Fourier Series & Fourier Transforms

The fundamental idea is that any **periodic function** (i.e. one that repeats after a particular interval) can be expressed as an **infinite sum of sine and cosine waves** of different amplitude and frequency. This can be generalized to an *integral* (**The Fourier Transform**), which has wide ranging applications for the solution of **differential equations** which occur in the Physical sciences. (e.g. spectral analysis of sound and light, conduction of heat, diffraction of waves …).

In essence, the **Fourier transform of a time varying signal will yield its harmonic content**, i.e. the frequencies which comprise it. A **Discrete Fourier Transform** (DFT). (which operates on signal amplitudes which are sampled at a fixed rate) can be efficiently implemented in a computer program via a **Fast Fourier Transform** (FFT).

Joseph Fourier
1768-1830

Fourier Series

Consider a function which is periodic after interval L $f(x+L)=f(x)$

Define the infinite series: $$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{2n\pi x}{L}$$

How do we find out what the coefficients a_n and b_n are?

Note the following trigonometric identities:

$$2\cos A\cos B=\cos(A+B)+\cos(A-B)$$
$$2\sin A\cos B=\sin(A+B)+\sin(A-B)$$
$$2\sin A\sin B=\cos(A-B)-\cos(A+B)$$

We firstly need to prove some integral results involving sine and cosines

Hence if n, m are integers, and $n\neq m$

$$I=\int_0^L 2\cos\frac{2n\pi x}{L}\cos\frac{2m\pi x}{L}dx$$
$$I=\int_0^L\left(\cos\frac{2(n+m)\pi x}{L}+\cos\frac{2(n-m)\pi x}{L}\right)dx$$
$$I=\frac{L}{2\pi}\left[\frac{1}{n+m}\sin\frac{2(n+m)\pi x}{L}+\frac{1}{n-m}\sin\frac{2(n-m)\pi x}{L}\right]_0^L$$
$$I=0$$

If $n=m$ $$I=\int_0^L 2\cos^2\frac{2n\pi x}{L}dx=\int_0^L\left(1+\cos\frac{4n\pi x}{L}\right)dx$$
$$I=\left[x+\frac{L}{4n\pi}\sin\frac{4n\pi x}{L}\right]_0^L=L$$

$$\cos^2 x=\tfrac{1}{2}(1+\cos 2x)$$

Hence: $$\int_0^L\cos\frac{2n\pi x}{L}\cos\frac{2m\pi x}{L}dx=\begin{cases}0 & n\neq m\\ \tfrac{1}{2}L & n=m\end{cases}$$

We can repeat this analysis for other combinations of sine and cosine terms

If n, m are integers, and $n\neq m$

$$I=\int_0^L 2\sin\frac{2n\pi x}{L}\cos\frac{2m\pi x}{L}dx$$
$$I=\int_0^L\left(\sin\frac{2(n+m)\pi x}{L}+\sin\frac{2(n-m)\pi x}{L}\right)dx$$
$$I=\frac{L}{2\pi}\left[-\frac{1}{n+m}\cos\frac{2(n+m)\pi x}{L}-\frac{1}{n-m}\cos\frac{2(n-m)\pi x}{L}\right]_0^L$$
$$I=0$$

$$2\cos x\sin x=\sin 2x$$

If $n=m$ $$I=\int_0^L 2\cos\frac{2n\pi x}{L}\sin\frac{2n\pi x}{L}dx=\int_0^L\sin\frac{4n\pi x}{L}dx$$
$$I=\left[-\frac{L}{4n\pi}\cos\frac{4n\pi x}{L}\right]_0^L=-\frac{L}{4n\pi}(1-1)=0$$

$$\therefore\int_0^L\sin\frac{2n\pi x}{L}\cos\frac{2m\pi x}{L}dx=0$$

If n, m are integers, and $n\neq m$

$$I=\int_0^L 2\sin\frac{2n\pi x}{L}\sin\frac{2m\pi x}{L}dx$$
$$I=\int_0^L\left(\cos\frac{2(n-m)\pi x}{L}-\cos\frac{2(n+m)\pi x}{L}\right)dx$$
$$I=\frac{L}{2\pi}\left[\frac{1}{n+m}\sin\frac{2(n+m)\pi x}{L}-\frac{1}{n-m}\sin\frac{2(n-m)\pi x}{L}\right]_0^L$$
$$I=0$$

