First order 'ordinary' differential equations (ODEs) are of the general form:

$$f\left(x, y, \frac{dy}{dx}\right) = 0$$

The goal is to find a *closed form* expression for *y(x)*

First order ODEs contain *only the first derivative* of the variable *y* so one expects *only one arbitrary constant*, which will result from integration, which must occur to remove the derivative.

Hence to solve the ODE, we will need to know one point on the (x,y) curve.

Case 1: ODE is *separable*

In this situation we can separate the x and y parts and then integrate both sides

$$y^2\frac{dy}{dx} - x^3 = 0$$
$$\int y^2 dy = \int x^3 dx$$
$$\tfrac{1}{3}y^3 = \tfrac{1}{4}x^4 + c$$
$$\boxed{y = \sqrt[3]{\tfrac{3}{4}x^4 + k}}$$

$$\frac{1}{y\sin x}\frac{dy}{dx} - 1 = 0$$
$$\frac{1}{y}\frac{dy}{dx} = \sin x$$
$$\int \tfrac{1}{y} dy = \int \sin x dx$$
$$\ln|y| = -\cos x + c$$
$$\boxed{y = Ae^{-\cos x}}$$

$$e^{-x}\cos y\frac{dy}{dx} - x^2 = 0$$
$$\cos y\frac{dy}{dx} = x^2 e^x$$
$$\int \cos y dy = \int x^2 e^x dx$$
$$\sin y = x^2 e^x - \int e^x(2x)dx$$
$$\sin y = x^2 e^x - \left\{2xe^x - \int e^x(2)dx\right\}$$
$$\sin y = e^x\left(x^2 - 2x + 2\right) + c$$
$$\boxed{y = \sin^{-1}\left\{e^x\left(x^2 - 2x + 2\right) + c\right\}}$$

Case 2: ODE is *linear*

$$a\frac{dy}{dx} + by = q(x)$$

The solution is the solution when $q(x) = 0$ ("The **Complimentary Function**") plus a "**Particular Integral**" which is typically something which has the same form as $q(x)$. If this doesn't work try $xq(x)$, $x^2q(x)$ etc...

$$a\frac{dy}{dx} + by = 0$$
$$a\frac{dy}{dx} = -by$$
$$\int \tfrac{1}{y} dy = -\tfrac{b}{a}\int dx$$
$$\ln|y| = -\tfrac{b}{a}x + c$$
$$\boxed{y = Ae^{-\frac{b}{a}x}}$$

i.e. complimentary function is of the form

$$y = Ae^{\lambda x}$$

Example: $3\frac{dy}{dx} - y = x^2;\ (0,1)$ is a solution

$\text{CF}: y = Ae^{\lambda x}$ Complimentary Function

$$\therefore 3\frac{dy}{dx} - y = 0$$
$$3A\lambda e^{\lambda x} - Ae^{\lambda x} = 0 \Rightarrow Ae^{\lambda x}(3\lambda - 1) = 0$$
$$\Rightarrow \lambda = \tfrac{1}{3}$$

$\text{CF}: y = Ae^{\frac{1}{3}x}$ The single arbitrary constant A appears in the complimentary function

$\text{PI}: y = ax^2 + bx$ Particular Integral

$$\therefore 3(2ax) - ax^2 - bx = x^2$$
$$x^2 : \quad a = 1 \quad \Rightarrow a = -1$$
$$x : \quad 6a - b = 0 \quad \Rightarrow b = -6$$

$$y = Ae^{\frac{1}{3}x} - x^2 - 6x$$
$$\therefore x = 0, y = 1 \therefore A = 1$$
$$\boxed{y = e^{\frac{1}{3}x} - x^2 - 6x}$$

Case 3: Using Integrating Factors

The method of *Integrating Factors* can be used to solve first order ODEs of the form:

$$\frac{dy}{dx} + yp(x) = q(x)$$

$$u = e^{\int p(x)dx} \quad \text{Integrating Factor}$$

$$\therefore \frac{du}{dx} = pu$$

Multiply both sides of the equation by the integrating factor:

$$u\frac{dy}{dx} + ypu = qu$$

Now: $$\frac{d}{dx}(uy) = u\frac{dy}{dx} + y\frac{du}{dx} = u\frac{dy}{dx} + ypu$$

$$\therefore \frac{d}{dx}(uy) = qu$$

$$\therefore uy = \int qudx$$

$$\therefore y = \tfrac{1}{u}\int q(x)u(x)dx$$

Hence a general solution can be found, assuming that the integrals

$e^{\int p(x)dx}$ and $\int q(x)e^{\int p(x)dx}dx$

can both be evaluated.

Example: $\frac{dy}{dx}\cos x + y\sin x = \tan x$

$$\frac{dy}{dx} + y\tan x = \tan x \sec x \leftarrow \text{Write in form } \frac{dy}{dx} + yp(x) = q(x)$$

$$u = e^{\int \tan xdx} \quad \text{Integrating Factor}$$

$$u = e^{\ln \sec x} = \sec x$$

$$\therefore y = \frac{1}{\sec x}\int \tan x \sec x \times \sec xdx \leftarrow y = \tfrac{1}{u}\int q(x)u(x)dx$$

$$y = \frac{1}{\sec x}\int \tan x \sec^2 xdx$$

$$z = \tan x$$

$$\frac{dz}{dx} = \sec^2 x \quad \therefore \sec^2 xdx = dz$$

$$\therefore \int \tan x \sec^2 xdx = \int zdz = \tfrac{1}{2}z^2 + c = \tfrac{1}{2}\tan^2 x + c$$

$$\therefore y = \frac{\sin^2 x}{2\cos x} + c\cos x$$