$
$$f^{-1}(x) = \tfrac{1}{a}\log_b x$$

When $b = e = 2.71828182845904523536028747 1...$ then

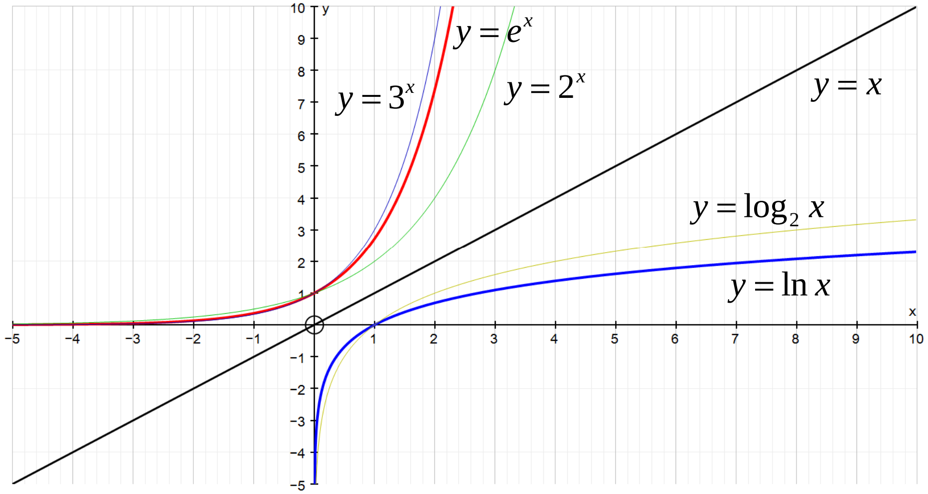
$$f'(x) = f(x) \qquad \frac{d}{dx}e^x = e^x$$

We can use the *chain rule* to generalize

$$y = e^{ax}$$
$$z = ax$$
$$y = e^z$$
$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$
$$\frac{d}{dx}e^{ax} = ae^{ax}$$

This basic result allows us to find the derivative of $y = \ln x$

$$y = \ln x$$
$$e^y = x$$
$$\frac{dy}{dx} = \frac{d}{dx}\ln x$$
$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = e^{-y} = \frac{1}{x}$$
$$\therefore \frac{d}{dx}\ln x = \frac{1}{x}$$



The *domain* of $y = \ln x$ is $0 < x < \infty$

To cover the full range of *x* we have a more general result $\frac{d}{dx}\ln|x| = \frac{1}{x}$

We can use the results above to differentiate more general expressions for exponential and logarithmic functions of *any* base

$$y = b^{ax}$$
$$\ln y = ax\ln b$$
$$\frac{d}{dy}\ln y = a\ln b\frac{d}{dy}x$$
$$\frac{1}{y} = a\ln b\frac{dx}{dy} = \frac{a\ln b}{dy/dx}$$
$$\frac{d}{dx}b^{ax} = ab^{ax}\ln b$$

$$y = b^{ax}$$
$$\ln y = ax\ln b$$
$$\frac{\ln y}{\ln b} = ax$$
$$\log_b y = ax$$
$$\therefore \log_b y = \frac{\ln y}{\ln b} \quad \text{i.e.}$$
$$\log_b x = \frac{\ln x}{\ln b}$$
$$\therefore \frac{d}{dx}\log_b x = \frac{1}{\ln b}\frac{d}{dx}\ln x$$
$$\therefore \frac{d}{dx}\log_b x = \frac{1}{x\ln b}$$

Integrals of exponential and logarithmic functions

Since integration is the inverse of differentiation:

$$\frac{d}{dx}b^{ax} = ab^{ax}\ln b$$

$$\int db^{ax} = a\ln b\int b^{ax}dx$$

$$\therefore \int b^{ax}dx = \frac{b^{ax}}{a\ln b} + c$$

$$\therefore \int e^{ax}dx = \tfrac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

$$\int d\ln|x| = \int \frac{1}{x}dx$$

$$\therefore \int \frac{1}{x}dx = \ln|x| + c$$

We can use the above results to prove a more general theorem:

$$\int \frac{f'(x)}{f(x)}dx$$

$$= \int \frac{1}{f}\frac{df}{dx}dx$$

$$= \int \frac{1}{f}df$$

$$\therefore \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$-\int \frac{\sin x}{\cos x}dx = -\int \tan x dx = \ln|\cos x| + c$$

$$\therefore \int \tan x dx = -\ln|\cos x| + c$$

$$f(x) = x^3 - x^2$$

$$f'(x) = 3x^2 - 2x$$

$$\int \frac{3x^2 - 2x}{x^3 - x^2}dx = \ln|x^3 - x^2| + c$$

We can use *integration by parts* to integrate the logarithm of x

$$\int 1\times\log_b dx = \tfrac{1}{\ln b}\int 1\times\ln dx$$

$$= \tfrac{1}{\ln b}x\ln x - \tfrac{1}{\ln b}\int x\left(\frac{d}{dx}\ln x\right)dx$$

$$= \tfrac{1}{\ln b}x\ln x - \tfrac{1}{\ln b}\int dx$$

$$\therefore \int \log_b dx = \frac{x(\ln x - 1)}{\ln b} + c$$

Summary

$$f(x) = b^{ax} \qquad b > 0$$

$$f^{-1}(x) = \tfrac{1}{a}\log_b x$$

$$\log_b x = \frac{\ln x}{\ln b}$$

$$e = 2.718281828459...$$

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

$$\frac{d}{dx}b^{ax} = ab^{ax}\ln b$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

$$\frac{d}{dx}\log_b|x| = \frac{1}{x\ln b}$$

$$\int e^{ax}dx = \tfrac{1}{a}e^{ax} + c$$

$$\int b^{ax}dx = \frac{b^{ax}}{a\ln b} + c$$

$$\int \log_b dx = \frac{x(\ln x - 1)}{\ln b} + c$$

$$\int \frac{1}{x}dx = \ln|x| + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$