$$\sin^2 x=\tfrac{1}{2}(1-\cos 2x)$$

If $n=m$ $$I=\int_0^L 2\sin\frac{2n\pi x}{L}\sin\frac{2n\pi x}{L}dx=\int_0^L\left(1-\cos\frac{4n\pi x}{L}\right)dx$$
$$I=\left[x-\frac{L}{4n\pi}\sin\frac{4n\pi x}{L}\right]_0^L=L$$

$$\int_0^L\sin\frac{2n\pi x}{L}\sin\frac{2m\pi x}{L}dx=\begin{cases}0 & n\neq m\\ \tfrac{1}{2}L & n=m\end{cases}$$

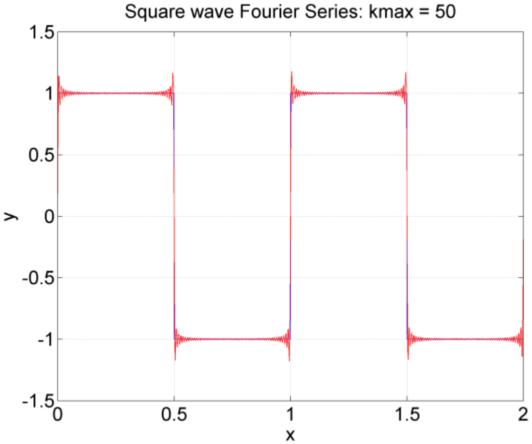
Returning to our **Fourier Series**...

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{L}$$

$$\int_0^L \cos\frac{2n\pi x}{L}\cos\frac{2m\pi x}{L}dx = \begin{cases} 0 & n \neq m \\ \frac{1}{2}L & n = m \end{cases}$$

$$\int_0^L \sin\frac{2n\pi x}{L}\sin\frac{2m\pi x}{L}dx = \begin{cases} 0 & n \neq m \\ \frac{1}{2}L & n = m \end{cases}$$

$$\int_0^L \sin\frac{2n\pi x}{L}\cos\frac{2m\pi x}{L}dx = 0$$



Hence:

$$\int_0^L f(x)\cos\frac{2m\pi x}{L}dx = a_0\int_0^L \cos\frac{2m\pi x}{L}dx + \sum_{n=1}^{\infty} a_n \int_0^L \cos\frac{2m\pi x}{L}\cos\frac{2n\pi x}{L}dx + \sum_{n=1}^{\infty} b_n \int_0^L \cos\frac{2m\pi x}{L}\sin\frac{2n\pi x}{L}dx$$

$$\int_0^L f(x)\cos\frac{2m\pi x}{L}dx = 0 + \tfrac{1}{2}La_m + 0$$

$$\therefore a_n = \tfrac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx$$

$$\int_0^L f(x)\sin\frac{2m\pi x}{L}dx = a_0\int_0^L \sin\frac{2m\pi x}{L}dx + \sum_{n=1}^{\infty} a_n \int_0^L \sin\frac{2m\pi x}{L}\cos\frac{2n\pi x}{L}dx + \sum_{n=1}^{\infty} b_n \int_0^L \sin\frac{2m\pi x}{L}\sin\frac{2n\pi x}{L}dx$$

$$\int_0^L f(x)\cos\frac{2m\pi x}{L}dx = 0 + 0 + \tfrac{1}{2}Lb_m$$

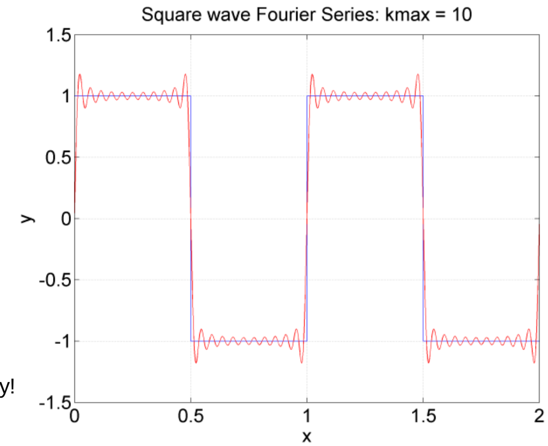
$$\therefore b_n = \tfrac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

$$\int_0^L f(x)dx = a_0\int_0^L dx + \sum_{n=1}^{\infty} a_n \int_0^L \cos\frac{2n\pi x}{L}dx + \sum_{n=1}^{\infty} b_n \int_0^L \sin\frac{2n\pi x}{L}dx$$

$$\int_0^L f(x)\cos\frac{2m\pi x}{L}dx = a_0 L + 0 + 0$$

$$\therefore a_0 = \tfrac{1}{L}\int_0^L f(x)dx$$

Example: Fourier Series of a square wave



Within the interval $0 \leq x \leq L$

$$f(x) = \begin{Bmatrix} 1 & 0 \leq x \leq \frac{1}{2}L \\ -1 & \frac{1}{2}L < x \leq L \end{Bmatrix}$$

$$a_0 = \tfrac{1}{L}\int_0^L f(x)dx$$

$$a_0 = \tfrac{1}{L}\int_0^{\frac{1}{2}L}(1)dx + \tfrac{1}{L}\int_{\frac{1}{2}L}^{L}(-1)dx$$

$$a_0 = \tfrac{1}{2} - \tfrac{1}{2} = 0$$

obvious really from symmetry!

$$a_n = \tfrac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx$$

$$a_n = \tfrac{2}{L}\int_0^{\frac{1}{2}L}\cos\frac{2n\pi x}{L}dx - \tfrac{2}{L}\int_{\frac{1}{2}L}^{L}\cos\frac{2n\pi x}{L}dx$$

$$a_n = \tfrac{2}{L}\frac{L}{2n\pi}\left[\sin\frac{2n\pi x}{L}\right]_0^{\frac{1}{2}L} - \tfrac{2}{L}\frac{L}{2n\pi}\left[\sin\frac{2n\pi x}{L}\right]_{\frac{1}{2}L}^{L}$$

$$a_n = 0$$

$$b_n = \tfrac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

$$b_n = \tfrac{2}{L}\int_0^{\frac{1}{2}L}\sin\frac{2n\pi x}{L}dx - \tfrac{2}{L}\int_{\frac{1}{2}L}^{L}\sin\frac{2n\pi x}{L}dx$$

$$b_n = \tfrac{2}{L}\frac{L}{2n\pi}\left[-\cos\frac{2n\pi x}{L}\right]_0^{\frac{1}{2}L} - \tfrac{2}{L}\frac{L}{2n\pi}\left[-\cos\frac{2n\pi x}{L}\right]_{\frac{1}{2}L}^{L}$$

$$b_n = \frac{1}{n\pi}\left(-\cos n\pi + 1 + \cos 2n\pi - \cos n\pi\right)$$

$$b_n = \begin{cases} \frac{1}{n\pi}\left(-1+1+1-1\right) & n \text{ even} \\ \frac{1}{n\pi}\left(1+1+1+1\right) & n \text{ odd} \end{cases} = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$

$$f(x) = \tfrac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\frac{2(2k+1)\pi x}{L}$$

i.e. replace n with an odd integer *2k + 1*

We can extend the ideas developed for the Fourier Series to define a **Fourier Transform**

$$f(x)=\sum_{n=-\infty}^{\infty} a_n\left(\cos\frac{2n\pi x}{L}+i\sin\frac{2n\pi x}{L}\right)$$

Define this *complex* series

$$f(x)=\sum_{n=-\infty}^{\infty} a_n e^{i\frac{2n\pi x}{L}}$$

← de-Moivre's Theorem $e^{ix}=\cos x+i\sin x$

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L} f(x)e^{-i\frac{2m\pi x}{L}}dx=\sum_{n=-\infty}^{\infty} a_n\int_{-\frac{1}{2}L}^{\frac{1}{2}L} e^{i\frac{2(n-m)\pi x}{L}}dx$$

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L} e^{i\frac{2(n-m)\pi x}{L}}dx=\begin{cases}0 & n\neq m\\ L & n=m\end{cases}$$

$$\therefore \int_{-\frac{1}{2}L}^{\frac{1}{2}L} f(x)e^{-i\frac{2m\pi x}{L}}dx=La_m$$

$$\therefore a_n=\tfrac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} f(x)e^{-i\frac{2n\pi x}{L}}dx$$

Complex Fourier Series

$$f(x)=\sum_{n=-\infty}^{\infty} a_n e^{i\frac{2n\pi x}{L}}$$

$$a_n=\tfrac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} f(x)e^{-i\frac{2n\pi x}{L}}dx$$

Hence:

$$f(x)=\sum_{n=-\infty}^{\infty} a_n e^{i\frac{2n\pi x}{L}}$$

$$f(x)=\sum_{n=-\infty}^{\infty} \tfrac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} f(x)e^{-i\frac{2n\pi x}{L}}dx\times e^{i\frac{2n\pi x}{L}}$$

$$k=\frac{2\pi n}{L},\quad \Delta k=\frac{2\pi}{L}$$

$$f(x)=\tfrac{1}{2\pi}\sum_{n=-\infty}^{\infty}\Delta k\int_{-\frac{1}{2}L}^{\frac{1}{2}L} f(x)e^{-ikx}dx\times e^{ikx}$$

$$L\to\infty$$

$$\Delta k\to 0 \quad \text{i.e.}\quad \Delta k\to dk$$

In this limit we can turn the infinite sum into an **integral**

$$f(x)\to\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)e^{-ikx}dx\right)e^{ikx}dk$$

Define the **Fourier Transform** of *f(x)*

$$\tilde{f}(k)=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

$$\therefore f(x)=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}dk$$

i.e. the **Inverse Fourier Transform**

For analysis of a spatial waveform we can associate k with the **wavenumber**

$$k=\frac{2\pi}{\lambda}$$

For a **time signal**

$$\tilde{f}(\omega)=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega x}dx$$

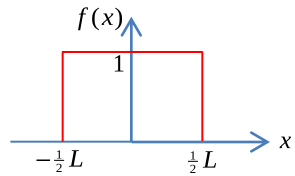
$$\therefore f(t)=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega x}dk$$

$$\omega=\frac{2\pi}{T}$$

Angular frequency → ω; Period → T

Fourier Transform of a 'top hat' function

$$f(x)=\begin{cases}1 & -\frac{1}{2}L\leq x\leq\frac{1}{2}L\\ 0 & |x|>\frac{1}{2}L\end{cases}$$



$$\tilde{f}(k)=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

$$\tilde{f}(k)=\tfrac{1}{\sqrt{2\pi}}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} e^{-ikx}dx$$

$$\tilde{f}(k)=\tfrac{1}{\sqrt{2\pi}}\left[-\frac{1}{ik}e^{-ikx}\right]_{-\frac{1}{2}L}^{\frac{1}{2}L}$$

$$\tilde{f}(k)=-\tfrac{1}{\sqrt{2\pi}}\frac{1}{ik}\left(e^{-ik\frac{1}{2}L}-e^{ik\frac{1}{2}L}\right)$$

$$\tilde{f}(k)=\tfrac{1}{\sqrt{2\pi}}\frac{2}{k}\frac{1}{2i}\left(e^{ik\frac{1}{2}L}-e^{-ik\frac{1}{2}L}\right)$$

$$\tilde{f}(k)=\tfrac{1}{\sqrt{2\pi}}\frac{2\times\frac{1}{2}L}{\frac{1}{2}kL}\sin\left(\tfrac{1}{2}kL\right)$$

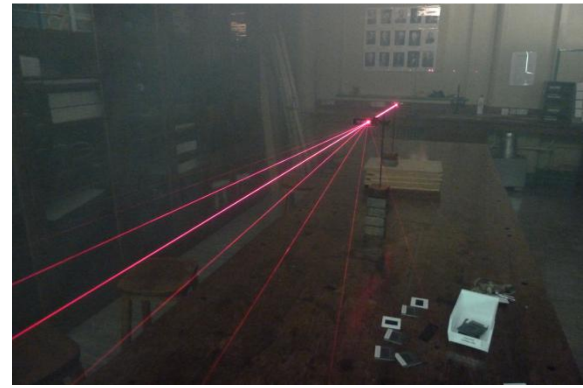
$$\tilde{f}(k)=\tfrac{L}{\sqrt{2\pi}}\operatorname{sinc}\left(\tfrac{1}{2}kL\right)$$

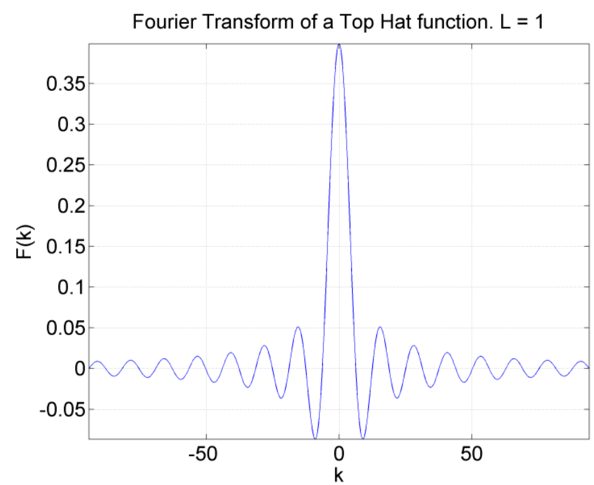
$$\operatorname{sinc}(x)=\frac{\sin x}{x}$$

This result is highly useful in **Optics**. Essentially, the far-field *Fraunhofer* **diffraction pattern** is the **Fourier Transform of the aperture function**

A **diffraction grating** illuminated by a laser will result in a pattern of bright lines angularly separated. The main line spacings are defined by the *separation of the thin slits* which comprise the grating.

The entire pattern will also have an *amplitude envelope* (given by a sinc function) which results from the finite width of the slit. If a slit is uniformly illuminated by the laser, we can model its aperture function by a 'top hat.'





Fourier Transform of a Gaussian function

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$$

$$\int_0^\infty e^{-ax^2}dx=\tfrac12\sqrt{\tfrac{\pi}{a}},\quad a>0$$

$$\int_{-\infty}^{\infty} f(x)dx=\tfrac{1}{\sigma\sqrt{2\pi}}2\int_0^\infty e^{-\frac{x^2}{2\sigma^2}}dx=\tfrac{2}{\sigma\sqrt{2\pi}}\times\tfrac12\times\sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}}=1$$

This **normalization** means the Gaussian function can be used as a *probability distribution*.

$$\tilde f(k)=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

$$\tilde f(k)=\tfrac{1}{2\sigma\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2}e^{-ikx}dx$$

$$\tilde f(k)=\tfrac{1}{2\sigma\pi}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2\sigma^2}x^2-ikx\right)dx$$

$$\tilde f(k)=\tfrac{1}{2\sigma\pi}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2\sigma^2}\left\{x^2+2i\sigma^2kx\right\}\right)dx$$

$$\tilde f(k)=\tfrac{1}{2\sigma\pi}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2\sigma^2}\left\{\left(x+i\sigma^2k\right)^2-i^2\sigma^2k^2\right\}\right)dx$$

$$\tilde f(k)=\tfrac{1}{\sqrt{2\pi}}e^{-\frac12\sigma^2k^2}\ \tfrac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2\sigma^2}\left(x+i\sigma^2k\right)^2\right)dx$$

$$z=x+i\sigma^2k,\ dz=dx$$

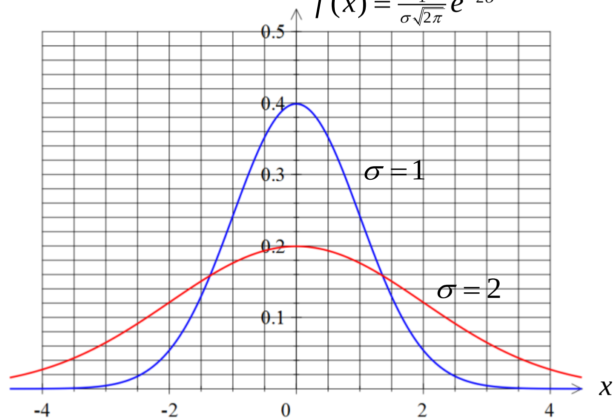
$$\tilde f(k)=\tfrac{1}{\sqrt{2\pi}}e^{-\frac12\sigma^2k^2}\ \underbrace{\tfrac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\tfrac{z^2}{2\sigma^2}\right)dz}_{=1}\ ^{*}$$

$$\tilde f(k)=\tfrac{1}{\sqrt{2\pi}}e^{-\frac12\sigma^2k^2}$$

So the **Fourier Transform of Gaussian function is *also* a Gaussian function.**

Note the **standard deviation** of the Gaussian and its Fourier Transform are *inversely related*. This property underpins the **Uncertainty Principle** in **Quantum Mechanics**, where the *wavefunction* of quantities such as **momentum** are related via a Fourier Transform to the wavefunction of **position**.

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$$



$$f(x)=\tfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$$

$$E[x]=\int_{-\infty}^{\infty}xf(x)dx$$

Since integrand is **odd**

$$E[x]=\tfrac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}xe^{-\frac{x^2}{2\sigma^2}}dx=0$$

$$V[x]=E[x^2]-\left(E[x]\right)^2$$

$$\int_0^\infty x^{2n}e^{-ax^2}dx=\frac{(2n-1)}{a^n2^{n+1}}\sqrt{\frac{\pi}{a}}$$

$$\therefore\int_0^\infty x^2e^{-ax^2}dx=\frac{1}{4a}\sqrt{\frac{\pi}{a}}$$

$$\therefore\int_{-\infty}^\infty x^2e^{-ax^2}dx=\frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

$$V[x]=\tfrac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}x^2e^{-\frac{x^2}{2\sigma^2}}dx$$

$$V[x]=\tfrac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}x^2e^{-\frac{x^2}{2\sigma^2}}dx$$

$$V[x]=\tfrac{1}{\sigma\sqrt{2\pi}}\times\frac{1}{2\times\frac{1}{2\sigma^2}}\sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}}$$

$$V[x]=\tfrac{\sigma^2}{\sigma\sqrt{2\pi}}\times\sigma\sqrt{2\pi}$$

$$V[x]=\sigma^2$$

$$\therefore\text{std}[x]=\Delta x=\sqrt{V[x]}=\sigma$$

So if the **standard deviation** of a Gaussian function is:

$$\Delta x=\sigma$$

The standard deviation of the Fourier Transform of the Gaussian (which is *also* a Gaussian) is

$$\Delta k=\frac{1}{\sigma}$$

This means: $\Delta x\Delta k=1$

In **Quantum Mechanics**, the uncertainties in position and momentum are related to the standard deviations of the probability density functions associated with their wavefunctions. (Actually the *modulus square* of the wavefunction – the **Born Interpretation**).

Now for a particle of wavevector k, its momentum p is:

$$p=\frac{h}{\lambda}=\frac{\hbar}{\lambda/2\pi}$$

de-Broglie relation

$$\therefore p=\hbar k$$

$$\therefore\Delta p\Delta x=\hbar\Delta k\Delta x=\hbar$$

$$\therefore\Delta p\Delta x=\hbar$$

The wavefunctions are not always Gaussian, so the **Uncertainty Principle** is actually an *inequality*

$$\Delta p\Delta x\geq\tfrac12\hbar$$

*Strictly speaking we ought to perform a *contour integral* here, but as in Riley, Hobson, Bence; *Mathematical Methods for Physics and Engineering*, pp436, we shall be cavalier and assume the high frequency oscillations caused by the imaginary terms are negligible!

Summary of Fourier Transform theorems

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$
$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}dk$$

Definition of the **Fourier Transform** and **Inverse Fourier Transform**

$$\Im\left[f(x)\right] = \tilde{f}(k)$$

'Operator' notation for 'find the Fourier Transform of the function *f(x)*'

$$\Im\left[f(x-a)\right] = e^{-iak}\tilde{f}(k)$$
$$\Im\left[e^{iax}f(x)\right] = \tilde{f}(k-a)$$

Fourier Shift Theorem. This is the algorithmic basis for many 'pitch shifting' signal processing effects

$$\Im\left[f(ax)\right] = \frac{1}{|a|}\tilde{f}\left(\tfrac{1}{a}k\right)$$

Scaling theorem

$$\Im\left[\frac{d^n}{dx^n}f(x)\right] = (ik)^n\tilde{f}(k)$$
$$\Im\left[x^n f(x)\right] = i^n\frac{d^n}{dk^n}\tilde{f}(k)$$

These theorems involving derivatives provide a powerful tool for solving **differential equations**. Applying a Fourier Transform will turn a differential equation into an *algebraic* one. Once solved, application of the Inverse Fourier Transform will yield the solution.

$$\Im\left[f(x)*g(x)\right] = \sqrt{2\pi}\,\tilde{f}(k)\tilde{g}(k)$$
$$\Im\left[f(x)g(x)\right] = \frac{1}{\sqrt{2\pi}}\tilde{f}(k)*\tilde{g}(k)$$

Convolution Theorem. This is very useful in signal processing. e.g. 'Pulse Compression' in Radar applications.

$$f(x)*g(x) = \int_{-\infty}^{\infty} f(z)g(x-z)dz$$

$$\int_{-\infty}^{\infty} f(x)g^*(x)dx = \int_{-\infty}^{\infty}\tilde{f}(k)\tilde{g}^*(k)dk$$
$$\int_{-\infty}^{\infty}\left|f(x)\right|^2dx = \int_{-\infty}^{\infty}\left|\tilde{f}(k)\right|^2dk$$

Parseval relations

$$f(x) = g(x)$$

i.e. the area under the **Power Spectrum** is the same as the integral of signal power. This must be true since energy is conserved!

Note the **Fourier Transform** is a special case of a general class of **Integral Transforms.** Another, which is popular in signal processing, is the **Laplace Transform**.

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

The Discrete Fourier transform and spectral analysis

If a signal *x(t)* is sampled at rate f_s, we can define a **digital filter**, which can be used to extract useful features, while perhaps inhibiting others.

$$y=\sum_{n=1}^{N} w_n x_n$$

y is the filter output, which combines *n* subsequent *weighted* samples of the signal.

A **Digital Fourier Transform (DFT)** is a *filter bank* with **filter weights**

$$w_{n,k}=\tfrac{1}{N}e^{-\frac{2\pi i(n-1)(k-1)}{N}}$$

The filter has a **frequency response** (i.e. the output vs frequency due to N samples of a pure sinusoid)

$$t_n=\frac{(n-1)}{f_s}$$

$$x_n=e^{2\pi i f t_n}$$

$$y_k(f)=\sum_{n=1}^{N} w_{n,k}x_n$$

$$y_k(f)=\tfrac{1}{N}\sum_{n=1}^{N} e^{-\frac{2\pi i(n-1)(k-1)}{N}+\frac{2\pi i f(n-1)}{f_s}}$$

This can be simplified by considering the sum of a geometric progression

$$a+ar+ar^2+.....+ar^{N-1}=\sum_{n=1}^{N} ar^{n-1}$$

$$a+ar+ar^2+.....+ar^{N-1}=a\frac{1-r^N}{1-r}$$

$$\therefore \sum_{n=1}^{N} ar^{n-1}=a\frac{1-r^N}{1-r}$$

$$y_k(f)=\tfrac{1}{N}\sum_{n=1}^{N}\left(e^{-\frac{2\pi i(k-1)}{N}+\frac{2\pi i f}{f_s}}\right)^{n-1} \quad \therefore y_k(f)=\tfrac{1}{N}\frac{1-e^{-2\pi i(k-1)+\frac{2\pi i f N}{f_s}}}{1-e^{-\frac{2\pi i(k-1)}{N}+\frac{2\pi i f}{f_s}}}=\frac{e^{\frac{\pi i f N}{f_s}-\pi i(k-1)}\left(e^{-\frac{\pi i f N}{f_s}+\pi i(k-1)}-e^{\frac{\pi i f N}{f_s}-\pi i(k-1)}\right)}{e^{\frac{\pi i f}{f_s}-\frac{\pi i(k-1)}{N}}\left(e^{\frac{\pi i(k-1)}{N}-\frac{\pi i f}{f_s}}-e^{\frac{\pi i f}{f_s}-\frac{\pi i(k-1)}{N}}\right)}$$

$$y_k(f)=e^{\frac{\pi i f N}{f_s}-\pi i(k-1)-\frac{\pi i f}{f_s}+\frac{\pi i(k-1)}{N}}\frac{\sin\left(\pi(k-1)-\frac{\pi f N}{f_s}\right)}{N\sin\left(\frac{\pi(k-1)}{N}-\frac{\pi f}{f_s}\right)}=e^{\pi i(k-1)\left(\frac{1}{N}-1\right)+\frac{\pi i f}{f_s}(N-1)}\frac{\sin\left(\pi(k-1)-\frac{\pi f N}{f_s}\right)}{N\sin\left(\frac{\pi(k-1)}{N}-\frac{\pi f}{f_s}\right)}$$

$$\therefore |y_k(f)|^2=\frac{\sin^2\left(\pi(k-1)-\frac{\pi f N}{f_s}\right)}{N^2\sin^2\left(\frac{\pi(k-1)}{N}-\frac{\pi f}{f_s}\right)}$$

Maximum response when:

$$\frac{\pi(k-1)}{N}-\frac{\pi f'}{f_s}=0$$

$$\therefore f'=\frac{(k-1)}{N}f_s$$

Let the outputs of the filter bank correspond to a desired range of frequencies, with a maximum response at *these* frequencies.

The filter frequency response is ambiguous (i.e. repeats) every frequency interval of f_s , so the maximum value can be *at most* the sample rate.

So for the DFT, define:

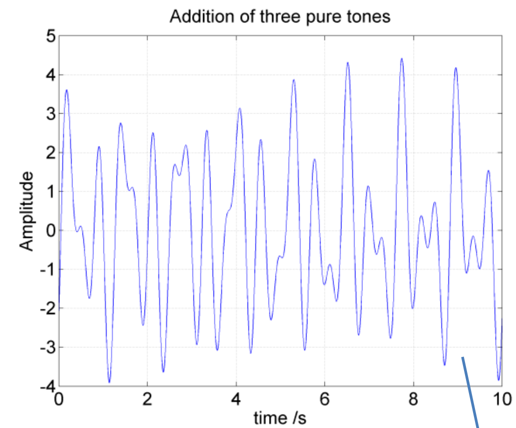
$$\frac{(k-1)}{N}=\frac{f}{f_s}$$

This is called the **Nyquist Criterion**

$$\therefore w_n(f)=\tfrac{1}{N}e^{-2\pi i(n-1)\frac{f}{f_s}}$$

$$\therefore y(f)=\tfrac{1}{N}\sum_{n=1}^{N} x_n e^{-2\pi i(n-1)\frac{f}{f_s}}$$

The latter is essentially a description of the DFT



Example signal – a sum of three sinusoids

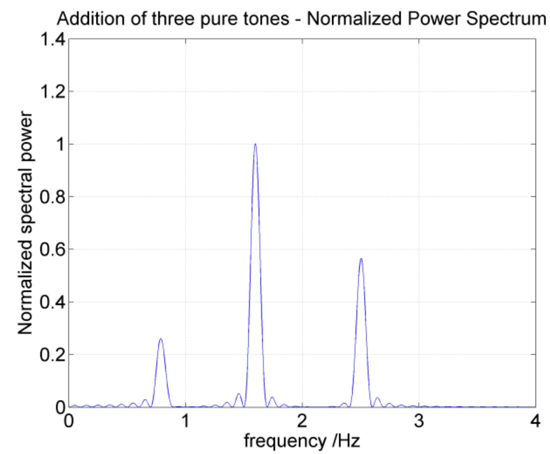
$$x(t)=\sum_{m=1}^{3} a_m\sin(2\pi f_m t-\phi_m)$$

$$\{a_m\}=\{1,2,1.5\}$$

$$\{f_m\}=\{0.8,1.6,2.5\}$$

$$\{\phi_m\}=\{0,\tfrac{\pi}{4},\tfrac{\pi}{6}\}$$

DFT



From the Power Spectrum it is easy to read off the component frequencies and their respective magnitudes.

Note if the Power Spectrum peaks need to be sharper, more samples N required.

If the sample rate is *increased*, the spectral peaks will *broaden*, so to sharpen them, you have to sample for *longer*